LECTURE #2
PROBABILITY REVIEW (I)

204528

Queueing Theory and Applications in Networks

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Outline

- Probability meaning
- Random Variable
- Discrete RV
- Continuous RV
- Multiple RVs

What is Probability ?

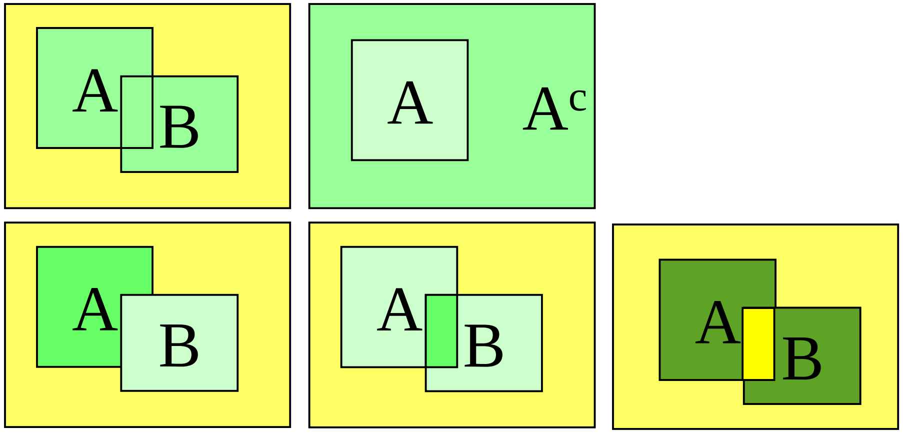
- Physical Property
  - Lottery
- Knowledge
  - Snow in Thailand
- Probability meaning
  - Situation cannot exactly replicate
  - But not chaotic (have a pattern)

Probability

- Definition
  - Logic of probability
- Axiom
  - Fact without proof
- Theorem
  - Derived from Definition, Axiom, or other Theorems

Probability Mathematics

- Set Theory
  - Set operation
  - Set properties

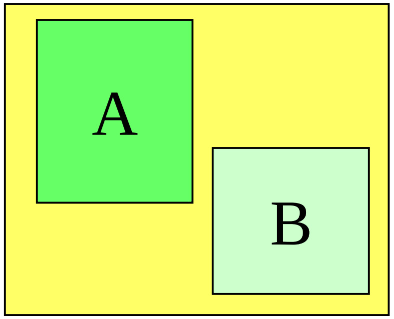


Important Set Properties

1. <u>Mutually Exclusive</u>

   $\mathbf{A_i \cap A_j = \varnothing}$ for $i \neq j$

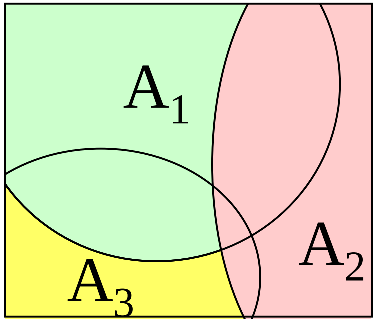
   $\mathbf{A \cap B = \varnothing} \rightarrow$ called **Disjoint** for only 2 sets



2. <u>Collectively Exhaustive</u>

   $$\mathbf{A_1 \cup A_2 \cup \ldots \cup A_n = S}$$

   $$\bigcup_{i=1}^{n} \mathbf{A_i = S}$$



Experiment

What is an Experiment?

- Method for finding some facts/conclusions

Give an example?

- For movie "**Mission Impossible - Fallout**", is it fun?

http://time.com/5349500/mission-impossible-fallout/

Mission Impossible

https://upload.wikimedia.org/wikipedia/en/3/3c/Missionimpossiblebluraybox set.jpg

Film	Year	Budget (M$)	Box off. gross (M $)	Cimema Score
Mission: Impossible	1996	80	458	B+
Mission: Impossible 2	2000	125	546	B
Mission: Impossible III	2006	150	398	A-
Mission: Impossible – Ghost Protocol	2011	145	695	A-
Mission: Impossible – Rogue Nation	2015	150	687	A-
Mission: Impossible – Fallout	2018	178	330 (1 month)	A

https://en.wikipedia.org/wiki/Mission:_Impossible_(film_series)

Experiment

What is an Experiment?
- Method for finding some facts/conclusions

Give an example?
- For movie "**Mission Impossible - Fallout**", is it fun?
- Stand in front of the theatre
- Ask audiences, fun or not?

Composition of an experiment
- Procedure
- Observation

Why experiment is needed?
- Uncertainty

https://variety.com/2018/film/reviews/mission-impossible-fallout-review-1202872043/

Experiment

- Concern about movie "**Mission Impossible - Fallout**" experiment
  - Should I ask man, women, or teenager?
  - Experience of the audiences
  - Knowledge of the audiences

- Complicated experiment →need **Model**
  - Real experiments: too complicate
  - Capture only the important part
  - Model Example:
    - Treat all audiences the same
    - Answer will only be like/dislike

Experiment

Same Procedure
but different Observations
→ Different Experiments

<u>Example:</u>

1. Flip a coin 3 times, Observe the sequence of heads/tails
2. Flip a coin 3 times, Observe # of heads

Definition in Probability

- **Outcome**
  - Any possible observation
- **Sample Space**
  - ***Finest-grain****:* each outcome is different
  - ***Mutually exclusive****:* if one outcome occurs, other will not occur
  - ***Collectively exhaustive****:* every outcome must be in the sample space
- **Event**
  - Set of outcomes (Must know all outcomes)
  - Event $\subset$ Sample Space

Event Examples

For an experiment:

Roll a dice, observe the shown numbers

Outcomes:

number = 1,2,3,4,5,6

Sample space:

S = {1,2,3,…,6}

Event examples:

E_1 = {number < 3} = {1,2}

E_2 = {number is odd} = {1,3,5}

Set VS. Probability

Set Algebra	Probability
Set	Event
Universal set	Sample space
Element	Outcome

Probability of Event, P[☺]

P[☺]
is a function that maps event
in the sample space to real number

https://www.123rf.com/

From experiment: Press a number on the ATM keypad

Outcomes:

number = 0,1,2,3,4,5,6,7,8,9

Sample space:

S = {0,1,2,…,9}

Event examples:

E1 = {number < 3} = {0,1,2}

E2 = {number is odd} = {1,3,5,7,9}

P[E_1] = 3/10

P[E_2] = 5/10 = 1/2

Probability Axioms

Axiom 1: For any event A, **$P[A] \geq 0$**

Axiom 2: **$P[S] = 1$**

Axiom 3: For events $A_1, A_2, \ldots, A_n$ of mutual exclusive events

$$P[A_1 \cup A_2 \cup \ldots \cup A_n] = P[A_1] + P[A_2] + \ldots + P[A_n]$$

Example Theorems

- **Theorem:** If A and B are disjoint, then

$$\mathbf{P[A \cup B] = P[A] + P[B]}$$

- **Theorem:** If $B = B_1 \cup B_2 \cup \ldots \cup B_n$ and $B_i \cap B_j = \emptyset$ for $i \neq j$, then

$$\mathbf{P[B]} = \sum_{i=1}^{n} \mathbf{P[B_i]}$$

Equally Likely

Theorem:

For an experiment with sample space $S=\{s_1,\ldots, s_n\}$ if each outcome is **equally likely**,

$$\mathbf{P[s_i] = 1/n \qquad 1 \le i \le n}$$

Consequences of Axioms

Theorem:

- $P[\varnothing] = 0$
- $P[A^c] = 1 - P[A]$
- For any A and B (not necessary disjoint)

  $P[A \cup B] = P[A] + P[B] - P[A \cap B]$
- If $A \subset B$, then $P[A] \leq P[B]$

Example

GROUP A	GROUP B	GROUP C	GROUP D	GROUP E	GROUP F	GROUP G	GROUP H
Uruguay	Spain	France	Croatia	Brazil	Sweden	Belgium	Colombia
Russia	Portugal	Denmark	Argentina	Switzerland	Mexico	England	Japan
Saudi Ar	IR Iran	Peru	Nigeria	Serbia	Korea Rep	Tunisia	Senegal
Egypt	Morocco	Australia	Iceland	Costa Rica	Germany	Panama	Poland

Assume: all teams are **equally likely** to win the game

What is the probability that **France** will be the champion ?

$$\frac{1}{32}$$

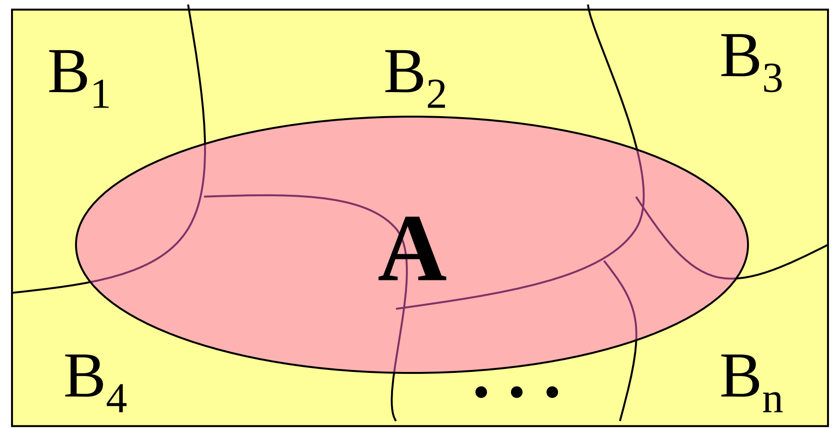
What is the probability that **Germany** will be the champion, given that **Sweden** and **Mexico** are withdrawn ?

$$\frac{1}{30} \quad \text{or} \quad \frac{1}{16}$$

A Useful Theorem

Let $B_1, B_2, \ldots, B_n$ be mutual exclusive events whose union equals sample space S

→ partition of S



For any event A

$$A = A \cap S = A \cap (B_1 \cup B_2 \cup \ldots \cup B_n)$$

$$P[A] = P[A \cap B_1] + P[A \cap B_2] + \ldots + P[A \cap B_3]$$

Theorem:

$$\mathbf{P[A] = \sum_{i=1}^{n} P[A \cap B_i]}$$

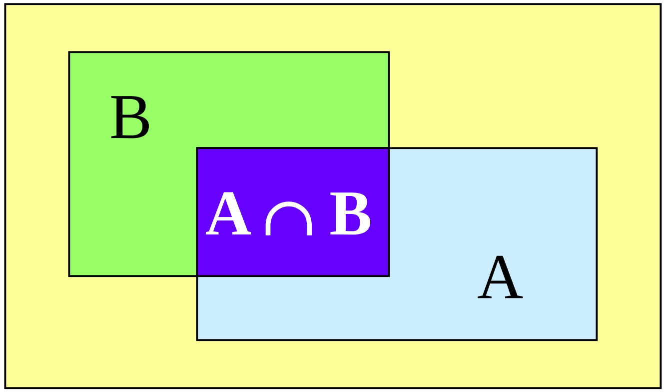
Conditional Probability

- In practice, it maybe impossible to find the precise outcome of an experiment
- However, if we know that Event B has occurred (the outcome of Event A is in set B)
  - Probability of A when B occurs can be described
  - Still don’t know P[A]

Conditional Probability

- **Notation:** **P[A|B]**
  - "Probability of A given B"
  - The condition probability of the event A given the occurrence of the event B
- **Definition:**

$$P[A|B] = \frac{P[AB]}{P[B]}$$



- **Example:**

Tiger Woods hits Hole-in-one

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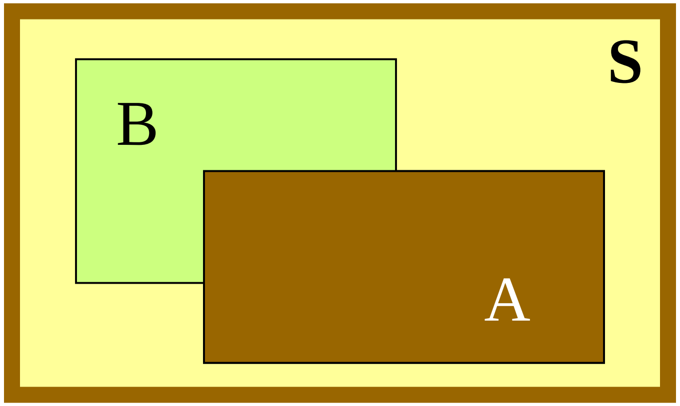
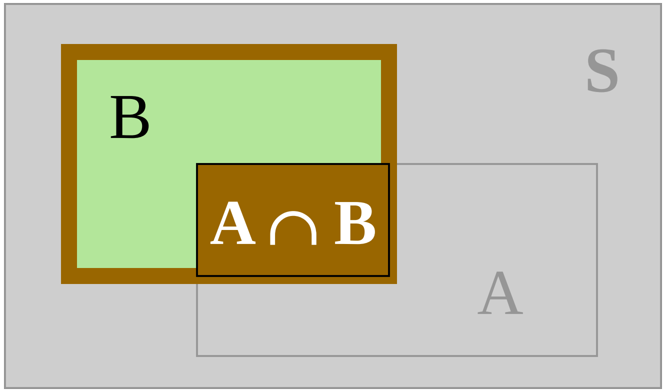
More Explanation

$$P[A|S] = \frac{P[AS]}{P[S]} = \frac{P[A]}{1} = P[A]$$

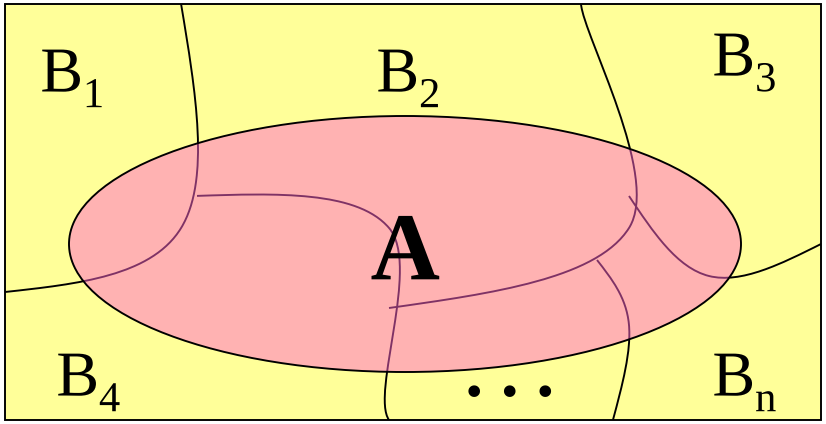


$$P[A|B] = \frac{P[AB]}{P[B]}$$



Law of Total Probability

- Let $B_1, B_2, \ldots, B_n$ be mutual exclusive events whose union equals sample space S
- $P[B_i] > 0$



Theorem: $P[A] = \sum_{i=1}^{n} P[A \cap B_i]$

$P[A] = P[A \cap B_1] + P[A \cap B_2] + \ldots$

$P[A] = P[A|B_1]P[B_1] + P[A|B_2]P[B_2] + \ldots$

Theorem: $P[A] = \sum_{i=1}^{n} P[A|B_i]P[B_i]$

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Bayes' Theorem

$$P[B|A] = \frac{P[BA]}{P[A]}$$

$$P[A|B] = \frac{P[AB]}{P[B]}$$

$$= \frac{P[A|B]P[B]}{P[A]}$$

Theorem: $$\mathbf{P[B|A] = \frac{P[A|B]P[B]}{P[A]}}$$

2 Independent Events

Definition: Event A and B are independent iff

$$\mathbf{P[AB] = P[A]P[B]}$$

$$P[A|B] = \frac{P[AB]}{P[B]}$$

$$= \frac{P[A]P[B]}{P[B]}$$

$$\mathbf{P[A|B] = P[A]}$$

$$\mathbf{P[B|A] = P[B]}$$

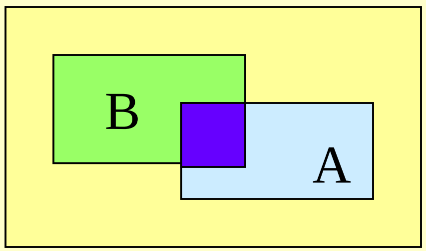
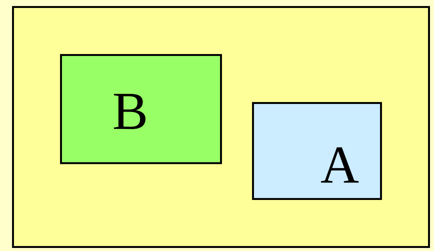
Independent Interpretation

$$P[A] \quad = \; 0.3$$

$$P[A|B] = 0.3$$

No matter event B occurs or not, event A is not affected

Independent VS. Disjoint

Independent	Disjoint
	
$P[AB] \neq 0$	$P[AB] = 0$
$P[A \cap B] = P[A] * P[B]$	$P[A \cup B] = P[A] + P[B]$

Note: Independent = Disjoint iff P[A]=0 or P[B]=0

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3 Independent Events

Definition: Event A_1,A_2 and A_3 are independent iff

1) A_1 and A_2 are independent
2) A_2 and A_3 are independent
3) A_1 and A_3 are independent
4) $P[A_1 \cap A_2 \cap A_3] = P[A_1]\,P[A_2]\,P[A_3]$

Why only number 4 is insufficient ?

Definition: Event A and B are independent iff

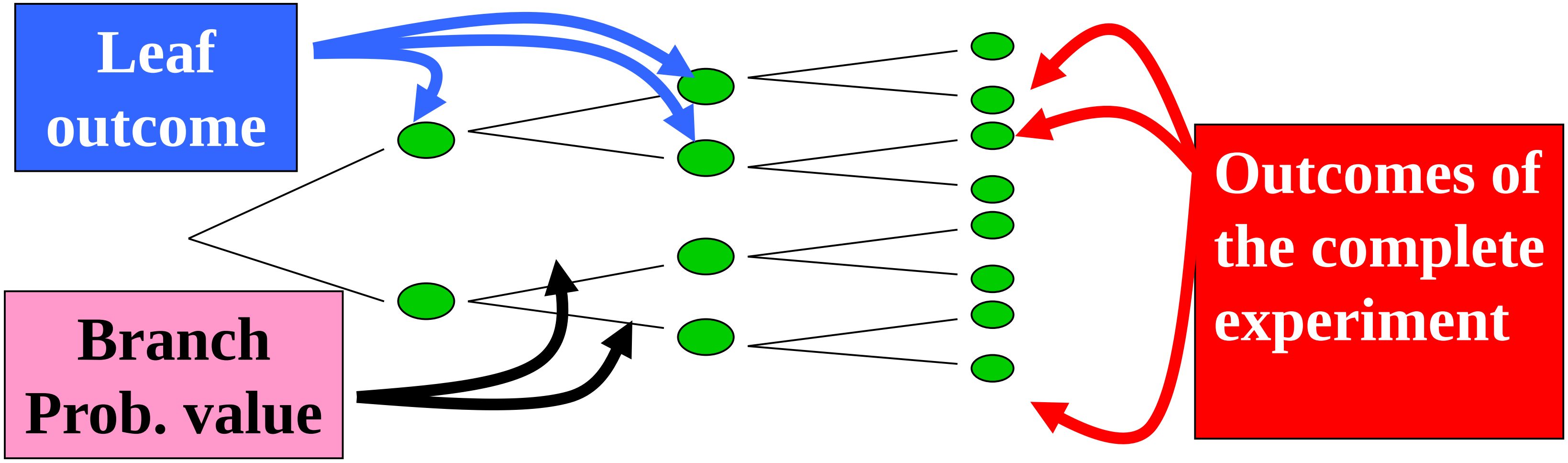
$$P[AB] = P[A]P[B]$$

Most Common Application

- **Assume that the events of separate experiments are independent**
- Example:
  - Assume that outcome of a coin toss is independent of the outcomes of all prior and all subsequent coin tosses
  - P[H] = P[T] = ½
  - P[HTH] = P[H]P[T]P[H] = 1/2*1/2*1/2 = 1/8

Sequential Experiments

- Experiment: in sequence
  subexperiments →subexperiments
- Each subexp. may depend on the previous one
- Represented by a **Tree Diagram**
- **Model Conditional Prob. → Sequential Experiment**



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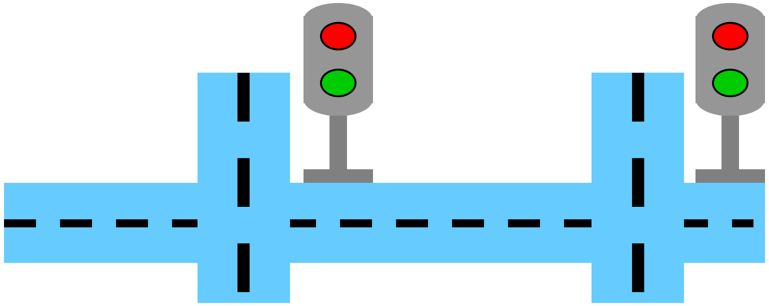
Sequential Example

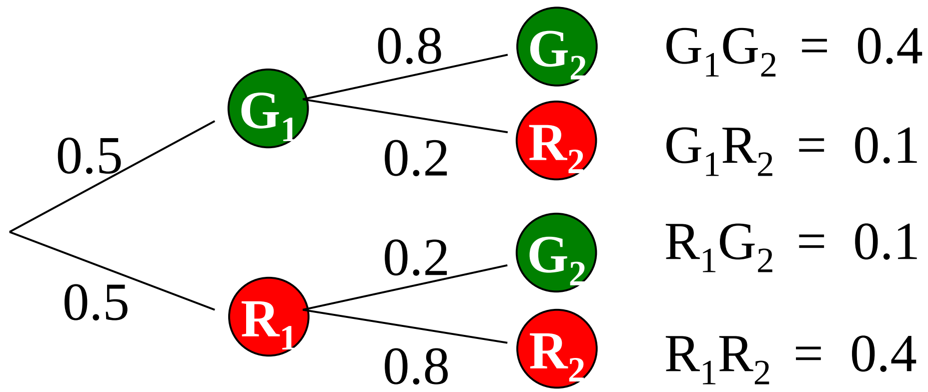


Timing coordination of 2 traffic lights

- P[the 2nd light is the same color as the 1st] = 0.8
- Assume 1st light is equally likely to be green or red

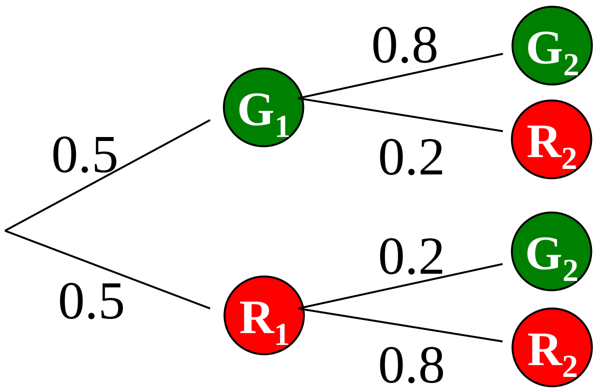
Find P[The second light is green] ?

Sequential Example



- $P[G_1] = P[R_1] = 0.5$
- $P[G_2G_1] = P[G_2|G_1]P[G_1] = (0.8)(0.5) = 0.4$

Sequential Example



$G_1G_2 = 0.4$

$G_1R_2 = 0.1$

$R_1G_2 = 0.1$

$R_1R_2 = 0.4$

P[The second light is green] ?

$$P[G_2] = P[G_2G_1] + P[G_2R_1] = 0.4 + 0.1 = 0.5$$

$$P[G_2] = P[G_2|G_1]P[G_1] + P[G_2|R_1]P[R_1]$$

Counting Method

Principle of Counting Method

If experiment A has **n** possible outcomes,
and experiment B has **k** possible outcomes,

→Then there are **nk** possible outcomes
when you perform both experiments

k-permutations

Theorem:

The number of **k**-permutations (ordered sequence) of **n** distinguishable objects is

$$(n)_k = \frac{n!}{(n-k)!}$$

Choose with replacement

Theorem: Given **n** distinguishable objects,

There are $\mathbf{n^k}$ ways to choose with replacement a sample of **k** objects

k-combination

Theorem:

The number of ways to choose **k** objects out of **n** distinguishable objects is

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}$$

Independent Trials

- Perform repeated trials
- **p** = a success probability
- **(1-p)** = a failure probability
- Each trial is independent
- $\mathbf{S_{k,n}}$ = the event that **k** successes in **n** trials

$$P[S_{k,n}] = \binom{n}{k} p^k (1-p)^{n-k}$$

Independent Trials: Example

- 3 trials with 2 successes
- 000 001 010 011 100 101 110 111
- How many way to choose 2 out of 3

$$= \binom{n}{k} = 3$$

- What is the probability of success for each way ?
- $p^2 * (1-p)$

$$P[S_{2,3}] = \binom{3}{2} p^2(1-p)^{3-2}$$

Independent Trials

Example: In the first round of a food contest, probability that a dish will pass the test is 0.8 .

From 10 candidates, what is the probability that x candidates will pass? P[x = 8]?

Solution:

A = {a dish passes the test}, P[A] = 0.8

Testing a dish is an independent trial

$$P[A_{x,10}] = \binom{10}{x}(0.8)^{x}(1-0.8)^{10-x}$$

$$P[A_{8,10}] = (45)(0.1678)(0.04) = 0.3$$

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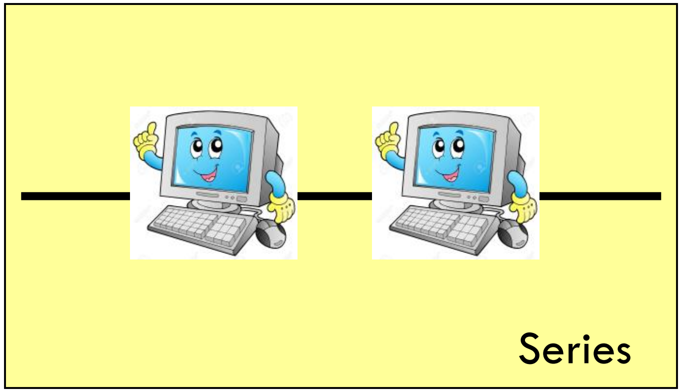
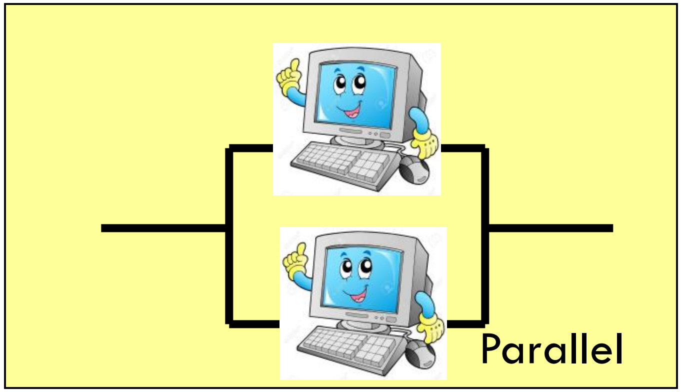
Independent Trials: Reliability





Let probability that a computer works = p

Series: $P[A] = P[A_1A_2] = p^2$

Parallel: P[B] = ?

$$P[B] = 1 - P[B^c]$$
$$= 1 - P[B_1^cB_2^c]$$
$$= 1 - (1 - p)^2$$

Outline

- Probability meaning
- Random Variable
- Discrete RV
- Continuous RV
- Multiple RVs

Random Variable

Random Variable

Experiment (Physical Model)

→ Compose of procedure & observation

→ From observation, we get outcomes

→ From all outcomes, we get a (mathematical) probability model called "Sample space"

→ From the model, we get P[A], $A \subset S$

Random Variable

From a probability model

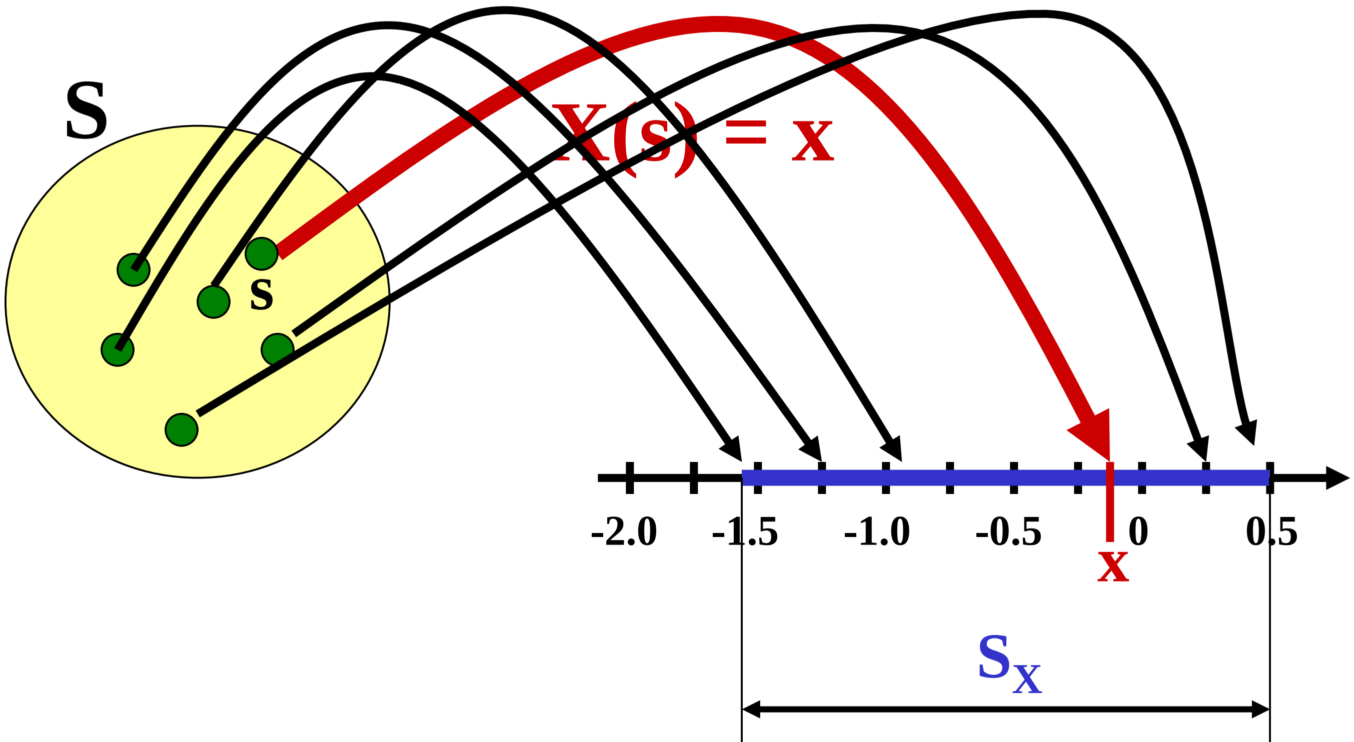
- Ex.: 2 traffic lights, observe the seq. of light

  $S = \{R_1R_2, R_1G_2, G_1R_2, G_1G_2\}$
- If assign a number to each outcome in S, each number that we observe is called "**Random Variable**"
- Observe the number of red light

  $S_X = \{0,1,2\}$

Random Variable

X is a function that maps each outcome, s, in S to a real number X(s), x



2 types of Random Variable

- Discrete Random Variable

  Example:

  X = # of shuttle-cocks used in one badminton game

  Y = # of people in a stand for a world cup soccer match

- Continuous Random Variable

  Example:

  Z = # of minutes for opening a web page

Discrete Random Variable

Definition:

- **X** is a **discrete random variable** if the range of **X** is countable

$$S_{\mathbf{x}} = \{x_1, x_2, \ldots\}$$

- **X** is a **finite random variable** if all values with nonzero probability are in the finite set

$$S_{\mathbf{x}} = \{x_1, x_2, \ldots, x_n\}$$

Why do we need a RV?

- For a probability model (experiment),
  the outcome in S can be in arbitrary form

- If we implement a Random Variable,
  we can calculate the average !

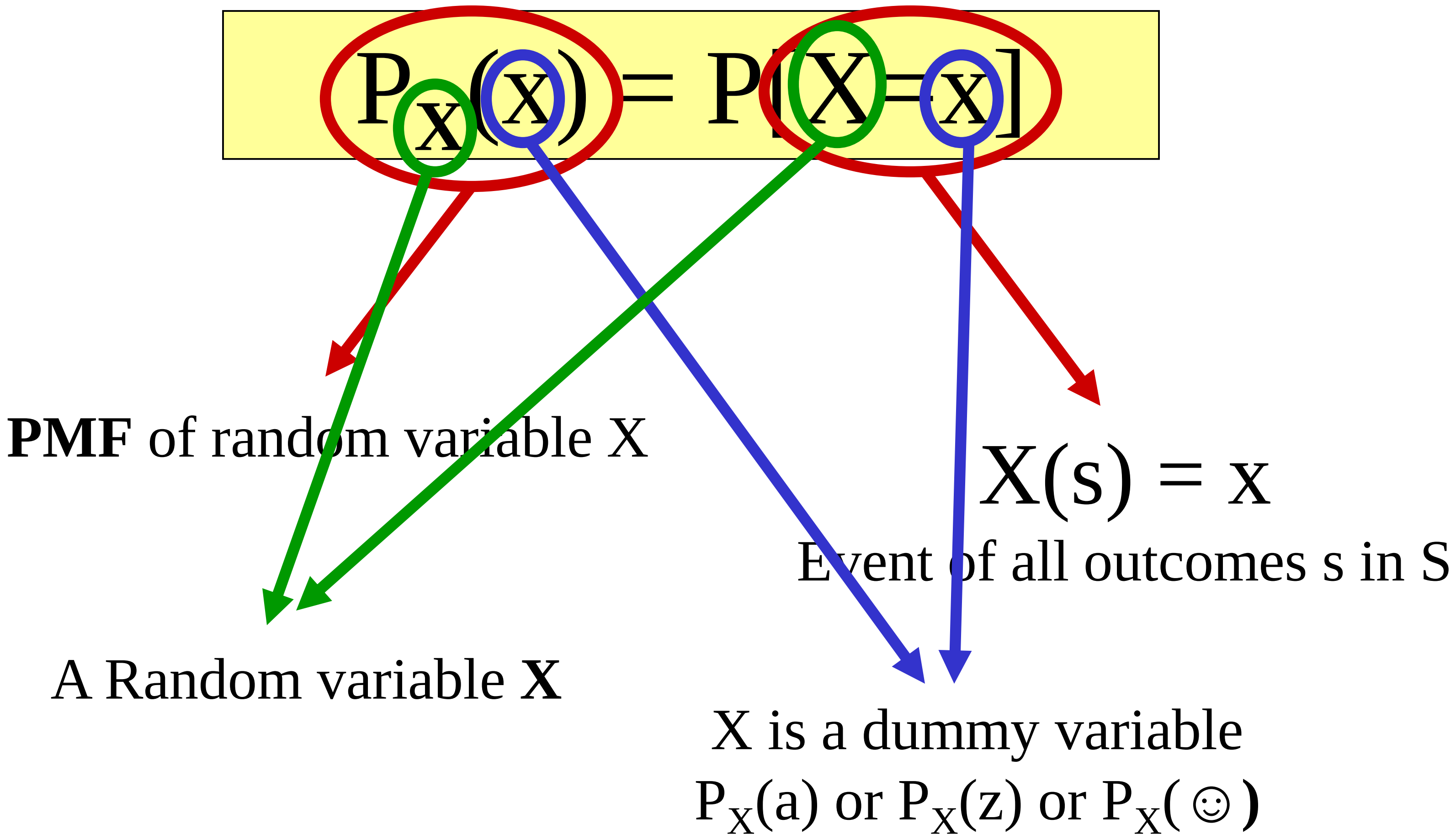
- In Probability, the average is called "**expected value**" of a random variable

Probability Mass Function

- For a (discrete) probability model, **P[A]** = [0,1]

- For a discrete random variable, the probability model is called a "**Probability Mass Function (PMF)**"

Probability Mass Function



PMF Example

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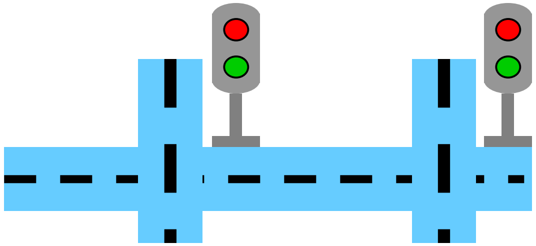


Example:

- 2 traffic lights, observe the seq. of light

  $S = \{ R_1R_2 , R_1G_2 , G_1R_2 , G_1G_2 \}$
- **Find PMF of T, the number of red light**

PMF Example

- T is a random variable of # of red light

→ Find $P_T(t)$

→ $P_T(t) = P[T = t]$

→ First, find probability for each t

→ Each outcome is equally likely → 1/4

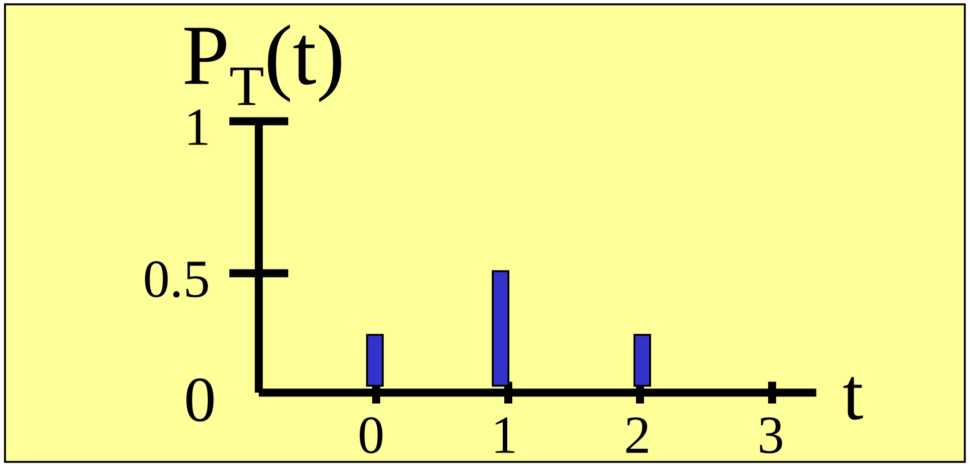
$$P[T=0] = P[\{G_1G_2\}] = 1/4$$

$$P[T=1] = P[\{R_1G_2, G_1R_2\}] = 2/4 = 1/2$$

$$P[T=2] = P[\{R_1R_2\}] = 1/4$$

PMF Example

$$P_T(t) = \begin{cases} 1/2 & t = 1 \\ 1/4 & t = 0, 2 \\ 0 & \text{Otherwise} \end{cases}$$



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PMF Theorem

Theorem: For a discrete random variable X with PMF $P_X(x)$ and Range S_X:

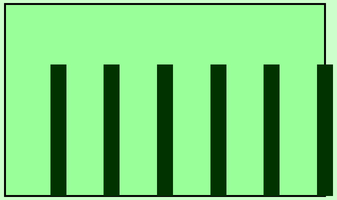
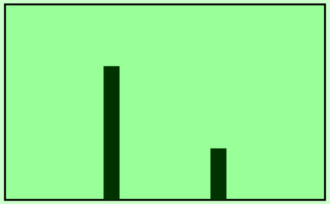
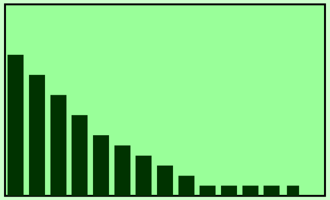
1) For any x, $P_X(x) \geq 0$
2) $\sum_{x \in S_X} P_X(x) = 1$
3) For event $B \subset S_X$, P[B], the probability that X is in the set B is

$$\mathbf{P[B] = \sum_{x \in B} P_X(x)}$$

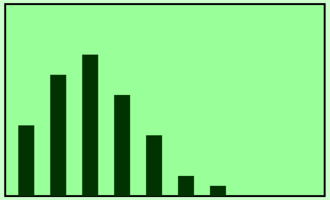
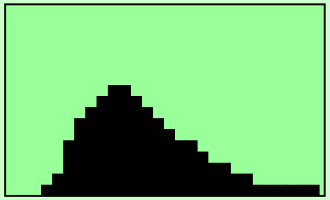
Useful Discrete RV

- Discrete Uniform Random Variable
- Bernoulli Random Variable
- Geometric Random Variable
- Binomial Random Variable
- Pascal Random Variable
- Poisson Random Variable

Useful Discrete RV

Uniform Equiprobable outcomes	$\begin{cases} 1/(j-k+1) & x = k, k+1, k+2, \ldots, j \\ 0 & \text{Otherwise} \end{cases}$	
Bernoulli Pass/Fail	$\begin{cases} 1-p & x = 0 \\ p & x = 1 \\ 0 & \text{Otherwise} \end{cases}$	
Geometric # tests until fail	$\begin{cases} p(1-p)^{x-1} & x = 1, 2, 3, \ldots \\ 0 & \text{Otherwise} \end{cases}$	

Useful Discrete RV

Binomial # fails in n tests	$\begin{cases} \binom{n}{x} p^x(1-p)^{n-x} & x=1,2,\ldots,n \\ 0 & \text{Otherwise} \end{cases}$	
Pascal # tests until k fails	$\begin{cases} \binom{x-1}{k-1} p^k(1-p)^{x-k} & x=k,k+1,\ldots \\ 0 & \text{Otherwise} \end{cases}$	
Poisson occurrence in a period	$\begin{cases} \dfrac{(\lambda T)^x e^{-(\lambda T)}}{x!} & x=0,1,2\ldots \\ 0 & \text{Otherwise} \end{cases}$