

Review Counting Principles

1) The Sum Rule:

If  $S_1$  and  $S_2$  are disjoint sets, then the number of members of  $S_1 \cup S_2$  is  $|S_1 \cup S_2| = |S_1| + |S_2|$

2) Inclusion-Exclusion Principle:

If  $S_1$  and  $S_2$  are arbitrary sets, then  $|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$

3) The Product Rule:

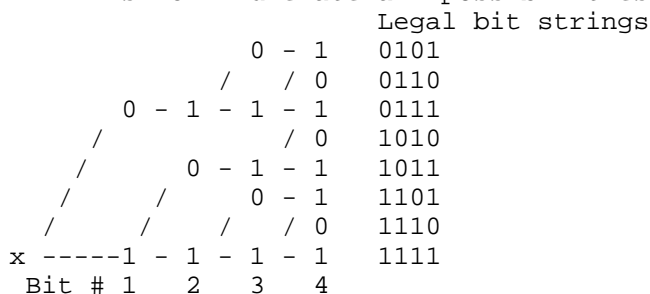
If  $S_1$  and  $S_2$  are sets, and  $S_1 \times S_2 = \{(s_1, s_2) : s_1 \text{ in } S_1 \text{ and } s_2 \text{ in } S_2\}$  is the Cartesian product of  $S_1$  and  $S_2$ , then  $|S_1 \times S_2| = |S_1| \times |S_2|$

If  $S_1, S_2, \dots, S_k$  are sets,  $S_1 \times S_2 \times \dots \times S_k$  the Cartesian product, then  $|S_1 \times \dots \times S_k| = |S_1| \times \dots \times |S_k|$

4) Tree diagrams

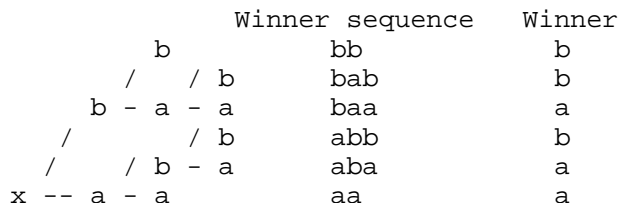
EX: How many bit strings of length 4 without two consecutive zeros?

Ans: 8; Enumerate all possibilities in a tree:



Tree Diagrams (formally) Draw a tree where the children of each node represent all the possible values of the next entry

EX: How many 2-out-of-3 game playoffs are there? Ans: 6



(5) Pigeonhole Principle:

If  $k+1$  or more objects (pigeons) are placed in  $k$  boxes (holes), then at least one box contains 2 or more objects.

(proof by contradiction: if each box had at most one object, there would only be  $k$  or fewer objects, a contradiction)

EX: In any group of 27 English words, at 2 begin with the same letter, since there are only 26 letters.



EX: How many different desserts can you make out of 4 scoops of ice cream, each of which may be chocolate (C), vanilla (V) or strawberry (S)?

Here are the 15 possibilities:

CCCC VVVV SSSS  
CCCV VVVC SSSC  
CCCS VVVS SSSV  
CCVS VVCS SSCV  
CCVV VVSS  
CCSS

Here is a more systematic way to get the answers: we will represent each dessert by a sequence of 4 stars (representing the 4 scoops) and 2 bars (dividing the stars into 3 groups: C, V and S). Here are some examples:

\*\*|\*|\* represents 2 Cs, 1 V and 1 S  
\*|\*\*|\* represents 1 C, 2 V's and 1 S  
\*|\*\*\*| represents 1 C, 3 V's and 0 S's  
|\*\*\*\*| represents 0 C, 4 V's and 0 S's  
||\*\*\*\* represents 0 C, 0 V's and 4 S's etc

The idea is that every sequence of 4 stars and 2 bars represents exactly one dessert. How many such sequences are there? The idea is that we take 6 possible positions (for 4 stars and 2 bars) and choose 2 of them for bars. There are  $C(6,2) = 6!/(2! 4!) = 15$  ways to do this.

Here is the general result:

Theorem: Suppose I have  $n$  types of objects ("flavors"). How many different sets ("desserts") consisting of  $r$  objects ("scoops") are there?

The answer is  $C(n+r-1, n-1)$ .

Proof: The idea is the same as before: each sequence of  $r$  stars ("scoops") and  $(n-1)$  bars represents a possible set. There are  $C(n+r-1, n-1)$  ways to pick  $n-1$  places out of  $r+n-1$  locations to put the bars.

Ex: If I have  $n=3$  flavors of ice cream, and make desserts of  $r=4$  scoops, there are  $C(n+r-1, n-1) = C(3+4-1, 3-1) = C(6, 2) = 16$  different desserts.

EX: How many anagrams are there of the word "mammal"?

Recall that an anagram is a distinct ordering of the letters.

Here are some smaller examples:

the word "the": The 6 anagrams are the, teh, eth, eht, het, hte

the word "see": The 3 anagrams are see, ese, ees

Here are different ways to try to solve this problem for the word "mammal", followed by the general result:

Solution 1: Pick 3 locations for the m's

Pick 2 of the remaining locations for the 2 a's

Pick the remaining location for l

By the product rule, the number of ways to pick locations is

$$\begin{aligned} & C(6,3) \quad \dots \text{ for the m's} \\ * & C(3,2) \quad \dots \text{ for the a's} \\ * & C(1,1) \quad \dots \text{ for the l} \\ & = 20*3*1 = 60 \end{aligned}$$

Solution 2: Pick 1 location for the l

Pick 3 of the remaining locations for the m's

Pick the remaining 2 locations for the a's

By the product rule, the number of ways to pick locations is

$$\begin{aligned} & C(6,1) \quad \dots \text{ for the l} \\ * & C(5,3) \quad \dots \text{ for the m's} \\ * & C(2,2) \quad \dots \text{ for the a's} \\ & = 6*10*1 = 60, \text{ the same answer (whew!)} \end{aligned}$$

Solution 3: Let us start by labeling the m's as m1,m2 and m3, and

and the a's as a1 and a2, so we can distinguish them.

So now we have 6 distinct symbols, m1,a1,m2,m3,a2,l,

and the number of ways to order them is 6!.

But clearly we have counted some ordering as distinct that we should not, so let's try to divide out by the number of multiple copies.

For example, consider all the orderings where the first

3 characters are m's, and the last three are a1,a2,l.

They are clearly  $3! = 6$  such orderings, since m1,m2,m3 can appear in the first three positions in any order, but yield the same anagram. This argument that we are counting each anagram  $3!$  times works no matter where the 3 m's appear, so we should divide the number of orderings by  $3!$  to account for the 3 m's.

Similarly, we should divide by  $2!$  to account for the two a's.

This yields  $6! / (3! 2!) = 60$ , the same answer (whew!)