

# LECTURE #9

## GENERAL MARKOV PROCESS IN EQUILIBRIUM

204528

Queueing Theory and  
Applications in Networks

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# Outline

- General Markov Process
- Erlangian Distribution
  - 2-stage Erlangian  $E_2$
  - r-stage Erlangian  $E_r$
- $M/E_r/1$

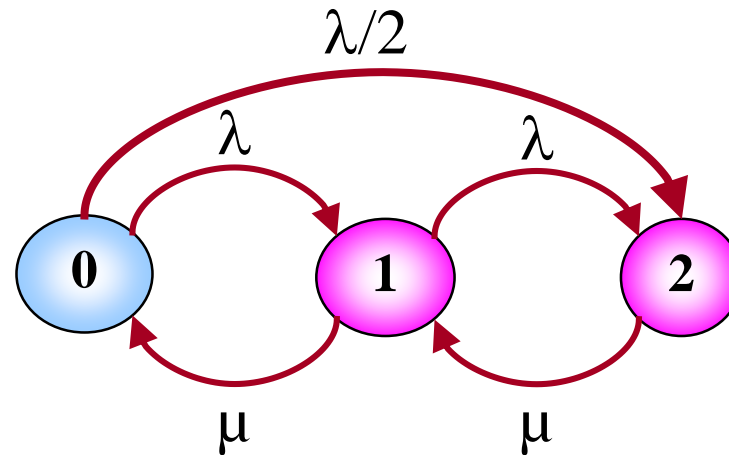
# General Markov Process

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- More general than birth-death process
  - Transition beyond nearest neighbors are allowed
  - But not too complicated
- We are interested in the Equilibrium Solutions

# Simple three-state Markov Chain

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- From Flow conservation
  - Prob. Flow in = Prob. Flow out

- For state 0:

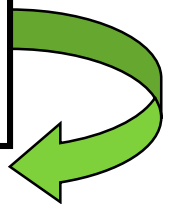
- For state 1:

- For state 2:

$$(3/2)\lambda p_0 = \mu p_1$$

$$(\lambda + \mu)p_1 = \lambda p_0 + \mu p_2$$

$$\mu p_2 = (\lambda/2)p_0 + \lambda p_1$$



# Simple three-state Markov Chain

- We need one more equation

$$p_0 + p_1 + p_2 = 1$$

- Solution

$$p_0 = [ 1 + 2(\lambda/\mu) + (3/2)(\lambda/\mu)^2 ]^{-1}$$

$$p_1 = 3/2(\lambda/\mu) p_0$$

$$p_2 = [ (1/2)(\lambda/\mu) + (3/2)(\lambda/\mu)^2 ] p_0$$

# Limiting Probability

- As  $t \rightarrow \infty$ 
  - We can find the  $p_k$
- How about prob. that an arriving customer finds the system in state  $E_k$  ?
  - It is equal to  $p_k$ , isn't it?

# D/D/1

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- Arrival is deterministic and occurs at every interval =  $\bar{t}$  sec
- Service time is also deterministic and identical for all customers =  $\bar{x}$  sec
- Let  
 $r_k =$  [prob. that an arriving customer finds the system in state  $E_k$ ]

# D/D/1

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- For stability of the system
  - It requires:  $\bar{x} < \bar{t}$
  - $\rho = \bar{x} / \bar{t}$
- No customer has to wait !
  - $r_0 = 1$  and  $r_k = 0$  for  $k \geq 1$
- Therefore
  - $p_0 = 1 - \rho$  and  $p_1 = \rho$
  - $p_k = 0$  for  $k \geq 2$



# D/D/1

- In summary
  - For D/D/1:  $r_k \neq p_k$
- How about the other systems?
  - Many contains the property of  $r_k = p_k$
  - Poisson arrival system is an example

# Poisson Arrivals

- For

$P_k(t)$  = Prob. that system is in state  $E_k$

$R_k(t)$  = Prob. that an arriving customer finds  
the system is in state  $E_k$

- Let  $A(t, t+\Delta t)$  = Event that arrival occurs in the interval  $(t, t+\Delta t)$

# Poisson Arrivals

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- $$\begin{aligned} R_k(t) &\stackrel{\Delta}{=} \lim_{\Delta t \rightarrow 0} P[ N(t) = k \mid A(t, t+\Delta t) ] \\ &= \lim_{\Delta t \rightarrow 0} \frac{P[ N(t) = k, A(t, t+\Delta t) ]}{P[A(t, t+\Delta t) ]} \\ &= \lim_{\Delta t \rightarrow 0} \frac{P[A(t, t+\Delta t) \mid N(t) = k] P[N(t) = k]}{P[A(t, t+\Delta t) ]} \\ &= \lim_{\Delta t \rightarrow 0} P[ N(t) = k ] \end{aligned}$$

$$P_k(t) = R_k(t)$$

# Erlangian Distribution $E_r$

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- A.K. Erlang proposed
  - The extreme simplicity of exponential distribution
  - Solving Markovian queueing systems
- The exponential distribution is not **always** appropriate for representing the true situation with regards to service times
- More general service distribution → Markov chain becomes complicated.



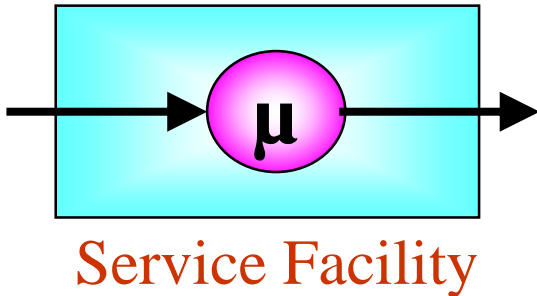
Agner Krarup Erlang  
Denmark  
From: Wikipedia

# Erlangian Distribution Er

- Erlang purposed
  - The decomposing the service time distribution into a collection of exponential distributions

# An exponential server

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- For an exponentially distributed service time

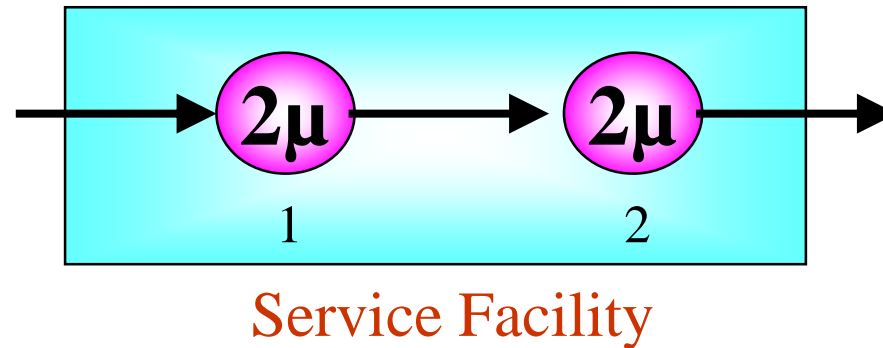
$$b(x) \triangleq \frac{dB(x)}{dx}$$
$$= \mu e^{-\mu x} \quad x \geq 0$$

- For Mean and Variance of the service time

$$E[x] = \frac{1}{\mu} \quad \sigma_b^2 = \frac{1}{\mu^2}$$

# A series of two exponential servers

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- A series of two small exponential servers
- Each internal server has parameter =  $2\mu$
- So the pdf of each server

$$h(y) = 2\mu e^{-2\mu y} \quad y \geq 0$$

# A series of two exponential servers

- For Mean and Variance of the service time

$$E[x] = \frac{1}{2\mu} \quad \sigma_h^2 = \frac{1}{(2\mu)^2}$$

- The functions of the system
  - Upon departure of customer, a new customer is allowed to enter the service facility
  - After finish stage 1 → enter stage 2
  - Only one customer is allowed in the service facility



# A series of two exponential servers

- What is the distribution of the time spent in the service facility ?
  - Sum of the two independent random variables
  - The convolution of the density function of the two random variables

# Convolution of two RVs

- $Y = X_1 + X_2$
- $F_Y(y) = P[Y \leq y] = P[X_1 + X_2 \leq y]$ 
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{y-x_2} f_{X_1, X_2}(x_1, x_2) dx_1 dx_2$$
$$= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{y-x_2} f_{X_1}(x_1) dx_1 \right] f_{X_2}(x_2) dx_2$$
$$= \int_{-\infty}^{\infty} F_{X_1}(y - x_2) f_{X_2}(x_2) dx_2$$

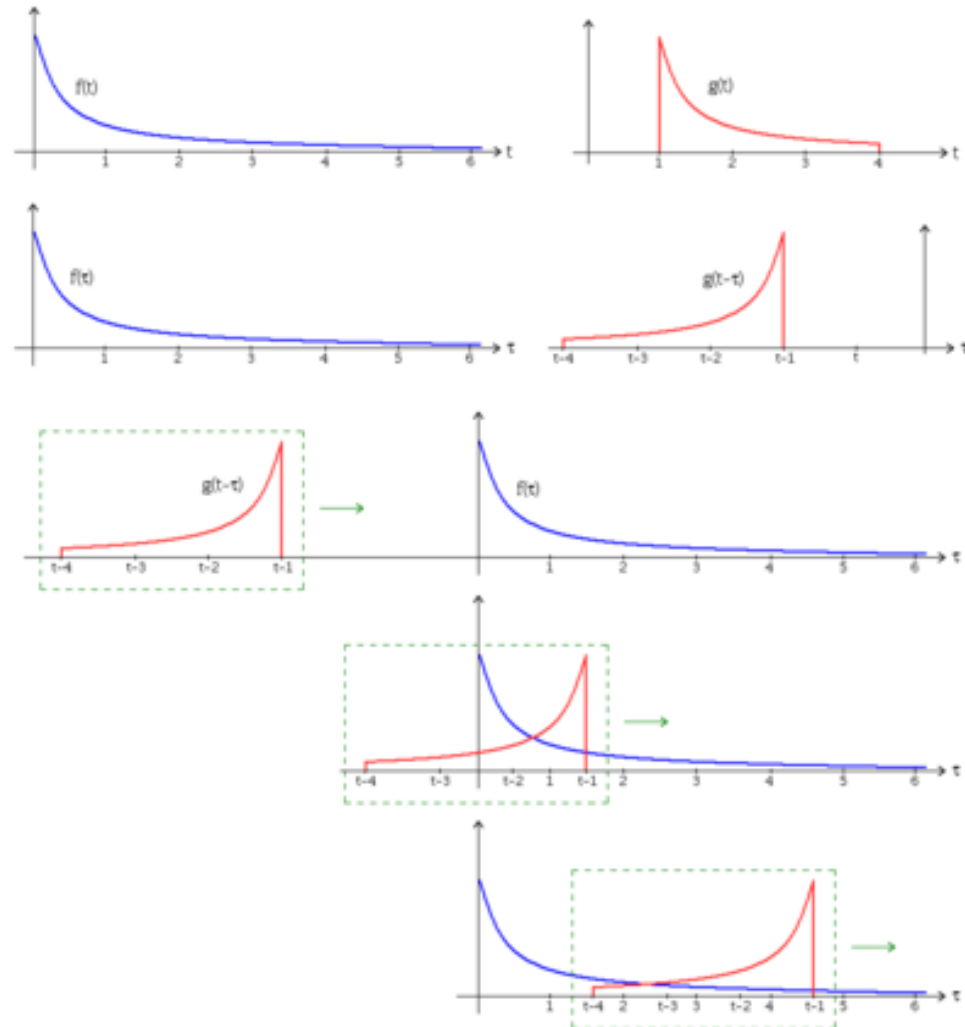
# Convolution of two RVs

- $F_Y(y) = \int_{-\infty}^{\infty} F_{X_1}(y - x_2) f_{X_2}(x_2) dx_2$
- $f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(y - x_2) f_{X_2}(x_2) dx_2$
- $f_Y(y) = f_{X_1}(y) \otimes f_{X_2}(y)$

# Convolution of two RVs

$$(f * g)(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau$$

- Express each function in terms of a dummy variable  $\tau$ .
- Reflect one of the functions:  $g(\tau) \rightarrow g(-\tau)$ .
- Add a time-offset,  $t$ , which allows  $g(t - \tau)$  to slide along the  $\tau$ -axis.
- Start  $t$  at  $-\infty$  and slide it all the way to  $+\infty$ . Wherever the two functions intersect, find the integral of their product.



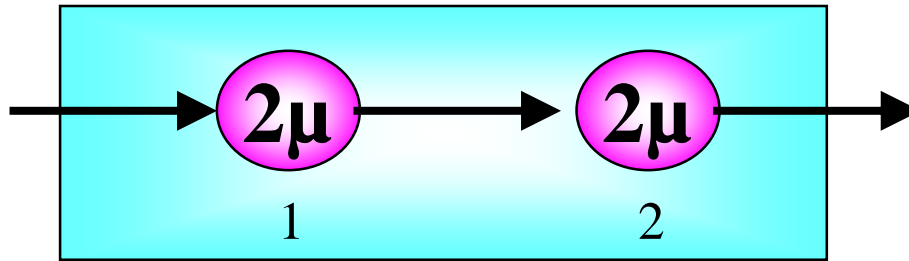
# Laplace transform in probability

- In probability, the Laplace transform is defined by means of an expectation value.
- If  $X$  is a random variable with probability density function  $f$ ,
  - then the Laplace transform of  $f$  is given by the expectation

$$\begin{aligned}(\mathcal{L}f)(s) &= E[e^{-sX}] . \\ &= \int_0^{\infty} e^{-sx} f_X(x) dx\end{aligned}$$

# Laplace transform of pdf

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Service Facility

- $B^*(s) \triangleq \int_0^{\infty} e^{-sx} b(x) dx$
- $H^*(s) \triangleq \int_0^{\infty} e^{-sy} h(y) dy$
- $B^*(s) = H^*(s) H^*(s) = [H^*(s)]^2$

# 2-stage Erlangian Server $E_2$

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- $h(y) = 2\mu e^{-2\mu y}$
- $H^*(s) \triangleq \int_0^{\infty} e^{-sy} h(y) dy$   
 $= \frac{2\mu}{s + 2\mu}$

- $B^*(s) = [H^*(s)]^2$   
 $= \left( \frac{2\mu}{s + 2\mu} \right)^2$

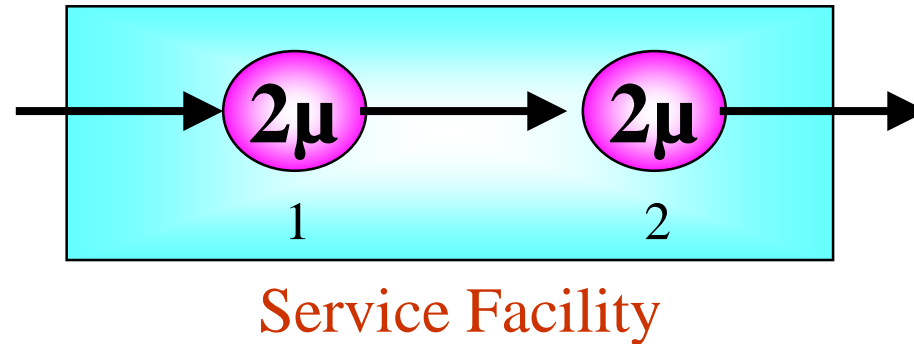
Inverse

$$b(x) = 2\mu(2\mu x) e^{-2\mu x}$$

For  $x \geq 0$

# A series of two exponential servers

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- For Mean and Variance of the service time

$$E[x] = 2E[y] = \frac{1}{\mu}$$

$$\sigma_b^2 = \sigma_h^2 + \sigma_h^2 = \frac{1}{2\mu^2}$$

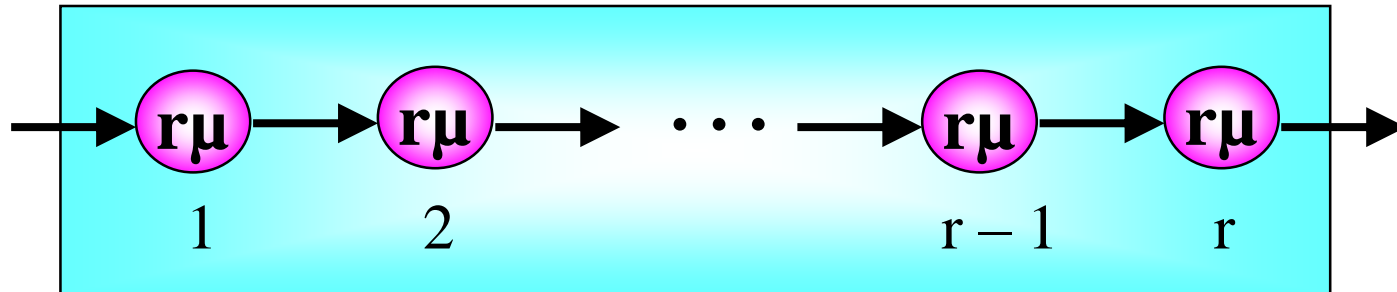


# A series of two exponential servers

- Mean time for single and two-stage are the same
  - By speed up the internal service 2 times
- Variance of the two-stage is one-half of the single server

# r-stage Erlangian server $E_r$

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Service Facility

- $h(y) = r\mu e^{-r\mu y} \quad y \geq 0$
- For Mean and Variance of each stage

$$E[y] = \frac{1}{r\mu}$$
$$\sigma_h^2 = \left( \frac{1}{r\mu} \right)^2$$

# r-stage Erlangian server Er

- For Mean and Variance of the service time

$$E[x] = rE[y] = \frac{1}{\mu}$$

$$\sigma_b^2 = r \left( \frac{1}{r\mu} \right)^2 = \frac{1}{r\mu^2}$$

# r-stage Erlangian server $E_r$

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- Solve for the pdf of the service time

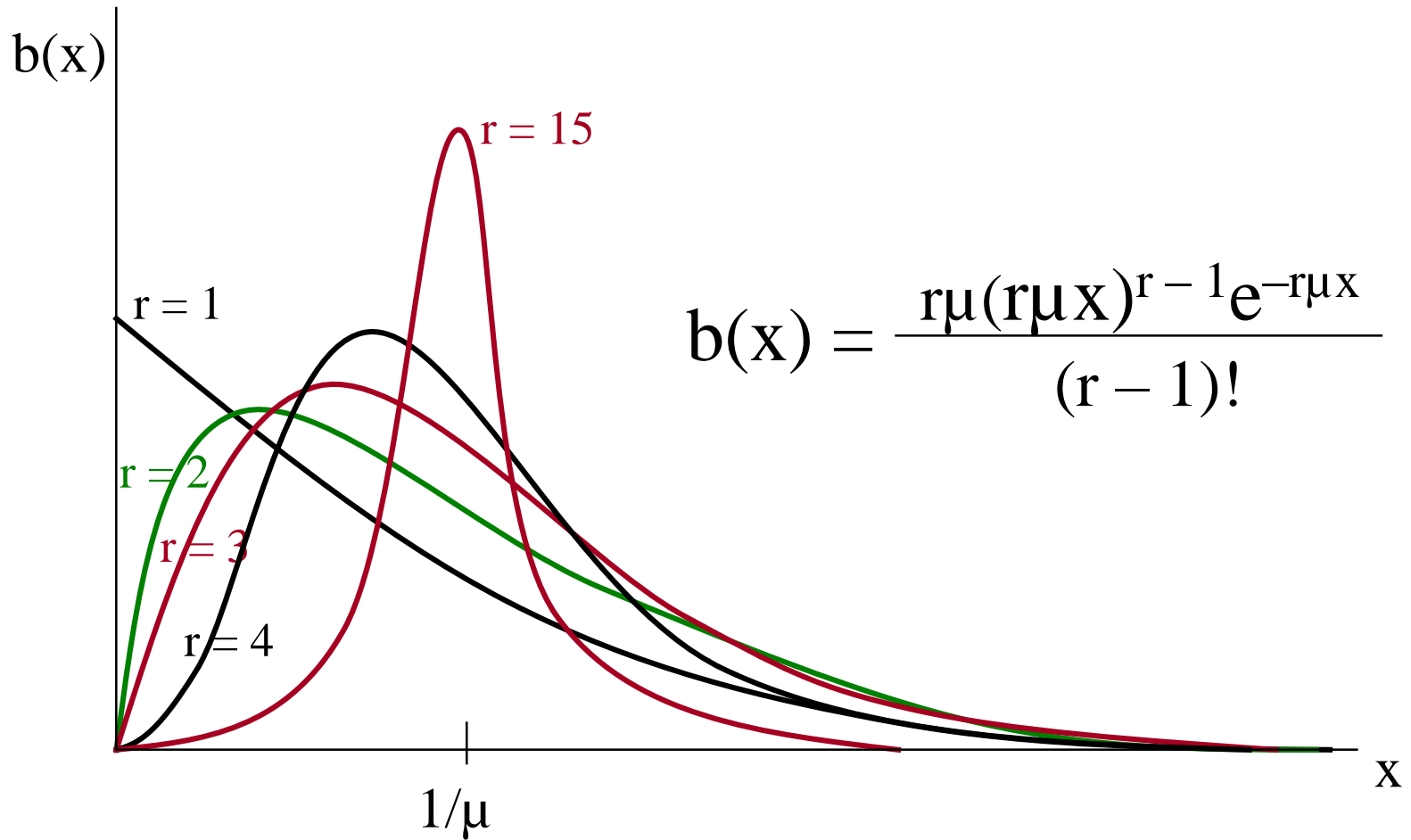
$$B^*(s) = \left( \frac{r\mu}{s + r\mu} \right)^r$$

$$b(x) = \frac{r\mu(r\mu x)^{r-1} e^{-r\mu x}}{(r-1)!} \quad x \geq 0$$

**Erlang Distribution**

# r-stage Erlangian distribution $E_r$

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# r-stage Erlangian distribution $E_r$

- Note
  - $r \uparrow \rightarrow$  pdf approaches Gaussian Distribution
  - Mean is constant
  - The standard deviation (width) shrinks by  $1/\sqrt{r}$
  - If  $r \rightarrow \infty$  then  
pdf  $\rightarrow$  unit impulse function at  $x = 1/\mu$

# M / E<sub>r</sub> / 1

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- Memoryless / Erlangian/1 server
- Define

$$a(t) = \lambda e^{-\lambda t} \quad t \geq 0$$

$$b(x) = \frac{r\mu(r\mu x)^{r-1} e^{-r\mu x}}{(r-1)!} \quad x \geq 0$$

# M / E<sub>r</sub> / 1

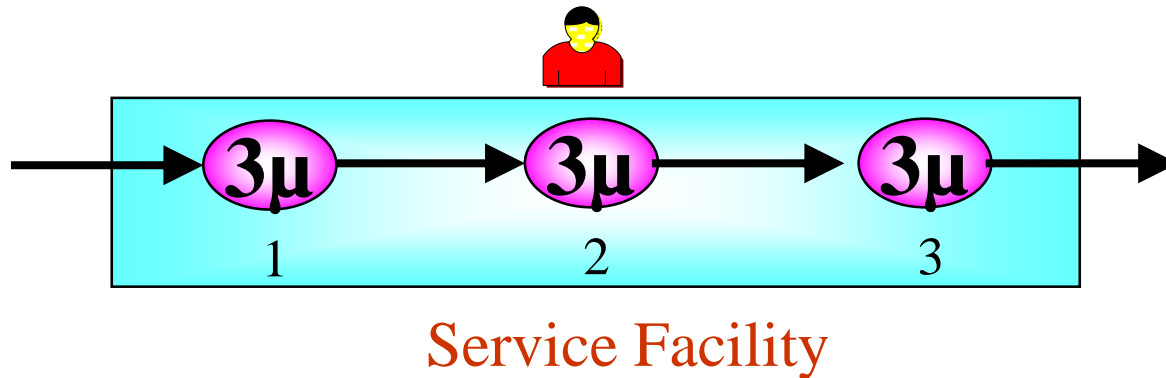
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- State variable = total number of service stage yet to be completed
- $k$  = # of customers
- Now we are in the  $i^{\text{th}}$  stage of service (  $1, r$  )
- $j$  = number of stages left in the total system
  - =  $(k - 1)r + (r - i + 1)$
  - =  $rk - i + 1$



# # of stages left to be completed

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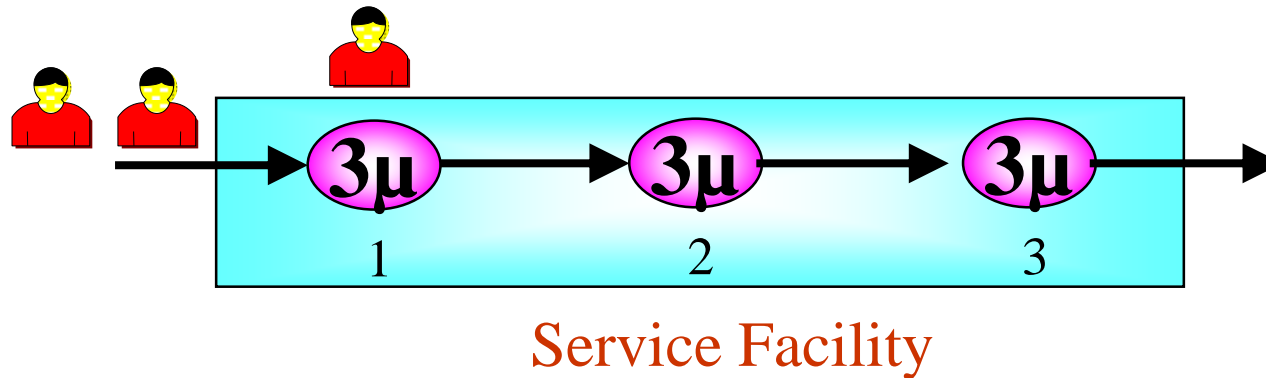


- $j = rk - i + 1$
- For  $r = 3$  stages, 1 customer, now in stage 2  
$$j = 3 * 1 - 2 + 1 = 2$$

= “there are **2 stages** left to be completed”

# # of stages left to be completed

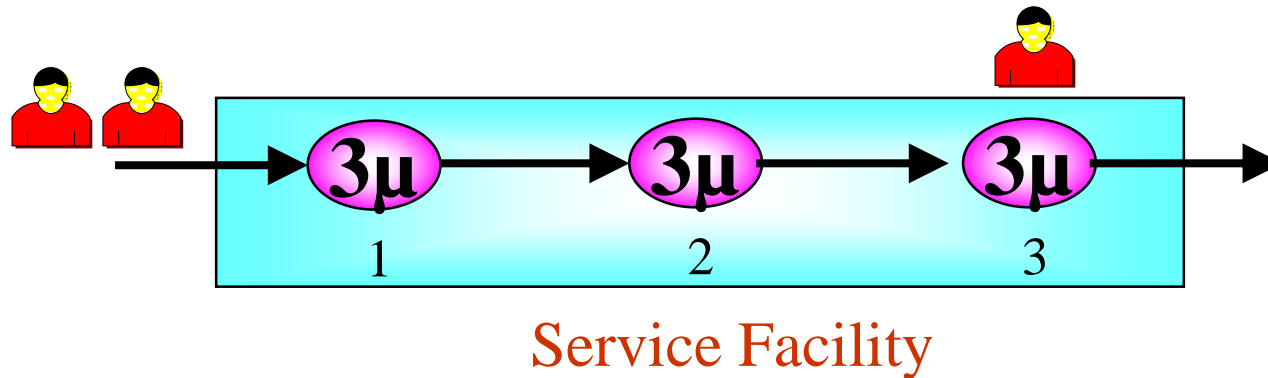
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- $j = rk - i + 1$
- For  $r = 3$  stages, 3 customers, now in stage 1  
 $j = 3*3 - 1 + 1 = 9$   
= “there are **9 stages** left to be completed”

# # of stages left to be completed

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- $j = rk - i + 1$
- For  $r = 3$  stages, 3 customers, now in stage 3  
$$j = 3 * 3 - 3 + 1 = 7$$

= “there are **7 stages** left to be completed”

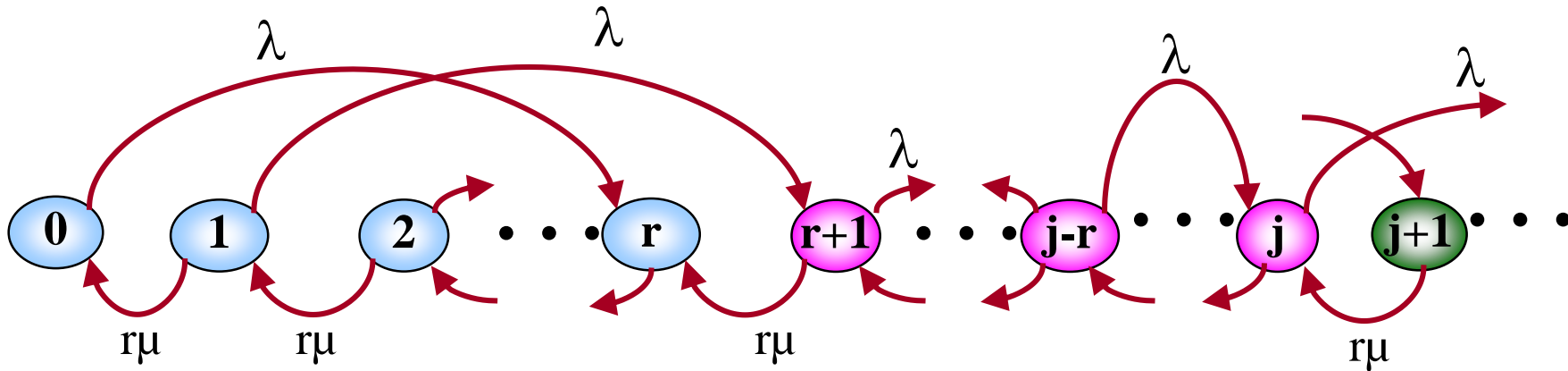
# M / E<sub>r</sub> / 1

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- $p_k$  = equilibrium prob. for the # of customers in the system
- $P_j = P[j \text{ stages in the system}]$
- $p_k = \sum_{j=(k-1)r+1}^{kr} P_j \quad k = 1, 2, 3, \dots$

# M / E<sub>r</sub> / 1 State diagram

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- $P_j = 0 \quad j < 0$
- $\lambda P_0 = r\mu P_1$
- $(\lambda + r\mu)P_j = \lambda P_{j-r} + r\mu P_{j+1} \quad j = 1, 2, \dots$

# M / E<sub>r</sub> / 1

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- Solved by using  $z$ -transform

$$P(z) = \sum_{j=0}^{\infty} P_j z^j$$

- Multiplied by  $z^j$  and sum

$$\sum_{j=1}^{\infty} (\lambda + r\mu) P_j z^j = \sum_{j=1}^{\infty} \lambda P_{j-r} z^j + \sum_{j=1}^{\infty} r\mu P_{j+1} z^j$$

$$(\lambda + r\mu) \left( \sum_{j=0}^{\infty} P_j z^j - P_0 \right) = \lambda z^r \sum_{j=1}^{\infty} P_{j-r} z^{j-r} + (r\mu/z) \sum_{j=1}^{\infty} P_{j+1} z^{j+1}$$

# M / E<sub>r</sub> / 1

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$$(\lambda + r\mu) [P(z) - P_0] = \lambda z^r P(z) + (r\mu/z) [P(z) - P_0 - P_1 z]$$

$$P(z) = \frac{r\mu P_0(1 - z)}{r\mu + \lambda z^{r+1} - (\lambda + r\mu)z}$$

Finding  $P_0$

$$P(1) = 1 = \frac{r\mu P_0}{r\mu - \lambda r}$$

$$P_0 = 1 - (\lambda/\mu) \rightarrow \rho$$

# M / E<sub>r</sub> / 1

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$$P(z) = \frac{r\mu(1 - \rho)(1 - z)}{r\mu + \lambda z^{r+1} - (\lambda + r\mu)z}$$

- Then invert the  $z$ -transform  
→ Distribution of # of the stages in the system



# $M / E_r / 1$ (For $r = 1$ )

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- For case  $r = 1$

$$\begin{aligned} P(z) &= \frac{\mu(1 - \rho)(1 - z)}{\mu + \lambda z^2 - (\lambda + \mu)z} \\ &= \frac{(1 - \rho)(1 - z)}{1 + \rho z^2 - (1 + \rho)z} \\ &= \frac{(1 - \rho)}{(1 - \rho z)} \end{aligned}$$

# M / E<sub>r</sub> / 1 (For r = 1)

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- Invert z-transform

$$P_k = (1 - \rho) \rho^k \quad k = 0, 1, 2, \dots$$

- $p_k = \sum_{j=(k-1)r+1}^{kr} P_j \quad k = 1, 2, 3, \dots$

- For  $r = 1$   $p_k = P_k \rightarrow M / M / 1$

# M / E<sub>r</sub> / 1 (For general r)

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$$\begin{aligned} P(z) &= \frac{r\mu(1-\rho)(1-z)}{r\mu + \lambda z^{r+1} - (\lambda + r\mu)z} \\ &= \frac{(1-\rho)}{(1-z/z_1)(1-z/z_2)\dots(1-z/z_r)} \\ &= (1-\rho) \sum_{i=1}^r \frac{A_i}{(1-z/z_i)} \\ A_i &= \prod_{\substack{n=1 \\ n \neq i}}^r \frac{1}{(1-z_i/z_n)} \end{aligned}$$

# M / E<sub>r</sub> / 1 (For general r)

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- Invert z-transform

$$P_j = (1 - \rho) \sum_{i=1}^r A_i (z_i)^{-j} \quad j = 1, 2, \dots, r$$

# $E_r / M / 1$

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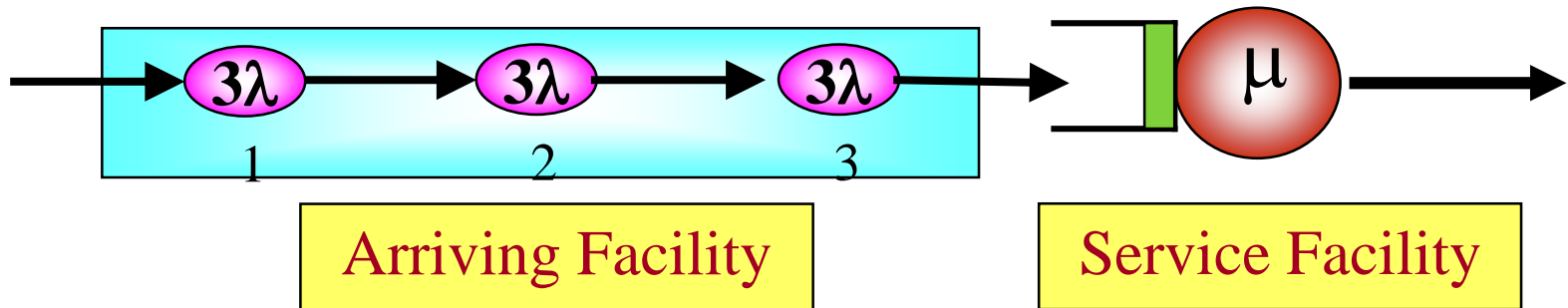
- Erlangian / Memoryless / 1 server
- Define

$$a(t) = \frac{r\lambda(r\lambda t)^{r-1}e^{-r\lambda t}}{(r-1)!} \quad t \geq 0$$

$$b(x) = \mu e^{-\mu x} \quad x \geq 0$$

# $E_r / M / 1$

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$$\text{Arriving : } a(t) = \frac{r\lambda(r\lambda t)^{r-1}e^{-r\lambda t}}{(r-1)!} \quad t \geq 0$$

$$\text{Service : } b(x) = \mu e^{-\mu x} \quad x \geq 0$$

# $E_r / M / 1$

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- State variable = total number of service stage yet to be completed
- $k$  = # of customers
- Now the arriving customer is in the  $i^{\text{th}}$  stage of service  $[1, r]$
- $j$  = total number of stages of arrival in the system  
=  $rk + i - 1$

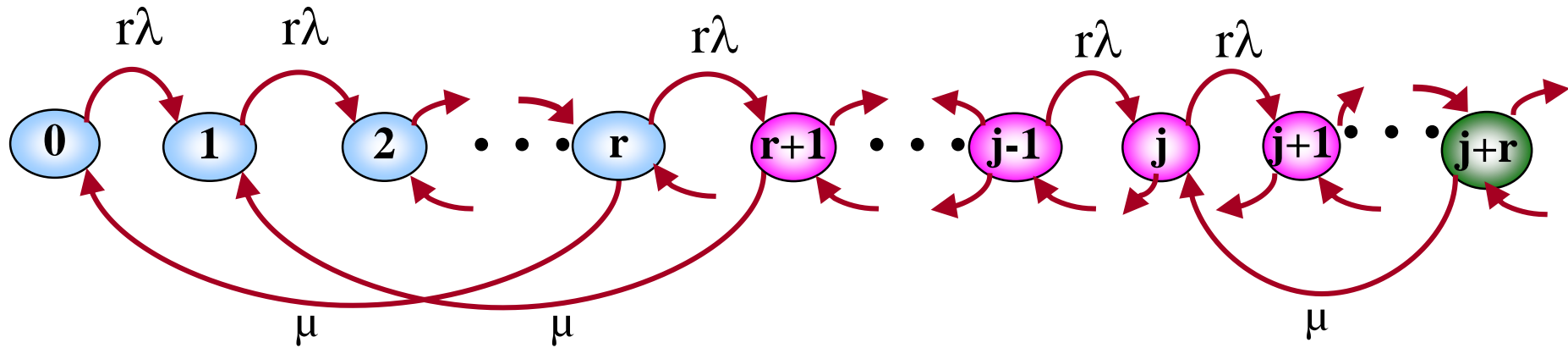
# $E_r / M / 1$

- $p_k$  = equilibrium prob. for the # of customers in the system
- $P_j$  = # of arrival stages in the system
- $p_k = \sum_{j=rk}^{r(k+1)-1} P_j$



# $E_r / M / 1$ State diagram

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- Each departure = remove “ $r$ ” stages of arrival
- $r\lambda P_0 = \mu P_r$
- $r\lambda P_0 = r\lambda P_{j-1} + \mu P_{j+r} \quad 1 \leq j \leq r-1$
- $(r\lambda + \mu)P_j = r\lambda P_{j-1} + \mu P_{j+r} \quad r \leq j$

# $E_r / M / 1$

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- Solved by using  $z$ -transform

$$P(z) = \sum_{j=0}^{\infty} P_j z^j$$

- Multiplied by  $z^j$  and sum
- For  $j \geq 1$

$$\sum_{j=1}^{\infty} (\mu + r\lambda) P_j z^j - \sum_{j=1}^{r-1} \mu P_j z^j = \sum_{j=1}^{\infty} r\lambda P_{j-1} z^j + \sum_{j=1}^{\infty} \mu P_{j+r} z^j$$

# $E_r / M / 1$

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$$P(z) = \frac{(1 - z^r) \sum_{j=1}^{r-1} P_j z^j}{r \rho z^{r+1} - (1 + r\rho)z^r + 1}$$

$$P(z) = \frac{(1 - z^r)(1 - 1/z_0)}{r(1 - z)(1 - z/z_0)}$$

$$P_j = \begin{cases} \frac{1}{r}(1 - z_0^{-j-1}) & 0 \leq j < r \\ \rho(z_0 - 1)z_0^{r-j-1} & j \geq r \end{cases}$$

# $E_r / M / 1$

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$$p_k = \begin{cases} 1 - \rho & k = 0 \\ \rho(z_0^r - 1)z_0^{-rk} & k > 0 \end{cases}$$