

# LECTURE #5

## MARKOV PROCESS

204528

Queueing Theory and  
Applications in Networks

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**Computer Engineering Department, Kasetsart University**

# Outline

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- Markov Processes
- Discrete Time Markov Chain
- Homogeneous, Irreducible, Transient/Recurrent, Periodic/Aperiodic
- Ergodic
- Stationary Probability
- Transient Behavior
- Birth-Death Process

# Markov Processes

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- $X(t)$  is a Markov Process if it satisfies the “**Markov (Memoryless) Property**”

$$\begin{aligned} P\{X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \dots, X(t_1) = x_1\} \\ = P\{X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n\} \end{aligned}$$

Where  $t_1 < t_2 < \dots < t_{n-1} < t_n < t_{n+1}$

- $X(t)$  only depends upon the current state
- The past history is summarized in the current state

# Definition

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- State Space (E)
  - Set of real numbers containing ranges of RV in a stochastic process

# From Markov Processes ...

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- *Discrete Time Markov Process:*  
State changes occur at **integer** points
- *Continuous Time Markov Process:*  
State changes occur at **arbitrarily** time

# From Markov Processes ...

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- ***Markov Chain:***  
Discrete state space Markov Process
- ***Discrete Time Markov Chain:***  
State (Discrete State) changes occur at integer points
- ***Continuous Time Markov Chain:***  
State (Discrete State) changes occur at arbitrarily time

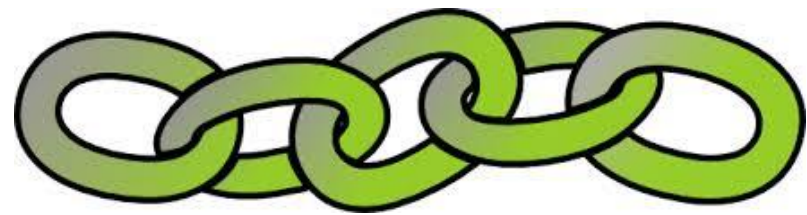
# Markov Chain Examples

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- Egg game
  - *Kai* has 10 eggs, *Ped* has 10 eggs
  - Toss a die
    - If {1,2,3} occurs → *Ped* gives one egg to *Kai*
    - If {4,5,6} occurs → *Kai* gives one egg to *Ped*
  - Game ends when 20-0 or 0-20 (*Kai* or *Ped* has all eggs)



# Markov Chain



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- The easiest stochastic process
  - (IID) Independent and Identically distributed RV
- A discrete IID process
  - Markov Chain
  - Lack of memory (Future is independent of the past given the present)
- Named for probabilist “A.A.Markov”
  - 1907 finite state Markov Chain
  - 1930 infinite state by A.N.Kolmogorov



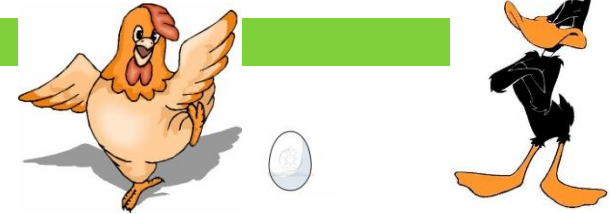
Andrey (Andrei)  
Andreyevich Markov  
Russia, Born:1856



# Markov Chain Examples

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- Egg game
  - *Kai* has 10 eggs, *Ped* has 10 eggs
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    - If {4,5,6} occurs → *Kai* gives one egg to *Ped*
  - Game ends when 20-0 or 0-20 (*Kai* or *Ped* has all eggs)
- Markov chain
  - The fifth play does not depend on the second play



# Discrete Time Markov Chains

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- One can stay in a *Discrete state (position)* and is permitted to change state at *Discrete time*.

# Discrete Time Markov Chains

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$$\begin{aligned} P\{X_n = j \mid X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\} \\ = P\{X_n = j \mid X_{n-1} = i_{n-1}\} \quad \text{Where } n = 1, 2, 3, \dots \end{aligned}$$

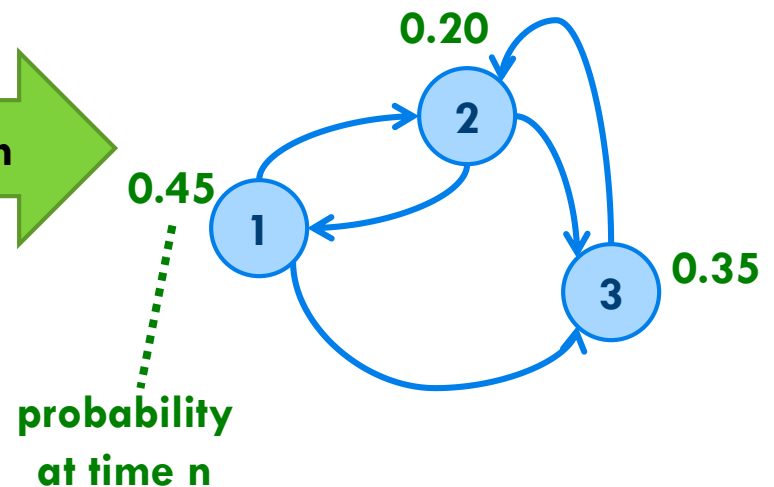
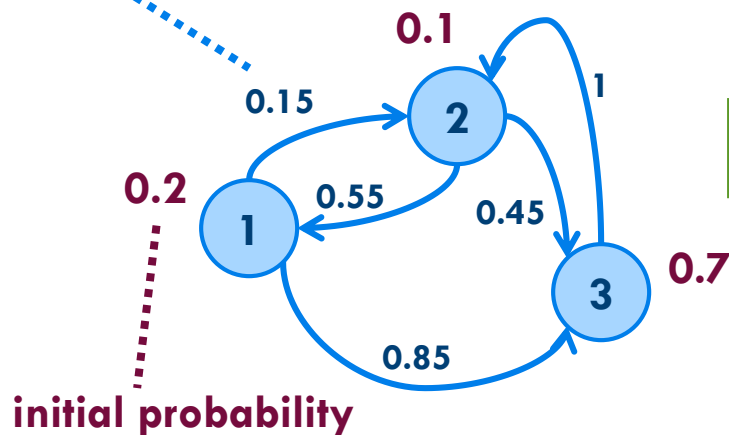
- $X_n$ : The system is in state  $j$  at time  $n$
- The system can begin at *state 0* with *initial probability*  $P[X_0 = x]$
- $P\{X_n = j \mid X_{n-1} = i_{n-1}\}$  is the *one-step transition probability*

# Discrete Time Markov Chains

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- From *initial probability* and *one-step transition probability*,
  - we can find *probability of being in various states at time  $n$*

one-step transition probability



# Homogeneous Markov Chain

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- If *transition probabilities* are independent of time  $n$ , it is called **Homogeneous Markov Chain**.
- Let 
$$p_{ij} \equiv P[X_n = j \mid X_{n-1} = i]$$
- We are in *state i* and going to be in *state j* in the next step
- The state transition prob. will only depend on
  - the *initial probability* and
  - the *transition probability*
  - regardless of transition time

# Markov Matrix

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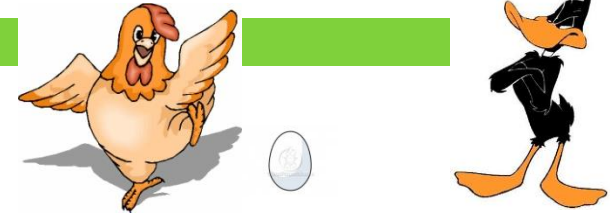
- A nonnegative square matrix
- Sum value of each row = 1

$$P = \begin{pmatrix} 0 & 0.3 & 0.7 \\ 0.1 & 0.2 & 0.7 \\ 0.8 & 0.2 & 0 \end{pmatrix}$$

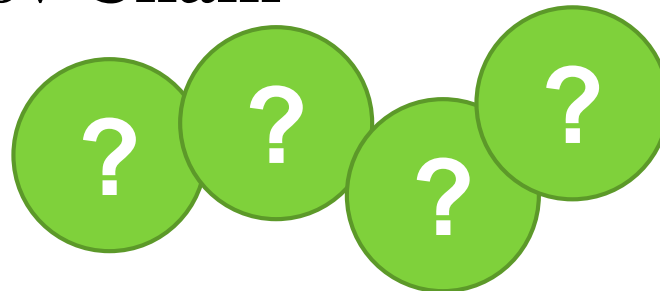
Sum = 1  
Sum = 1  
Sum = 1

# Example #0 Egg Game

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- Egg game
  - *Kai* has 10 eggs, *Ped* has 10 eggs
  - Toss a die
    - If {1,2,3} occurs  $\rightarrow$  *Ped* gives one egg to *Kai*
    - If {4,5,6} occurs  $\rightarrow$  *Kai* gives one egg to *Ped*
  - Game ends when 20-0 or 0-20 (*Kai* or *Ped* has all eggs)
- **Draw a Markov Chain**



# Example #1: Jump Game

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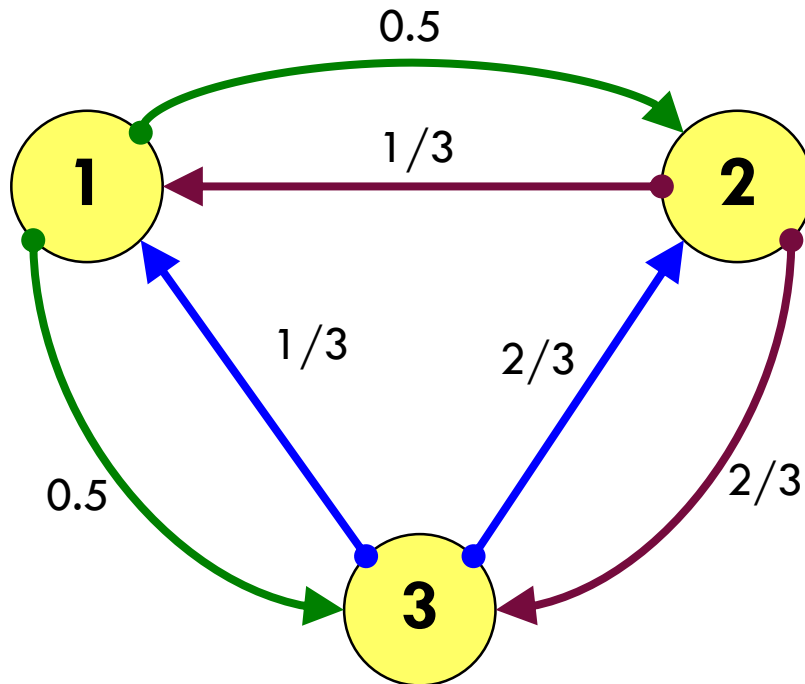


- A boy will jump among Plate#1, 2, & 3
- Every time he will change the plate
  - When he is on Plate #1, uses a coin tossing mechanism
    - If “head”  $\rightarrow$  he jumps to Plate #2
    - If “tail”  $\rightarrow$ he jumps to Plate #3
  - After that, uses a die tossing mechanism
    - If  $\{1,2\}$   $\rightarrow$  he will jump back to plate #1
    - Otherwise  $\rightarrow$ he will jump to another plate
- **Draw a Markov Chain**



# Example #1: Jump Game

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$$P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 1/3 & 0 & 2/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

Markov Matrix

State space  $E = \{1, 2, 3\}$

Random variable  $X_n = 1$  or  $2$  or  $3$

# Example #2: Weather Forecast

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- Observe the weather in an area
  - Typically longer periods of rainy or dry(sunshine) days
  - **Rain** and **sunshine** are same relative frequency over the entire year
- Sometimes claimed that the best way to predict tomorrow's weather
  - Guess that, tomorrow is as same as today
  - If assume predicting will **be correct in 75%** of the cases (regardless rain or sunshine)
- **Draw a Markov Chain**

O. Häggström (2002) *Finite Markov Chains and Algorithmic Applications*. CU Press, Cambridge

# Example #2: Weather Forecast

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- The weather can be easily modeled by a **Markov chain**

- The state space consists of the two states

(1 = rain) and (2 = sunshine)

- The transition matrix is

$$\mathbf{P} = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}$$

- In areas where sunshine is much more common than rain such as Bangkok,

- more realistic transition matrix would be

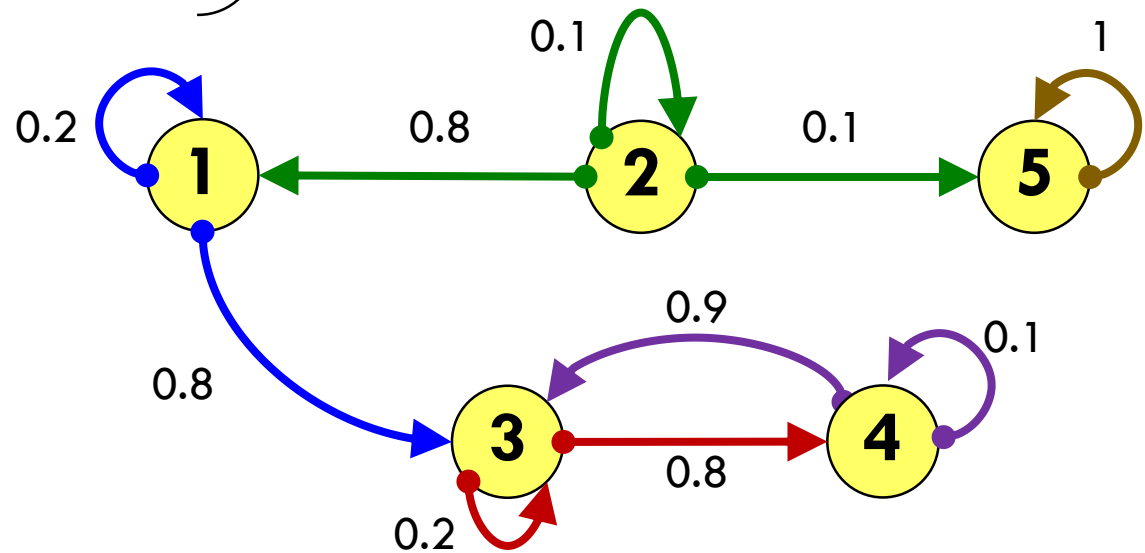
$$\mathbf{P} = \begin{pmatrix} 0.30 & 0.70 \\ 0.15 & 0.85 \end{pmatrix}$$

# Example #3

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$$P = \begin{pmatrix} 0.2 & 0 & 0.8 & 0 & 0 \\ 0.8 & 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

From the Markov Matrix,  
draw a Markov Chain



# Multistep Transition

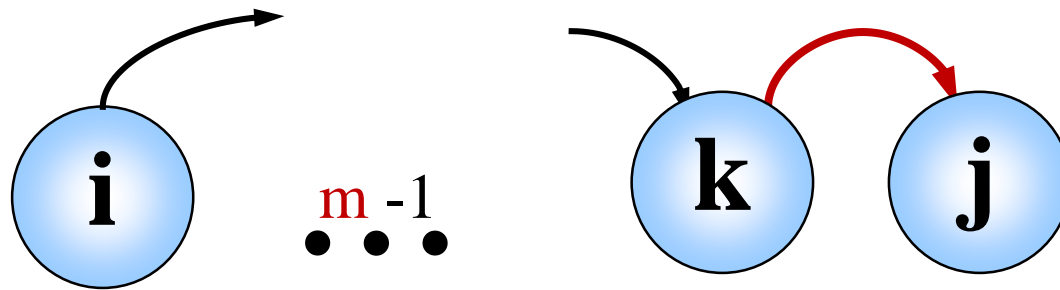
## Homogeneous Markov Chain

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- $m$ -step transition probabilities are

$$p_{ij}^{(m)} \equiv P[X_{n+m} = j \mid X_n = i]$$

$$= \sum_{\forall k} p_{ik}^{(m-1)} p_{kj} \quad m = 2, 3, \dots$$



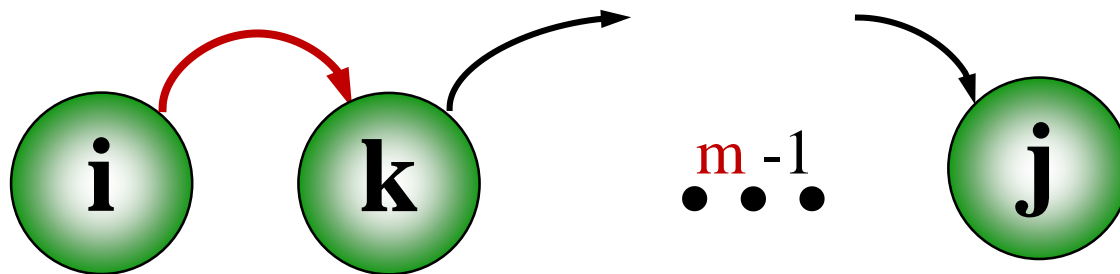
# Multistep Transition

## Homogeneous Markov Chain

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$$p_{ij}^{(m)} \equiv P[X_{n+m} = j \mid X_n = i]$$

$$= \sum_{\forall k} p_{ik} p_{kj}^{(m-1)} \quad m = 2, 3, \dots$$



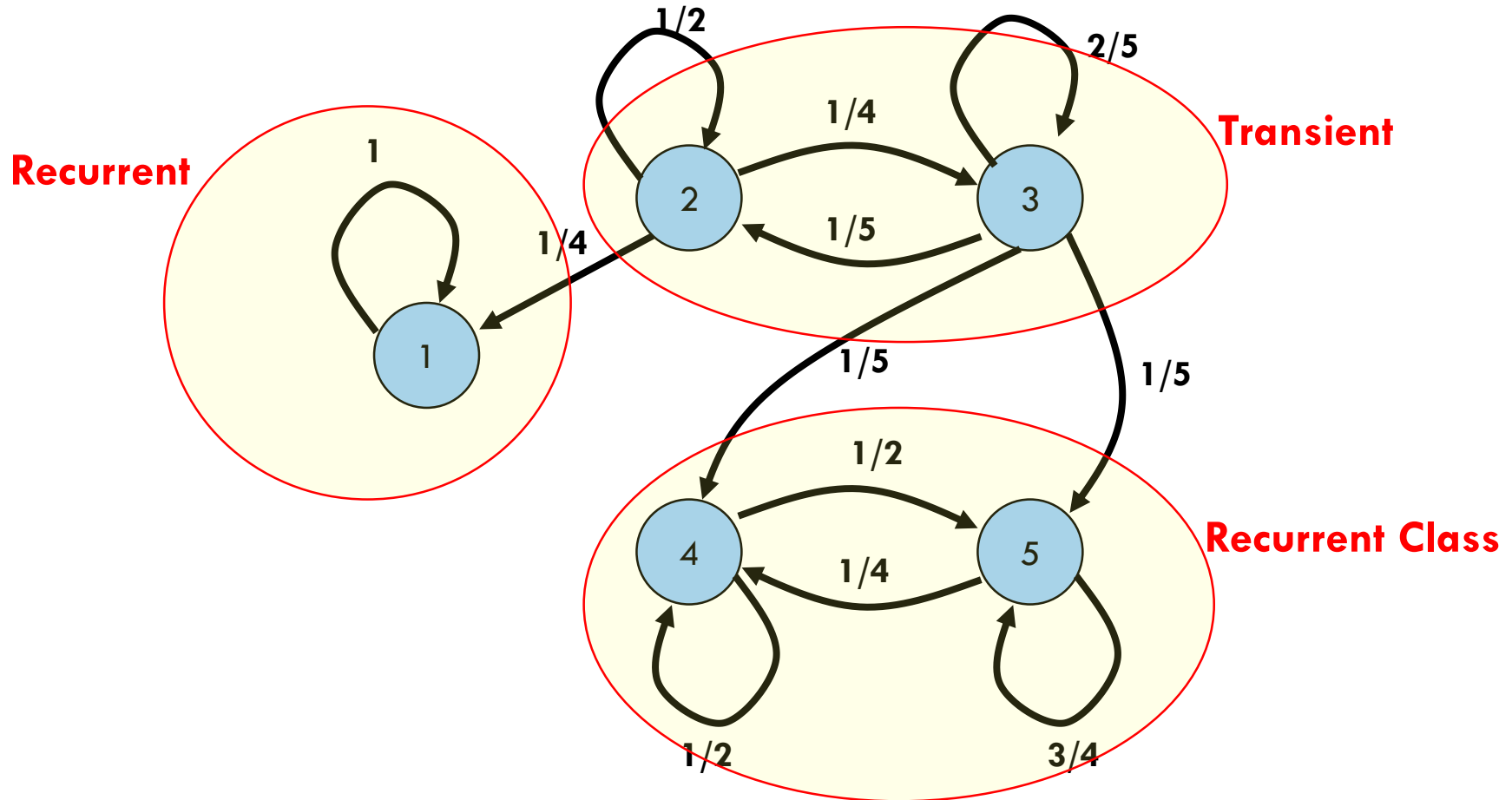
# Type of States

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- Transient State
  - Starting @ state E, it will eventually leave state E and never return
- Recurrent State
  - Starting @ state E, it will **continuously** reoccur (come back to state E)

# Transient or Recurrent States

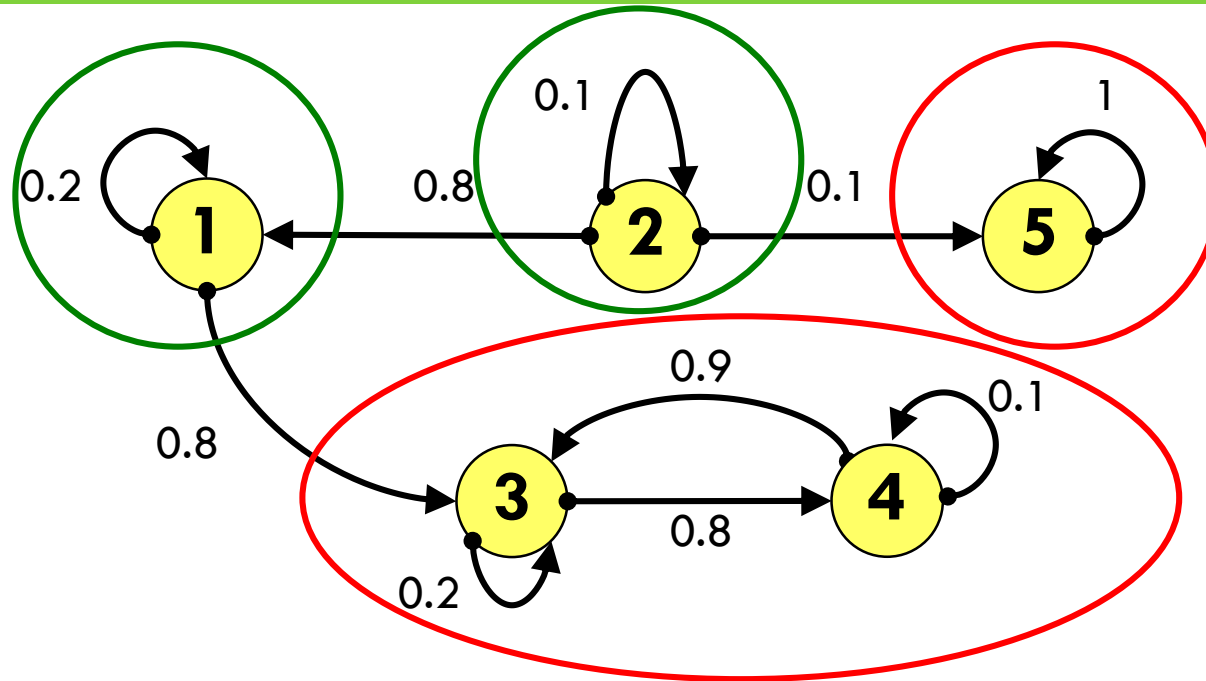
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# Example 3 (Revisit)

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Identify

Transient states = ?     {1} , {2}

Recurrent states = ?     {5} , {3,4}

# Transient or Recurrent States

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- $f_j^{(n)} = P[\text{the process first returns to state } \mathbf{j} \text{ after leaving state } \mathbf{j} \text{ in } \mathbf{n} \text{ steps}]$
- $f_j = P[\text{the process returns to state } \mathbf{j} \text{ after leaving state } \mathbf{j}]$

$$f_j = \sum_{n=1}^{\infty} f_j^{(n)}$$

- $M_j = \text{Mean recurrence time of state } \mathbf{j}$

$$M_j = \sum_{n=1}^{\infty} n f_j^{(n)}$$

# Transient or Recurrent States

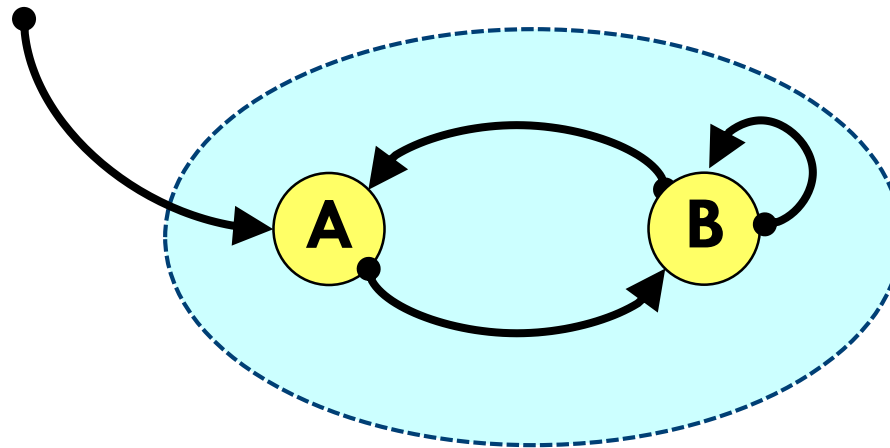
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- If  $f_j < 1$ 
  - State  $E_j$  is called “**Transient State**”
- If  $f_j = 1$ 
  - State  $E_j$  is called “**Recurrent State**”
    - If  $M_j = \infty$ 
      - State  $E_j$  is called “**Recurrent Null State**”
    - If  $M_j < \infty$ 
      - State  $E_j$  is called “**Recurrent Nonnull State**”

# A Closed Set

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- A set that once the Markov chain has entered the set  $\rightarrow$  it cannot leave the set

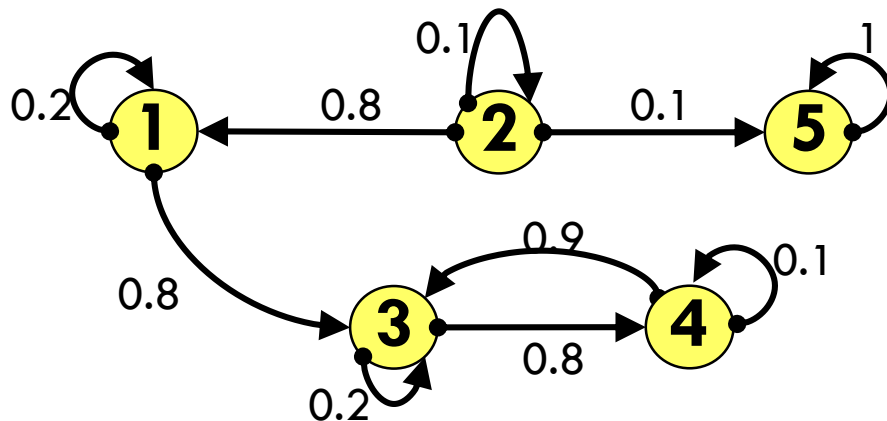


# Example 3 (Revisit)

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$$P = \begin{pmatrix} 0.2 & 0 & 0.8 & 0 & 0 \\ 0.8 & 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Identify is this a closed set ?

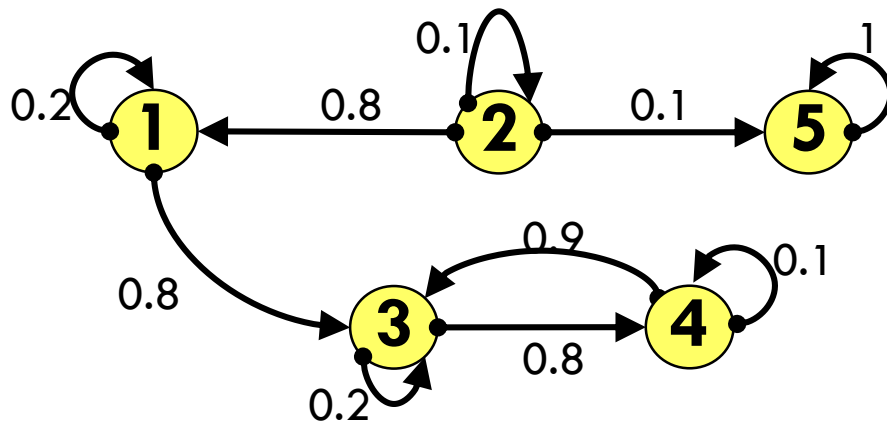


- {1} **Not Closed**
- {1,2} **Not Closed**
- {1,2,3} **Not Closed**
- {2,3} **Not Closed**
- {4,5} **Not Closed**
- {3,4,5}
- {5}
- {1,2,3,4,5}

# Example 3 (Revisit)

30

$$P = \begin{pmatrix} 0.2 & 0 & 0.8 & 0 & 0 \\ 0.8 & 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



Identify is this a closed set ?

- {1} **Not Closed**
- {1,2} **Not Closed**
- {1,2,3} **Not Closed**
- {2,3} **Not Closed**
- {4,5} **Not Closed**
- {3,4,5} **Closed**
- {5} **Closed**
- {1,2,3,4,5} **Closed**

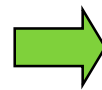
# A Closed Set

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P =

0.2	0	0.8	0	0
0.8	0.1	0	0	0.1
0	0	0.2	0.8	0
0	0	0.9	0.1	0
0	0	0	0	1

$\{1,2,3,4,5\}$  = Closed Set



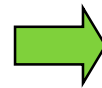
It can be reduced to a smaller closed set  $\{3,4,5\}$

$\{3,4,5\}$  = Closed Set



It can be reduced to a smaller closed set  $\{3,4\}$  and  $\{5\}$

$\{3,4\}$  = Closed Set  
 $\{5\}$  = Closed Set



It can **NOT** be reduced to any smaller closed set

# Irreducible Set

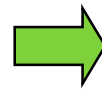
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P =

0.2	0	0.8	0	0
0.8	0.1	0	0	0.1
0	0	0.2	0.8	0
0	0	0.9	0.1	0
0	0	0	0	1

## Reducible Set

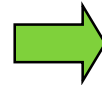
$\{1,2,3,4,5\}$  = Closed Set



It can be reduced to a smaller closed set  $\{3,4,5\}$

## Reducible Set

$\{3,4,5\}$  = Closed Set



It can be reduced to a smaller closed set  $\{3,4\}$  and  $\{5\}$

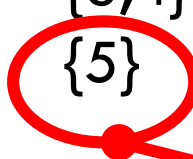
## Irreducible Set

$\{3,4\}$  = Closed Set

$\{5\}$  = Closed Set



It can NOT be reduced to any smaller closed set

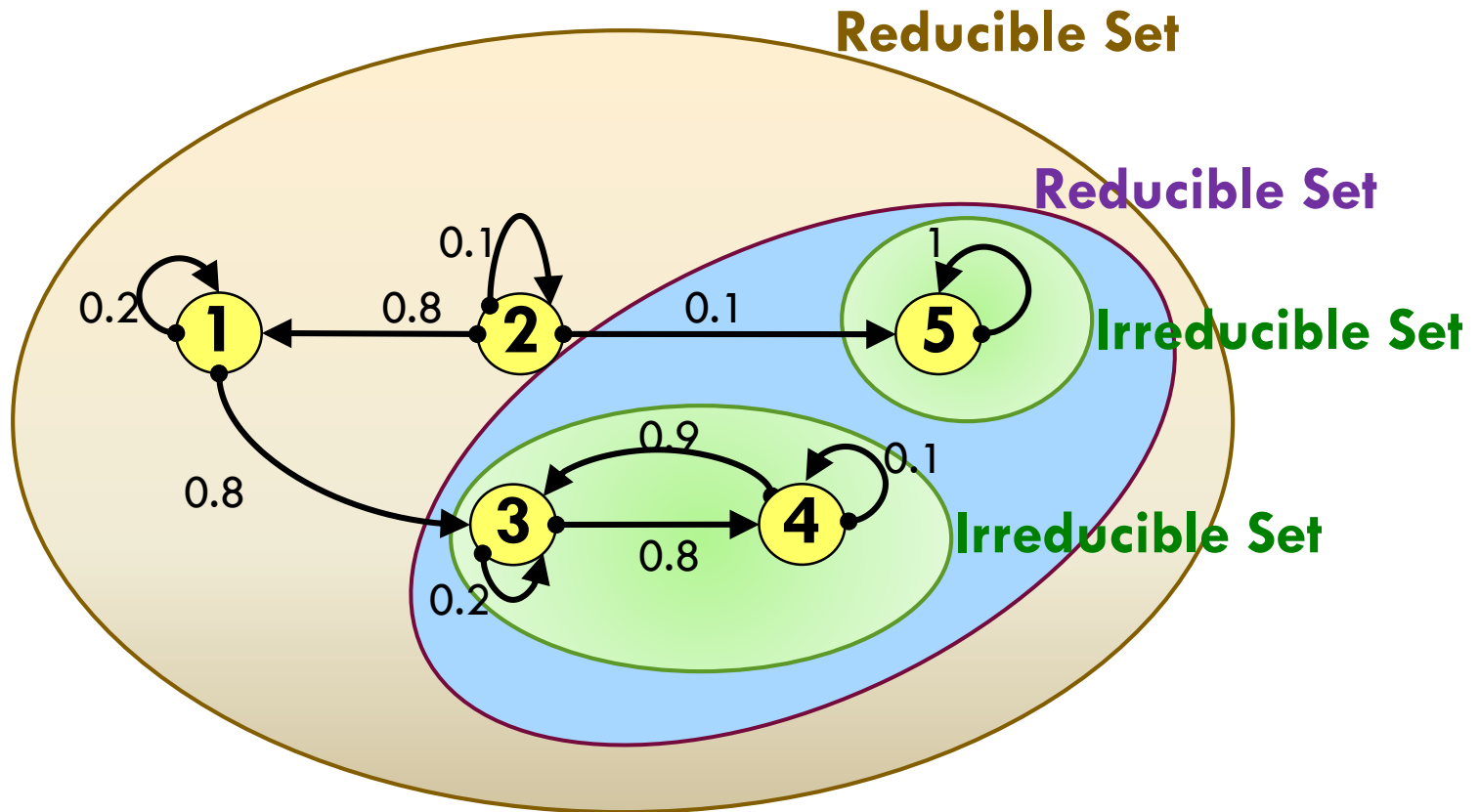


**Absorbing State**



# Irreducible Set

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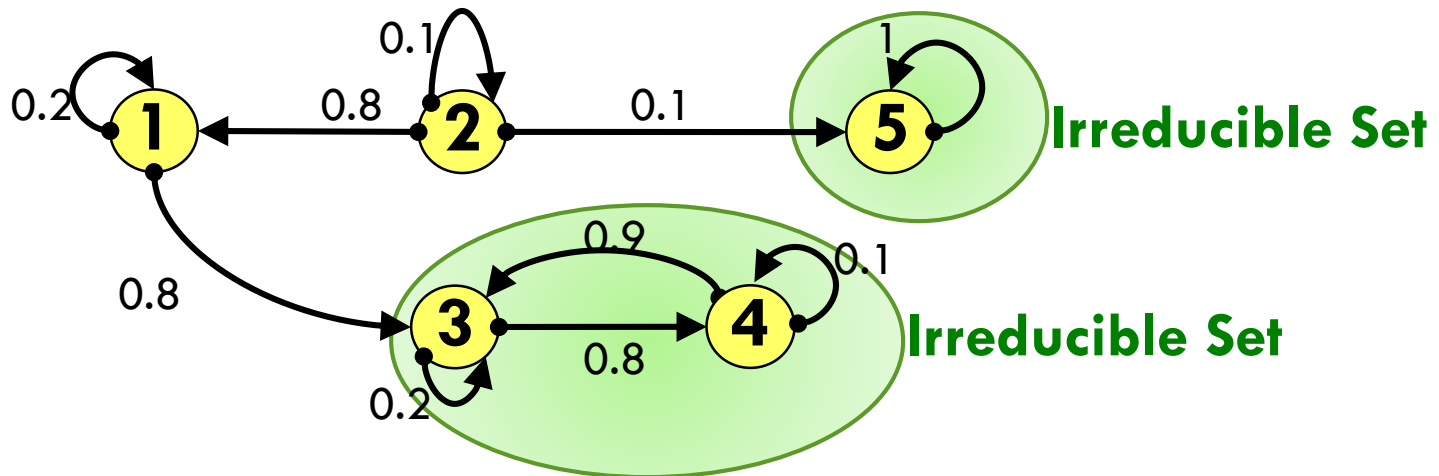


# Irreducible Markov Chain

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- A Markov Chain is *irreducible* if every state can be reached from every other state in a *finite* number of steps.

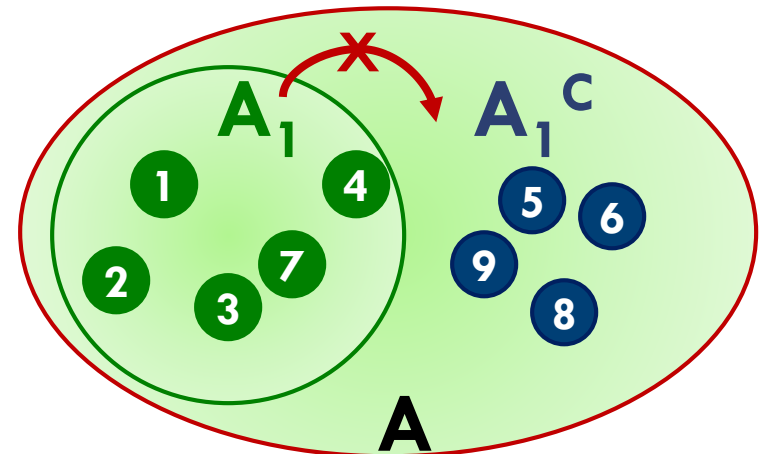
$$p_{ij}^{(m_0)} > 0 \quad \text{for } m_0 = \text{integer}$$



# Not Irreducible Markov Chain

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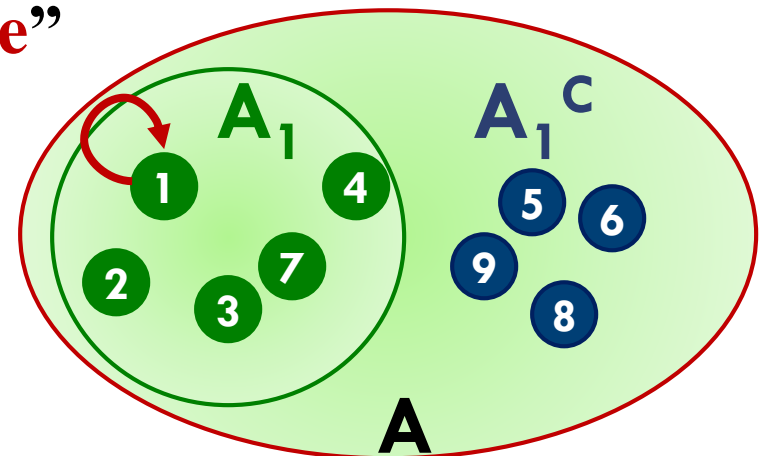
- Case 1
  - For  $A$  = set of all states in a Markov chain
  - $A_1 \subset A$
  - If no one-step transition from state  $A_1$  to  $A_1^c$
  - $A_1$  is defined as “**Closed**”



# Not Irreducible Markov Chain

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- Case 2
  - For  $A$  = set of all states in a Markov chain
  - $A_1 \subset A$
  - If  $A_1$  consists of one or more state  $E_i$  that once get in state  $E_i$ , the process cannot move to any other states
  - $E_i$  is called “**Absorbing State**”
  - $p_{ii} = 1$



# Note on Irreducible

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- All states within an irreducible set are of the same classification
  - If one state is transient  $\rightarrow$  all transient
  - If one state is recurrent  $\rightarrow$  all recurrent
- Irreducible sets
  - $\rightarrow$  Communication between states
  - $\rightarrow$  Communication must be both ways
  - $\rightarrow$  But does not have to be in one step

# Periodic or Aperiodic

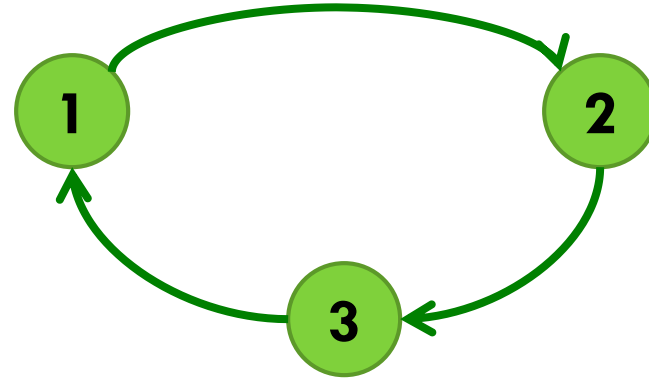
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- Recurrent State
- Let  $\beta = \text{integer}$
- If the **only possible steps** that the process returns to state  $E_i$  are  $\beta, 2\beta, 3\beta, \dots$ 
  - If  $\beta > 1$  and  $\beta$  is the largest integer
    - State  $E_i$  is called “**Periodic**”
    - The **recurrence time** for state  $E_j$  has period  $\beta$
  - If  $\beta = 1$ 
    - State  $E_i$  is called “**Aperiodic**”

# Example

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$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$



- Starting from state 1
  - It returns to state 1 for every 3 steps
  - Same as other two states.
- The chain is therefore **Periodic**

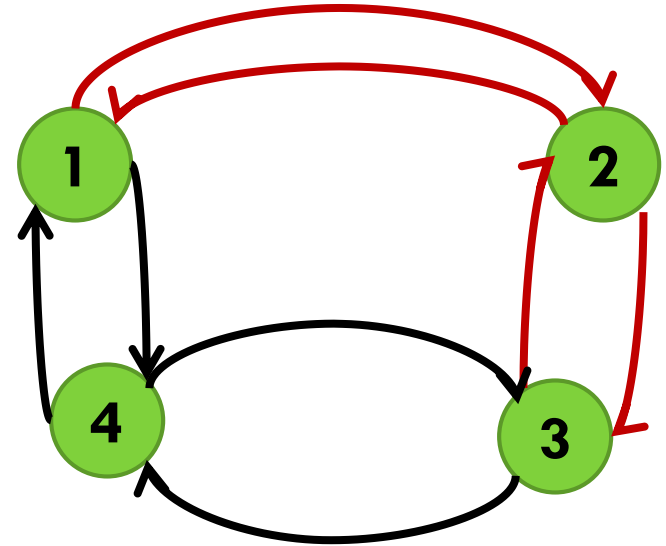
Modified from:

[http://www.wikicoursenote.com/wiki/Again\\_on\\_Markov\\_Chain](http://www.wikicoursenote.com/wiki/Again_on_Markov_Chain)

# Example

40

$$\mathbf{P} = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{pmatrix}$$



- Starting from state 1
  - It returns to state 1 for every **multiple of 2 steps**
  - Same as other states.
- The chain is therefore **Periodic**

Modified from:

[http://www.wikicoursenote.com/wiki/Again\\_on\\_Markov\\_Chain](http://www.wikicoursenote.com/wiki/Again_on_Markov_Chain)



# Ergodicity

- $E_j = \text{Ergodic}$  if
  - $E_j = \text{Aperiodic}$  and *Recurrent Nonnull*  
(recurrent nonnull)
  - $f_j = 1$ ,  $M_j < \infty$ , and  $\beta = 1$   
(recurrent) (Aperiodic)
- A Markov Chain is **ergodic**
  - If **all** states of a Markov Chain are **ergodic**
  - If number of states is **finite** and **all** states of a Markov Chain are **aperiodic**, and **irreducible**

# Theorem 1

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- The states of **an irreducible** Markov Chain are either
  - all transient or
  - all recurrent nonnull or
  - all recurrent null
- If periodic, then all states have the same period  $\beta$

# Definition

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- Let  $\pi_j^{(n)} = P[\text{finding the system in state } E_j \text{ at the } n^{\text{th}} \text{ step}]$   
$$\pi_j^{(n)} = P[X_n = j]$$
- Let  $\pi_j = \text{Stationary Probability}$   
=  $P[\text{being in state } j \text{ at arbitrarily time}]$   
= The limiting state probabilities

# Theorem 2

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- In an irreducible and aperiodic, homogeneous Markov Chain,
- the limiting state probabilities  $[\pi_j]$  always exist and are independent of the initial state probability distribution  $[\pi_j^{(0)}]$

$$\pi_j = \lim_{n \rightarrow \infty} \pi_j^{(n)}$$

# Theorem 2

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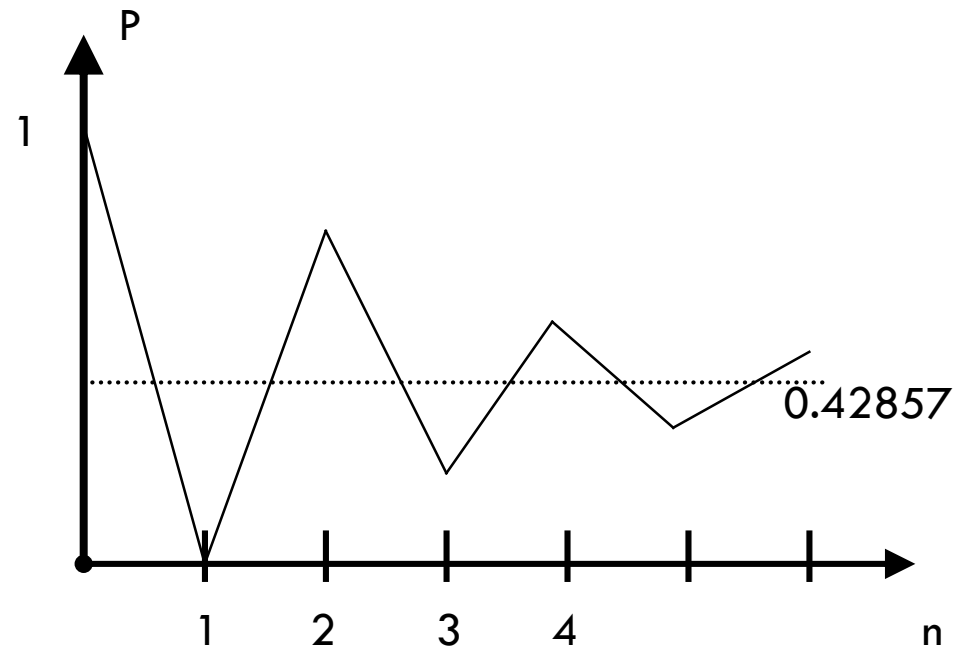
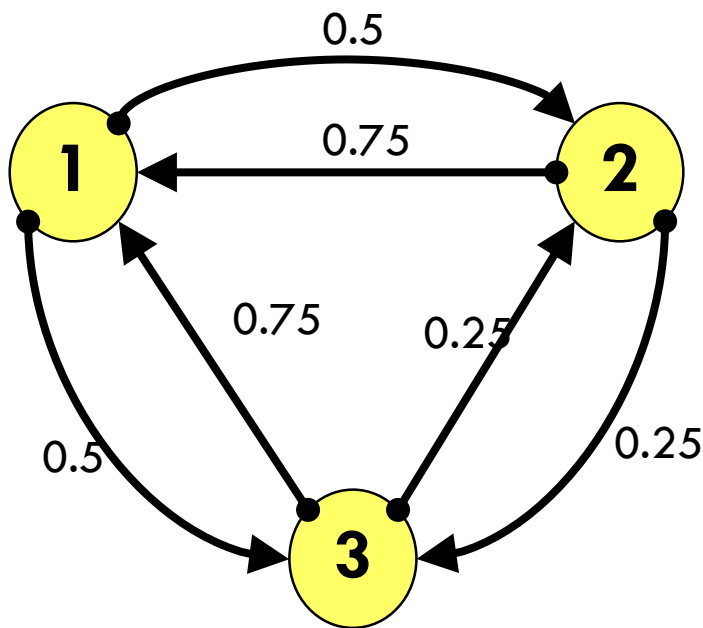
- **Either**
- **Case (a)**
  - All states are transient or
  - All states are recurrent null
    - ➔  $\pi_j = 0 \quad \forall j$
    - ➔ No stationary distribution exist
- **Or Case (b)**
  - All states are recurrent nonnull
    - ➔  $\pi_j > 0 \quad \forall j$
    - ➔ Stationary distribution exist ➔  $\pi_j = 1 / M_j$

# Steady-State Behavior

## (Limiting condition)

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$P[\text{be in state 1 @ time } n \mid \text{in state 1 @ } t = 0]$



$$\lim_{n \rightarrow \infty} \Pr\{X_n = 1 \mid X_0 = 1\} = 0.42857$$

$$\lim_{n \rightarrow \infty} \Pr\{X_n = 1 \mid X_0 = 2\} = 0.42857$$

Same as  
Anan Phonphoem

Dept. of Computer Engineering, Kasetsart University, Thailand

# To solve for $\pi_j$

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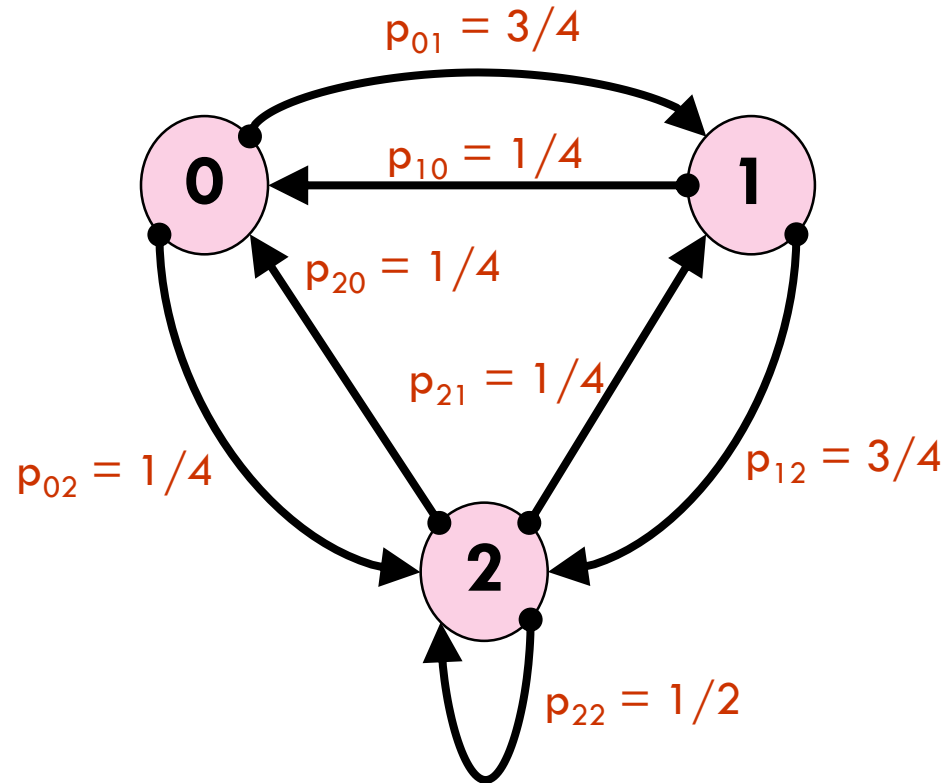
Balance Equations:  $\pi_j = \sum_i \pi_i p_{ij}$   
(Linear dependency)

Normalization condition:  $1 = \sum_i \pi_i$

# Markov Chain Example

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- Driving from town to town





# Markov Chain Example

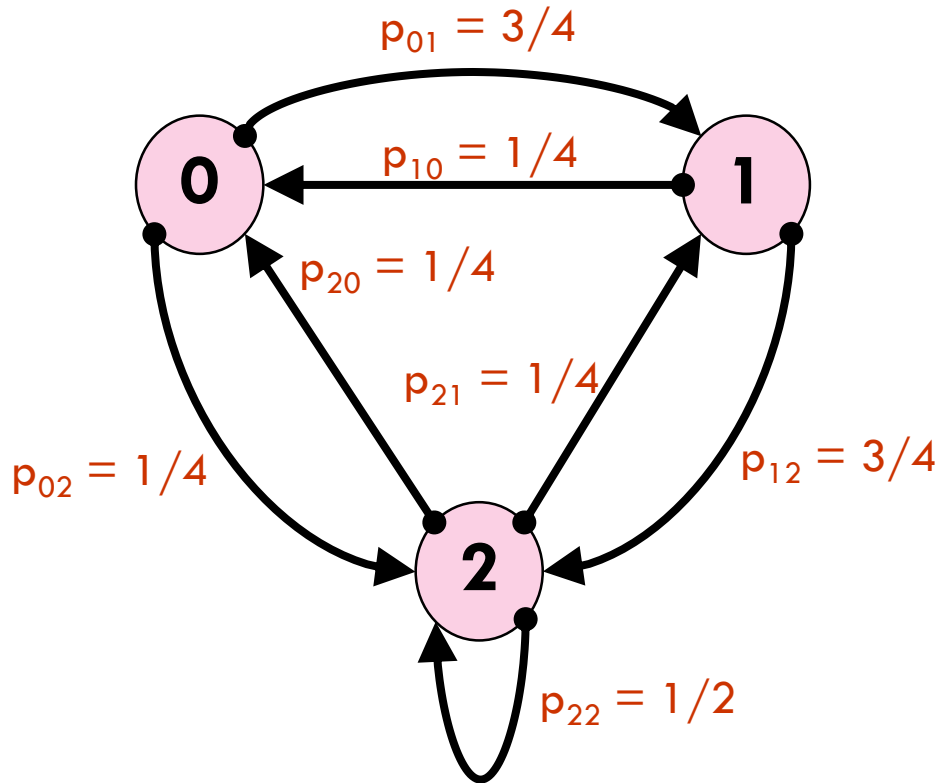
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- Let  $P =$  Transition probability matrix  
 $= [p_{ij}]$
- Let  $\pi = [\pi_0, \pi_1, \pi_2, \dots]$
- From Balance equation

$$\pi = \pi P$$

# Markov Chain Example

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$$P = \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/4 & 0 & 3/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

# Markov Chain Example

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$$\pi = \pi \mathbf{P}$$
$$[\pi_0, \pi_1, \pi_2] = [\pi_0, \pi_1, \pi_2] \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/4 & 0 & 3/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$\pi_0 = 0 \quad \pi_0 + 1/4 \pi_1 + 1/4 \pi_2$$

$$\pi_1 = 3/4 \pi_0 + 0 \quad \pi_1 + 1/4 \pi_2$$

$$\pi_2 = 1/4 \pi_0 + 3/4 \pi_1 + 1/2 \pi_2$$

$$\mathbf{1} = \pi_0 + \pi_1 + \pi_2$$

# Markov Chain Example

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**Solution:**

$$\begin{aligned}\pi_0 &= 0.20 \\ \pi_1 &= 0.28 \\ \pi_2 &= 0.52\end{aligned}$$

- This is the stationary (equilibrium) state probability
- This is the ergodic Markov Chain
  - Finite number of states
  - Irreducible (recurrent)

# Transient Behavior

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- We want to know the probability of finding the process in state  $E_j$  at time  $n$
- $\pi^{(n)} = [\pi_0^{(n)}, \pi_1^{(n)}, \pi_2^{(n)}, \dots]$
- From Transition Probability  $\mathbf{P}$ 
  - We can calculate:
$$\pi^{(1)} = \pi^{(0)}\mathbf{P}$$
$$\pi^{(n)} = \pi^{(n-1)}\mathbf{P}$$
  - By recursive:
$$\pi^{(n)} = \pi^{(0)}\mathbf{P}^n$$

# Transient Behavior

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- From stationary probability:  $\pi = \lim_{n \rightarrow \infty} \pi^{(n)}$

- From  $\pi^{(n)} = \pi^{(n-1)}\mathbf{P}$

$$\lim_{n \rightarrow \infty} \pi^{(n)} = \lim_{n \rightarrow \infty} \pi^{(n-1)} \mathbf{P}$$

$$\pi = \pi \mathbf{P}$$

- Note: The solution  $\boldsymbol{\pi}$  is independent of  $\pi^{(0)}$

# Transient Behavior

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$$\pi^{(0)} = [1, 0, 0]$$

n	0	1	2	3	$\infty$
$\pi_0^{(n)}$	1	0	0.25	0.187	0.20
$\pi_1^{(n)}$	0	0.75	0.062	0.359	0.28
$\pi_2^{(n)}$	0	0.25	0.688	0.454	0.52

$$\pi^{(0)} = [0, 0, 1]$$

n	0	1	2	3	$\infty$
$\pi_0^{(n)}$	0	0.25	0.187	0.203	0.20
$\pi_1^{(n)}$	0	0.25	0.313	0.266	0.28
$\pi_2^{(n)}$	1	0.50	0.500	0.531	0.52

# HW: Steady-State Behavior

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- By using EXCEL (or others), show the graph of Steady-State behavior
- Show number of steps

