Introduction to Mathematical Modeling of Signals and Systems

Mathematical Representation of Signals

- Signals represent or encode information
  - In communications applications the information is almost always encoded
  - In the probing of medical and other physical systems, where signals occur naturally, the information is not purposefully encoded
  - In human speech we create a waveform as a function of time when we force air across our vocal cords and through our vocal tract

A microphone has converted the sound pressure from the vocal tract into an electrical signal that varies over time, $t$
• Signals, such as the above speech signal, are continuous functions of time, and denoted as a *continuous-time signal*

• The independent variable in this case is time, \( t \), but could be another variable of interest, e.g., position, depth, temperature, pressure

• The mathematical notation for the speech signal recorded by the microphone might be \( s(t) \)

• In order to process this signal by computer means, we may *sample* this signal at regular interval \( T_s \), resulting in

\[
s[n] = s(nT_s)
\]  

(1.2)

• The signal \( s[n] \) is known as a *discrete-time signal*, and \( T_s \) is the *sampling period*

  – Note that the independent variable of the sampled signal is the integer sequence \( n \in \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \)

  – Discrete-time signals can only be evaluated at integer values

![Samples of a Speech Waveform: \( s[n] = s(nT_s) \) ![Image of a speech waveform with sample index \( n \) ranging from 0 to 200, signal values ranging from -2000 to 2000.]}
• The speech waveform is an example of a one-dimensional signal, but we may have more that one dimension

• An image, say a photograph, is an example of a two-dimensional signal, being a function of two spatial variables, e.g. \( p(x, y) \)

• If the image is put into motion, as in a movie or video, we now have a three-dimensional image, where the third independent variable is time, \((x, y, t)\)
  
  – Note: movies and videos are shot in frames, so actually time is discretized, e.g., \( t \rightarrow nT_s \) (often \( 1/T_s = 30 \) fps)

• To manipulate an image on a computer we need to sample the image, and create a two-dimensional discrete-time signal

\[
p[m, n] = p(m\Delta x, n\Delta y)
\]  
(1.3)

where \( m \) and \( n \) takes on integer values, and \( \Delta x \) and \( \Delta y \) represent the horizontal and vertical sampling periods respectively

### Mathematical Representation of Systems

• In mathematical modeling terms a system is a function that transforms or maps the input signal/sequence, to a new output signal/sequence

\[
y(t) = T_c\{x(t)\}
\]

\[
y[n] = T_d\{x[n]\}
\]  
(1.4)

where the subscripts \( c \) and \( d \) denote continuous and discrete system operators
• Because we are at present viewing the system as a pure mathematical model, the notion of a system seems abstract and distant

• Consider the microphone as a system which converts sound pressure from the vocal tract into an electrical signal

• Once the speech waveform is in an electrical waveform format, we might want to form the square of the signal as a first step in finding the energy of the signal, i.e.,

\[ y(t) = [x(t)]^2 \]  

(1.5)

The squarer system also exists for discrete-time signals, and in fact is easier to implement, since all we need to do is multiply each signal sample by itself.
\[ y[n] = (x[n])^2 = x[n] \cdot x[n] \]  \hspace{1cm} (1.6)

- If we send \( y[n] \) through a second system known as a digital filter, we can form an estimate of the signal energy
  - This is a future topic for this course

**Thinking About Systems**

- Engineers like to use block diagrams to visualize systems
- Low level systems are often interconnected to form larger systems or subsystems
- Consider the squaring system

\[
\begin{align*}
x(t) & \rightarrow T\{ \} \rightarrow y(t) \\
(\cdot)^2 & \rightarrow y(t)
\end{align*}
\]

- The ideal sampling operation, described earlier as a means to convert a continuous-time signal to a discrete-times signal is represented in block diagram form as an ideal C-to-D converter

\[
\begin{align*}
x(t) & \rightarrow \text{Ideal C-to-D Converter} \rightarrow x[n] = x(nT_s) \\
T_s & \text{A system parameter that specifies the sample spacing}
\end{align*}
\]
• A more complex system, depicted as a collection of subsystem blocks, is a system that records and then plays back an audio source using a compact disk (CD) storage medium.

• The optical disk reader shown above is actually a high-level block, as it is composed of many lower-level subsystems, e.g.,
  – Laser, on a sliding carriage, to illuminate the CD
  – An optical detector on the same sliding carriage
  – A servo control system positions the carriage to follow the track over the disk
  – A servo speed control to maintain a constant linear velocity as 1/0 data is read from different portions of the disk
  – more ...

The Next Step
• Basic signals, composed of linear combinations of trigonometric functions of time will be studied next
• We also consider complex number representations as a means to simplify the combining of more than one sinusoidal signal
Sinusoids

• A general class of signals used for modeling the interaction of signals in systems, are based on the trigonometric functions sine and cosine

• The general mathematical form of a single sinusoidal signal is

\[ x(t) = A \cos(\omega_0 t + \phi) \]  

(2.1)

where \( A \) denotes the amplitude, \( \omega_0 \) is the frequency in radians/s (radian frequency), and \( \phi \) is the phase in radians

– The arguments of \( \cos(\ ) \) and \( \sin(\ ) \) are in radians

• We will spend considerable time working with sinusoidal signals, and hopefully the various modeling applications presented in this course will make their usefulness clear

Example: \( x(t) = 10 \cos[2\pi(440)t - 0.4\pi] \)

• The pattern repeats every \( 1/440 = 0.00227 = 2.27 \text{ms} \)

• This time interval is known as the period of \( x(t) \)
The text discusses how a tuning fork, used in tuning musical instruments, produces a sound wave that closely resembles a single sinusoid signal.

- In particular the pitch \( A \) above middle \( C \) has an oscillation frequency of 440 hertz

### Review of Sine and Cosine Functions

- Trigonometric functions were first encountered in your K–12 math courses.

- The typical scenario to explain sine and cosine functions is depicted below

\[
\sin \theta = \frac{y}{r} \quad \Rightarrow \quad y = r \sin \theta
\]
\[
\cos \theta = \frac{x}{r} \quad \Rightarrow \quad x = r \cos \theta
\]

- The right-triangle formed in the first quadrant has sides of length \( x \) and \( y \), and hypotenuse of length \( r \)

- The angle \( \theta \) has cosine defined as \( x/r \) and sine defined as \( y/r \)

- The above graphic also shows how a point of distance \( r \) and angle \( \theta \) in the first quadrant of the \( x-y \) plane is related.
to the $x$ and $y$ coordinates of the point via $\sin(\ )$ and $\cos(\ )$, e.g.,

$$ (x, y) = (r \cos \theta, r \sin \theta) \quad (2.2) $$

- Moving beyond the definitions and geometry interpretations, we now consider the signal/waveform properties

- The function plots are identical in shape, with the sine plot shifted to the right relative to the cosine plot by $\pi/2$
- This is expected since a well known trig identity states that

$$ \sin \theta = \cos(\theta - \pi/2) \quad (2.3) $$

- We also observe that both waveforms repeat every $2\pi$ radians; read period $= 2\pi$
- Additionally the amplitude of each ranges from -1 and 1

- A few key function properties and trigonometric identities
are given in the following tables

Table 2.1: Some sine and cosine properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equivalence</td>
<td>$\sin \theta = \cos(\theta - \pi/2)$ or $\cos \theta = \sin(\theta + \pi/2)$</td>
</tr>
<tr>
<td>Periodicity</td>
<td>$\cos(\theta - 2\pi k) = \cos \theta$, when $k$ is an integer; holds for sine also</td>
</tr>
<tr>
<td>Evenness of cosine</td>
<td>$\cos(-\theta) = \cos \theta$</td>
</tr>
<tr>
<td>Oddness of sine</td>
<td>$\sin(-\theta) = -\sin \theta$</td>
</tr>
</tbody>
</table>

Table 2.2: Some trigonometric identities

<table>
<thead>
<tr>
<th>Number</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\sin^2 \theta + \cos^2 \theta = 1$</td>
</tr>
<tr>
<td>2</td>
<td>$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$</td>
</tr>
<tr>
<td>3</td>
<td>$\sin 2\theta = 2\sin \theta \cos \theta$</td>
</tr>
<tr>
<td>4</td>
<td>$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$</td>
</tr>
<tr>
<td>5</td>
<td>$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$</td>
</tr>
<tr>
<td>6</td>
<td>$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$</td>
</tr>
<tr>
<td>7</td>
<td>$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$</td>
</tr>
</tbody>
</table>

- For more properties consult a math handbook
• The relationship between sine and cosine show up in calculus too, in particular

\[
\frac{d\sin\theta}{d\theta} = \cos\theta \quad \text{and} \quad \frac{d\cos\theta}{d\theta} = -\sin\theta \tag{2.4}
\]

– This says that the slope at any point on the sine curve is the cosine, and the slope at any point on the cosine curve is the negative of the sine

Example: Prove Identity #6 Using Identities #1 and #2
• If we add the left side of 1 to the right side of 2 we get

\[
2\cos^2\theta = 1 + \cos 2\theta
\]

or \[
\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta) \tag{2.5}
\]

Example: Find an expression for \(\cos 8\theta\) in terms of \(\cos 9\theta\), \(\cos 7\theta\), and \(\cos \theta\) using #5
• Let \(\alpha = 8\theta\) and \(\beta = \theta\), then write out #5 under both sign choices

\[
\cos(8\theta + \theta) = \cos 8\theta \cos \theta - \sin 8\theta \sin \theta
\]

\[
+ \cos(8\theta - \theta) = \cos 8\theta \cos \theta + \sin 8\theta \sin \theta
\]

\[
\cos 9\theta + \cos 7\theta = 2\cos 8\theta \cos \theta \tag{2.6}
\]

or

\[
\cos 8\theta = \frac{\cos 9\theta + \cos 7\theta}{2\cos \theta} \tag{2.7}
\]
Review of Complex Numbers

• See Appendix A of the text for more information

• A complex number is *an ordered pair of real numbers*\(^1\) denoted \(z = (x, y)\)
  
  – The first number, \(x\), is called the real part, while the second number, \(y\), is called the imaginary part

  – For algebraic manipulation purposes we write \((x, y)\) = \(x + iy = x + jy\) where \(i = j = \sqrt{-1}\); electrical engineers typically use \(j\) since \(i\) is often used to denote current

  **Note:** \(\sqrt{-1} \times \sqrt{-1} = -1 \Rightarrow j \times j = -1\)

• The *rectangular form* of a complex number is as defined above,

  \[ z = (x, y) = x + jy \]

• The corresponding *polar form* is

  \[ z = re^{j\theta} = r\angle\theta = |z|e^{j\text{arg}z} \]

---

• We can plot a complex number as a vector \((x, y)\)

\[
\begin{align*}
\text{Tail} & \quad 0 \\
\text{Head} & \quad z = x + jy \\
\text{Length} & \quad r = |z| \\
\text{Direction} & \quad \theta = \arg(z)
\end{align*}
\]

**Example:** \(z = 2 + j5, z = 4 - j3, z = -5 + j0, z = -3 - j3\)
Example: \( z = 2 \angle 45^\circ, z = 3 \angle 150^\circ, \& z = 3 \angle -80^\circ \)

- For complex numbers \( z_1 = x_1 + jy_1 \) and \( z_2 = x_2 + jy_2 \) we define/calculate
  
  \[
  z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \quad \text{(sum)}
  \]
  
  \[
  z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \quad \text{(difference)}
  \]
  
  \[
  z_1 z_2 = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + y_1 x_2) \quad \text{(product)}
  \]
  
  \[
  \frac{z_1}{z_2} = \frac{(x_1 x_2 + y_1 y_2) - j(x_1 y_2 - y_1 x_2)}{x_2^2 + y_2^2} \quad \text{(quotient)}
  \]
\[ |z_1| = \sqrt{x_1^2 + y_1^2} \text{ (magnitude)} \]
\[ \angle z_1 = \tan^{-1}\left(\frac{y_1}{x_1}\right) \text{ (angle)} \]
\[ z_1^* = x_1 - jy_1 \text{ (complex conjugate)} \]

MATLAB is also consistent with all of the above, starting with the fact that \( i \) and \( j \) are predefined to be \( \sqrt{-1} \)

- To convert from polar to rectangular we can use simple trigonometry to show that
  \[ x = r \cos \theta \]
  \[ y = r \sin \theta \]  \hspace{1cm} (2.24)

- Similarly we can show that rectangular to polar conversion is
  \[ r = \sqrt{x^2 + y^2} \]
  \[ \theta = \tan^{-1}\left(\frac{y}{x}\right), \text{ note add } \pm \pi \text{ outside Q1 & Q4} \]  \hspace{1cm} (2.25)
Example: Rect to Polar and Polar to Rect

- Consider $z_1 = 2 + j5$
  - In MATLAB we simply enter the numbers directly and then need to use the functions \texttt{abs()} and \texttt{angle()} to convert
    
    $\texttt{>> z1 = 2 + j*5}$
    
    $z1 = 2.0000e+00 + 5.0000e+00i$
    
    $\texttt{>> [abs(z1) angle(z1)]}$
    
    $\texttt{ans} = 5.3852e+00 \quad 1.1903e+00 \% \text{mag \& phase in rad}$
  - Using say a TI-89 calculator is similar

- Consider $z_2 = 2 \angle 45^\circ$
  - In MATLAB we simply enter the numbers directly as a complex exponential
    
    $\texttt{>> z2 = 2*exp(j*45*pi/180)}$
    
    $z2 = 1.4142e+00 + 1.4142e+00i$
– Using the TI-89 we can directly enter the polar form using the angle notation or using a complex exponential

Example: Complex Arithmetic

• Consider \( z_1 = 1 + j7 \) and \( z_2 = -4 - j9 \)

• Find \( z_1 + z_2 \)
  
  \[
  \begin{align*}
  \gg z1 &= 1+j*7; \\
  \gg z2 &= -4-j*9; \\
  \gg z1+z2 \\
  \end{align*}
  \]

  \[
  \text{ans} = -3.0000e+00 - 2.0000e+00i
  \]

– Using the TI-89 we obtain
• Find $z_1 z_2$
  >> z1*z2

  \[
  \text{ans} = 5.9000e+01 - 3.7000e+01i
  \]

– Using the TI-89 we obtain

  \[
  \text{ans} = -6.9072e-01 - 1.9588e-01i
  \]

• Find $z_1 / z_2$
  >> z1/z2

  \[
  \text{ans} = -6.9072e-01 - 1.9588e-01i
  \]

**Euler’s Formula:** A special mathematical result, of special importance to electrical engineers, is the fact that

\[
e^{j\theta} = \cos \theta + j \sin \theta \quad (2.26)
\]
• Turning (2.26) around yields (inverse Euler formulas)

\[
\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad \text{and} \quad \cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}
\]  
(2.27)

• It also follows that

\[
z = x + jy = r \cos \theta + jr \sin \theta
\]  
(2.28)

**Sinusoidal Signals**

• A general sinusoidal function of time is written as

\[
x(t) = A \cos(\omega_0 t + \phi) = A \cos(2\pi f_0 t + \phi)
\]  
(2.29)

where in the second form \( \omega_0 = 2\pi f_0 \)

• Since \( |\cos \theta| \leq 1 \) it follows that \( x(t) \) swings between \( \pm A \), so the amplitude of \( x(t) \) is \( A \)

• The phase shift in radians is \( \phi \), so if we are given a sine signal (instead of the cosine version), we see via the equivalence property that

\[
x(t) = A \sin(\omega_0 t + \phi') = A \cos(\omega_0 t + \phi' - \pi/2)
\]  
(2.30)

which implies that \( \phi = \phi' - \pi/2 \)

• Engineers often prefer the second form of (2.8) where \( f_0 \) is the oscillation frequency in cycles/s.

\[
\frac{\omega_0 \text{ (rad/s)}}{2\pi \text{ (rad)}} = f_0 \text{ (sec}^{-1})
\]
Example: $x(t) = 20 \cos[2\pi(40)t - 0.4\pi]$

- Clearly, $A = 20$, $f_0 = 40$ cycles/s, and $\phi = -0.4\pi$ rad

- Since this signal is periodic, the time interval between maxima, minima, and zero crossings, for example, are identical

Relation of Frequency to Period

- A signal is periodic if we can write
  \[
x(t + T_0) = x(t)
  \]
  where the smallest $T_0$ satisfying (2.10) is the period

- For a single sinusoid we can relate $T_0$ to $f_0$ by considering
  \[
x(t + T_0) = x(t) = A \cos(\omega_0(t + T_0) + \phi) = A \cos(\omega_0 t + \phi)
  \]
  \[
  \cos(\omega_0 t + \phi + \omega_0 T_0) = \cos(\omega_0 t + \phi)
  \]

- From the periodicity property of cosine, equality is maintained if $\cos(\theta \pm 2\pi k) = \cos(\theta)$, so we need to have
So we see that $T_0$ and $f_0$ are reciprocals, with the units of $T_0$ being time and the units of $f_0$ inverse time or cycles per second, as stated earlier.

- In honor of Heinrich Hertz, who first demonstrated the existence of radio waves, cycles per second is replaced with Hertz (Hz)
Example: \( 5 \cos(2\pi f_0 t) \) with \( f_0 = 200, 100, \) and \( 0 \) Hz

- The inverse relationship between time and frequency will be explored throughout this course.
Phase Shift and Time Shift

- We know that the phase shift parameter in the sinusoid moves the waveform left or right on the time axis.
- To formally understand why this is, we will first form an understanding of time-shifting in general.
- Consider a triangularly shaped signal having piecewise continuous definition:

\[ s(t) = \begin{cases} 
2t, & 0 \leq t \leq 1/2 \\
\frac{1}{3}(4-2t), & 1/2 \leq t \leq 2 \\
0, & \text{otherwise}
\end{cases} \]  

Now we wish to consider the signal \( x_1(t) = s(t - 2) \).

- As a starting point we note that \( s(t) \) is active over just the interval \( 0 \leq t \leq 2 \), so with \( t \to t - 2 \) we have

\[ 0 \leq (t - 2) \leq 2 \Rightarrow 2 \leq t \leq 4 \]  

which means that \( x_1(t) \) is active over \( 2 \leq t \leq 4 \).

- The piecewise definition of \( x_1(t) \) can be obtained by direct substitution of \( t - 2 \) everywhere \( t \) appears in (2.34)
In summary we see that the original signal $s(t)$ is moved to the right by 2 s

**Example: Plot $s(t + 1)$**

- With $t \to t + 1$ we expect that the signal will shift to the left by one second
• The new equations are obtained as before
  \[ 0 \leq (t + 1) \leq 2 \Rightarrow -1 \leq t \leq 1 \]  
  so
  \[
s(t + 1) = \begin{cases} 
    2(t + 1), & 0 \leq (t + 1) \leq 1/2 \\
    \frac{1}{3} (4 - 2(t + 1)), & 1/2 \leq (t + 1) \leq 2 \\
    0, & \text{otherwise}
  \end{cases}
\]
  \[ (2.38) \]

  \[
  = \begin{cases} 
    2t + 2, & -1 \leq t \leq -1/2 \\
    \frac{1}{3} (2 - 2t), & -1/2 \leq t \leq 1 \\
    0, & \text{otherwise}
  \end{cases}
\]

• Modeling time shifted signals shows up frequently

• In general terms we say that
  \[ x_1(t) = s(t - t_1) \]  
  is *delayed* in time relative to \( s(t) \) if \( t_1 > 0 \), and *advanced* in time relative to \( s(t) \) if \( t_1 < 0 \)

• A cosine signal has positive peak located at \( t = 0 \)

• If this signal is delayed by \( t_1 \) the peak shifts to the right and the corresponding phase shift is negative

• Consider \( x_0(t) = A \cos(\omega_0 t) \)
\[ x_0(t - t_1) = A \cos[\omega_0(t - t_1)] = A \cos[\omega_0 t - \omega_0 t_1] \quad (2.40) \]

which implies that in terms of phase shift we have \( \phi = -\omega_0 t_1 \)

- For a given phase shift we can turn the above analysis around and solve for the time delay via

\[ t_1 = -\frac{\phi}{\omega_0} = -\frac{\phi}{2\pi f_0} \quad (2.41) \]

- Since \( T_0 = 1/f_0 \), we can also write the phase shift in terms of the period

\[ \phi = -2\pi f_0 t_1 = -2\pi \left( \frac{t_1}{T_0} \right) \quad (2.42) \]

- An important point to note here is that both cosine and sine are mod \( 2\pi \) functions, meaning that phase is only unique on a \( 2\pi \) interval, say \( (-\pi, \pi] \) or \( (0, 2\pi] \)

**Example:** Suppose \( t_1 = 10 \text{ ms} \) and \( T_0 = 3 \text{ ms} \)

- Direct substitution into (2.21) results in

\[ \phi = -2\pi \left( \frac{10}{3} \right) = -\frac{20}{3}\pi = -6.6667\pi \quad (2.43) \]

- We need to reduce this value modulo \( 2\pi \) to the interval \( (-\pi, \pi] \) by adding (or subtracting as needed) multiples of \( 2\pi \)

- The result is the reduced phase value
\[ \phi = -\frac{20}{3}\pi + 6\pi = -\frac{20 + 18}{3}\pi = -\frac{2}{3}\pi = -0.6667\pi \quad (2.44) \]

- Does this result make sense?
- A time delay of 10 ms with a period of 3 ms means that we have delayed the sinusoid three full periods plus 1 ms
- A 1 ms delay is 1/3 of a period, with half of a period corresponding to \( \pi \) rad, so a delay of 1/3 period is a phase shift of \(-\frac{2}{3}\pi = -0.6667\pi\); agrees with the above analysis

![Diagram showing sinusoids with delays](image)

- The value of phase shift that lies on the interval \(-\pi < \phi \leq \pi\) is known as the *principle value*

**Sampling and Plotting Sinusoids**

- When plotting sinusoidal signals using computer tools, we are also faced with the fact that only a discrete-time version
of

\[ x(t) = A \cos(2\pi f_0 t + \phi) \]

may be generated and plotted

- This fact holds true whether we are using MATLAB, C, Mathematica, Excel, or any other computational tool
- When \( t \to nT_s \) we need to realize that sample spacing needs to be small enough relative to the frequency \( f_0 \) such that when plotted by connecting the dots (linear interpolation), the waveform picture is not too distorted
  - In Chapter 4 we will discuss sampling theory, which will tell us the maximum sample spacing (minimum sampling rate which is \( 1/T_s \)), such that the sequence \( x[n] = x(nT_s) \) can be used to perfectly reconstruct \( x(t) \) from \( x[n] \)
- For now we are more concerned with having a good plot appearance relative to the expected sinusoidal shape
- A reasonable plot can be created with about 10 samples per period, that is with \( T_s \approx 1/(10f_0) = T_0/10 \)
- We will now consider several MATLAB example plots

```matlab
>> t = 0:1/(5*3):1; x = 15*cos(2*pi*3*t-.5*pi);
>> subplot(311)
>> plot(t,x,'.-'); grid
>> xlabel('Time in seconds')
>> ylabel('Amplitude')
>> t = 0:1/(10*3):1; x = 15*cos(2*pi*3*t-.5*pi);
>> subplot(312)
```
Sampling and Plotting Sinusoids

```matlab
>> plot(t,x,'.-'); grid
>> xlabel('Time in seconds')
>> ylabel('Amplitude')
>> t = 0:1/(50*3):1; x = 15*cos(2*pi*3*t-.5*pi);
>> subplot(313)
>> plot(t,x,'.-'); grid
>> xlabel('Time in seconds')
>> ylabel('Amplitude')
>> print -depsc -tiff sampled_cosine.eps
```

\( f_0 = 3 \text{ Hz}, \ A = 15, \ \phi = -\pi/2 \)

![Graph showing sampling and plotting sinusoids](#)

- 5 Samples per period
- 10 Samples per period
- 50 Samples per period

\( T_s = \frac{T_0}{5} \)

\( T_s = \frac{1}{10} T_0 \)

\( T_s = \frac{1}{50} T_0 \)
Complex Exponentials and Phasors

Modeling signals as pure sinusoids is not that common. We typically have more than one sinusoid present. Manipulating multiple sinusoids is actually easier when we form a complex exponential representation.

Complex Exponential Signals

- Motivated by Euler’s formula above, and the earlier definition of a cosine signal, we define the complex exponential signal as

\[
   z(t) = Ae^{j(\omega_0 t + \phi)}
\]  

(2.44)

where \(|z(t)| = A\) and \(\angle z(t) = \arg\{z(t)\} = \omega_0 t + \phi\)

- Note that using Euler’s formula

\[
   z(t) = Ae^{j(\omega_0 t + \phi)}
   = A \cos(\omega_0 t + \phi) + jA \sin(\omega_0 t + \phi)
\]  

(2.45)

- We see that the complex sinusoid has amplitude \(A\), phase shift \(\phi\), and frequency \(\omega_0\) rad/s

  - Note in particular that

\[
   \text{Re}\{z(t)\} = A \cos(\omega_0 t + \phi)
   \]

\[
   \text{Im}\{z(t)\} = A \sin(\omega_0 t + \phi)
\]  

(2.46)
– The result of (2.46) is what ultimately motivates us to consider the complex exponential signal

• We can always write

\[
x(t) = \text{Re}\{Ae^{j(\omega_0t + \phi)}\} = A \cos(\omega_0t + \phi)
\] (2.47)

The Rotating Phasor Interpretation

• Complex numbers in polar form can be easily multiplied as

\[
z_3 = \underbrace{r_1e^{j\theta_1}}_{z_1} \cdot \underbrace{r_2e^{j\theta_2}}_{z_2} = r_1r_2e^{j(\theta_1 + \theta_2)}
\] (2.48)

• For the case of

\[
z(t) = Ae^{j(\omega_0t + \phi)}
\] (2.49)
we can write

\[ z(t) = Ae^{j\phi} \cdot e^{j\omega_0 t} = Xe^{j\omega_0 t} \]  (2.50)

where \( X = Ae^{j\phi} \)

- The complex amplitude \( X \) is called the *phasor*, as it is the gain and phase value applied to the time varying component \( e^{j\omega_0 t} \) to form \( z(t) \)
  - This is common terminology is electrical engineering circuit theory

- The time varying term \( e^{j\omega_0 t} \) has unit magnitude and rotates counter clockwise in the complex plane at a rate of \( \omega_0 \) rad/s (\( f_0 \) rotations/s)
  - The time duration for one rotation is the period \( T_0 = 1/f_0 \)

- The combination (product) of the fixed phasor \( X \) and \( e^{j\omega_0 t} \) results in a *rotating phasor*
  - For positive frequency \( \omega_0 \) the rotation is counter clockwise, and for negative frequency the rotation is clockwise

![Rotating Phasors](image-url)
Example: \( z(t) = \exp[2\pi t - \pi/4] \)

- Plot a series of snap shots of the rotating phasor when \( T_s = 1/8 \) (note \( T_0 = 1 \) s)

```matlab
%//.......................................................................%
% A script file for generating a sequence of rotating
% phasor snap shots
%
%
%//.......................................................................%

% get the focus of figure window #1 or create
% if not created.
figure(1)
clf(1); % clear figure window #1

A = 1.0; f0 = 1; phi = -pi/4;
N = 8; % create 8 vector plots
Ts = 1/8;
for n = 0:N-1
    subplot(4,2,n+1)
t = 0:1/200:1;
plot(cos(2*pi*t),sin(2*pi*t),'k:')
hold on
z = A*exp(j*(2*pi*f0*n*Ts+phi));
plot([0,real(z)],[0,imag(z)],'LineWidth',1)
% inside sprintf creates a formatted string
title(sprintf('Time = %1.4f s',n*Ts));
axis(1.1*[-A A -A A]); axis equal;
plot(real(z),imag(z),'r.','MarkerSize',18)
hold off
end```
Complex Exponentials and Phasors

Time = 0.0000 s

Time = 0.1250 s

Time = 0.2500 s

Time = 0.3750 s

Time = 0.5000 s

Time = 0.6250 s

Time = 0.7500 s

Time = 0.8750 s
• The inverse Euler formulas can be used to see that a cosine signal is composed of positive and negative frequency exponentials

\[ A \cos(\omega_0 t + \phi) = A \left( \frac{e^{j(\omega_0 t + \phi)} + e^{-j(\omega_0 t + \phi)}}{2} \right) \]

\[ = \frac{1}{2} X e^{j\omega_0 t} + \frac{1}{2} X^* e^{-j\omega_0 t} \]

\[ = \frac{1}{2} z(t) + \frac{1}{2} z^*(t) \]

\[ = \text{Re}\{z(t)\} \]

**Phasor Addition**

We often have to deal with multiple sinusoids. When the sinusoids are at the same frequency, we can derive a formula of the form

\[ \sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = A \cos(\omega_0 t + \phi) \]  \hspace{1cm} (2.52)

At present we have only the trig identities to aid us, and this approach becomes very *messy* for large \( N \).

**Phasor Addition Rule**

• We know that when complex numbers are added we must add real and imaginary parts separately

• Consider the sum
The above is valid since the real and imaginary parts add independently, that is

\[ \text{Re}\left\{ \sum_{k=1}^{N} X_k \right\} = \sum_{k=1}^{N} \text{Re}\{X_k\} \]  \hspace{1cm} (2.54)

and the same holds for the imaginary part

• Secondly, a real sinusoid can always be written in terms of a complex sinusoid via

\[ A \cos(\omega_0 t + \phi) = \text{Re}\{A e^{j(\omega_0 t + \phi)}\} \]  \hspace{1cm} (2.55)

**Proof:**

\[
\sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = \sum_{k=1}^{N} \text{Re}\{A_k e^{j(\omega_0 t + \phi_k)}\}
\]

Follows from (2.54)

\[
= \text{Re}\left\{ \sum_{k=1}^{N} A_k e^{j\phi_k} e^{j\omega_0 t} \right\}
\]

\[
= \text{Re}\{(A e^{j\phi})e^{j\omega_0 t}\}
\]

\[
= \text{Re}\{A e^{j(\omega_0 t + \phi)}\}
\]

\[
= A \cos(\omega_0 t + \phi)
\]
Example: Phasor Addition Rule in Action

- Consider the sum
  \[ x(t) = x_1(t) + x_2(t) \]
  \[ = 4.5 \cos(30\pi t + 35\pi/180) \]
  \[ + 7.2 \cos(30\pi t + 80\pi/180) \]

- The frequency of the sinusoids is 15 Hz
- Using phasor notation we can write that
  \[ x_1(t) = \text{Re}\{4.5e^{j35\pi/180} \cdot e^{j30\pi t}\} \]
  \[ x_2(t) = \text{Re}\{7.2e^{j80\pi/180} \cdot e^{j30\pi t}\} \]

so in the phasor addition rule

\[ X_1 = 4.5e^{j35\pi/180}, \quad X_2 = 7.2e^{j80\pi/180} \]

- We perform the complex addition and conversion back to polar form using the TI-89

\[ X = X_1 + X_2 = 4.93645 + j9.67171 \]
\[ = 10.8587e^{j62.9602\pi/180} \]
• Finally,
\[ x(t) = 10.86 \cos(30\pi t + 62.9602\pi / 180) \quad (2.60) \]

• We can check this by directly plotting the waveform in MATLAB

```matlab
>> t = 0:1/(50*15):0.2;
>> x1 = 4.5*cos(30*pi*t+35*pi/180);
>> x2 = 7.2*cos(30*pi*t+80*pi/180);
>> x = x1+x2; %plot using hold and line styles
```

• The measured amplitude, 10.822, is close to the expected value
• The location of the peak can be converted to phase via

\[ \phi = -2\pi \left( \frac{-11.67}{66.67} \right) = 1.10 \text{ rad} = 63.02^\circ \]  \hspace{1cm} (2.61)

---

**Summary of Phasor Addition**

• When we need to form the sum of sinusoids at the same frequency, we obtain the final amplitude \( A \) and phase \( \phi \) via

\[ X = X_1 + X_2 + \cdots + X_N = A e^{j\phi} \]  \hspace{1cm} (2.62)

where \( X_k = A_k e^{j\phi_k} \) and

\[ x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) \]  \hspace{1cm} (2.63)

\[ = A \cos(\omega_0 t + \phi) \]

---

**Example:**

\[ x(t) = \text{Re} \left\{ 3 e^{j \left( \frac{2\pi f_0 t + \pi}{2} \right)} + 5 e^{j \left( \frac{2\pi f_0 t - \pi}{4} \right)} + (3 + j2) e^{j2\pi f_0 t} \right\} \]

• Find \( X = X_1 + X_2 + X_3 \)
• From the given \( x(t) \) we observe that

\[ X_1 = 3 e^{j\frac{\pi}{2}}, \quad X_2 = 5 e^{-j\frac{\pi}{4}}, \quad X_3 = 3 + j2 \]
• To perform the complex addition we will work step-by-step
• To add complex numbers we convert to rectangular form
  \[ X_1 = 3 \cos \frac{\pi}{2} + j3 \sin \frac{\pi}{2} = j3 \]
  \[ X_2 = 5 \cos \left(-\frac{\pi}{4}\right) + j5 \sin \left(-\frac{\pi}{4}\right) = 3.5355 - j3.5355 \]
  \[ X_3 = 3 + j2 \]
• Now,
  \[ X = (j3) + (3.5355 - j3.5355) + (3 + j2) \]
  \[ = (6.5355 + j1.4645) \]
• For use in the phasor sum formula we likely need the answer in polar form
  \[ X = \sqrt{6.5355^2 + 1.4645^2} \angle \arctan \frac{1.4645}{6.5355} \]
  \[ = 6.6976 \angle 0.2204 = 6.6976 e^{j0.2204} \]

**Physics of the Tuning Fork**

The tuning fork signal generation example discussed earlier was important because it is an example of a physical system that when struck, produces nearly a pure sinusoidal signal.

– By pure we mean a signal composed of a single frequency sinusoid, no other sinusoids at other frequencies, say harmonics (multiples of \( f_0 \)) are present
Equations from Laws of Physics

A 2-D model of the tuning fork is shown below.

- When struck the vibration of the metal tine moves air molecules to produce a sound wave.
- Hooke’s law from physics (springs, etc.) says that the force to restore the tine back to its original $x = 0$ position is the same as the original deformation (striking force), except for a sign change,

$$ F = -kx \quad (2.64) $$

where $k$ is the material stiffness constant.
- The acceleration produced by the restoring force (Newton’s second law) is
\[ F = ma = \frac{d^2x}{dt^2} \quad (2.65) \]

- To balance the two forces (sum is zero), we must have

\[ m \frac{d^2x}{dt^2} = -kx(t) \quad (2.66) \]

**General Solution to the Differential Equation**

- To solve this equation we can actually guess the solution by inserting a test function of the form \( x(t) = \cos \omega_0 t \)

\[
\frac{d^2 x(t)}{dt^2} = \frac{d}{dt}(-\omega_0 \sin \omega_0 t) \\
= -\omega_0^2 \cos \omega_0 t
\]

- We now plug this result into (2.66) to obtain

\[
m \frac{d^2 x}{dt^2} = -kx(t) \quad (2.67)
\]

\[-m(\omega_0^2 \cos \omega_0 t) = -k \cos \omega_0 t \]

which tells us that we must have

\[-m \omega_0^2 = -k \quad (2.68)\]

so it must be that

\[\omega_0 = \pm \sqrt{\frac{k}{m}} \quad (2.69)\]
• This tells us that the oscillation frequency of the tuning fork is related to the ratio of the stiffness constant to the mass
  – Greater stiffness means a higher oscillation frequency
  – Greater mass means a lower oscillation frequency
• In terms of a real sinusoid the sound wave, to within a phase shift constant is of the form
  \[ x(t) = A \cos \left( \sqrt{\frac{k}{m}} t + \phi \right) \]  \hspace{1cm} (2.70)
• The sound produced by the 440Hz tuning fork was captured using MATLAB on a PC with a sound card and microphone
  – The results were converted to double precision and saved in a .mat file along with a time axis vector
    
    >> load tuningfork
    >> plot(t,x)
• How pure is the signal produced by the tuning fork?
• In Chapter 3 of the text we begin a study of *spectrum representation*.
• The zoom of the captured signal looks like a single sinusoid, but spectral analysis can be more revealing.
• Consider the use of MATLAB’s power spectral density function `psd(x, Nfft, fsamp)` (Detail comes later Chapter)

\[
\text{>> } \text{psd}(x, 2^{12}, 8000)
\]
A time-frequency plot can be obtained using the MATLAB’s spectrogram function (Detail comes later Chapter)

\[
>> \text{spectrogram}(x, 2^{10}, 50, [], 8000)
\]

Listening to Tones

To play the tuning fork sound on the PC speakers using Matlab we type

\[
>> \text{sound}(x, 8000)
\]

where the second argument sets the sampling frequency for playback
Time Signals: More Than Formulas

• The signal modeling of this chapter has focused on single sinusoids

• In practice real signals are far more complex, even a multiple sinusoids model is only an approximation

• Modeling still has great value in system design