01204312
Probability Theory and Statistics

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1^{st} Semester 2018
(Aug – Dec)

Lecture #6
Multiple Discrete Random Variable

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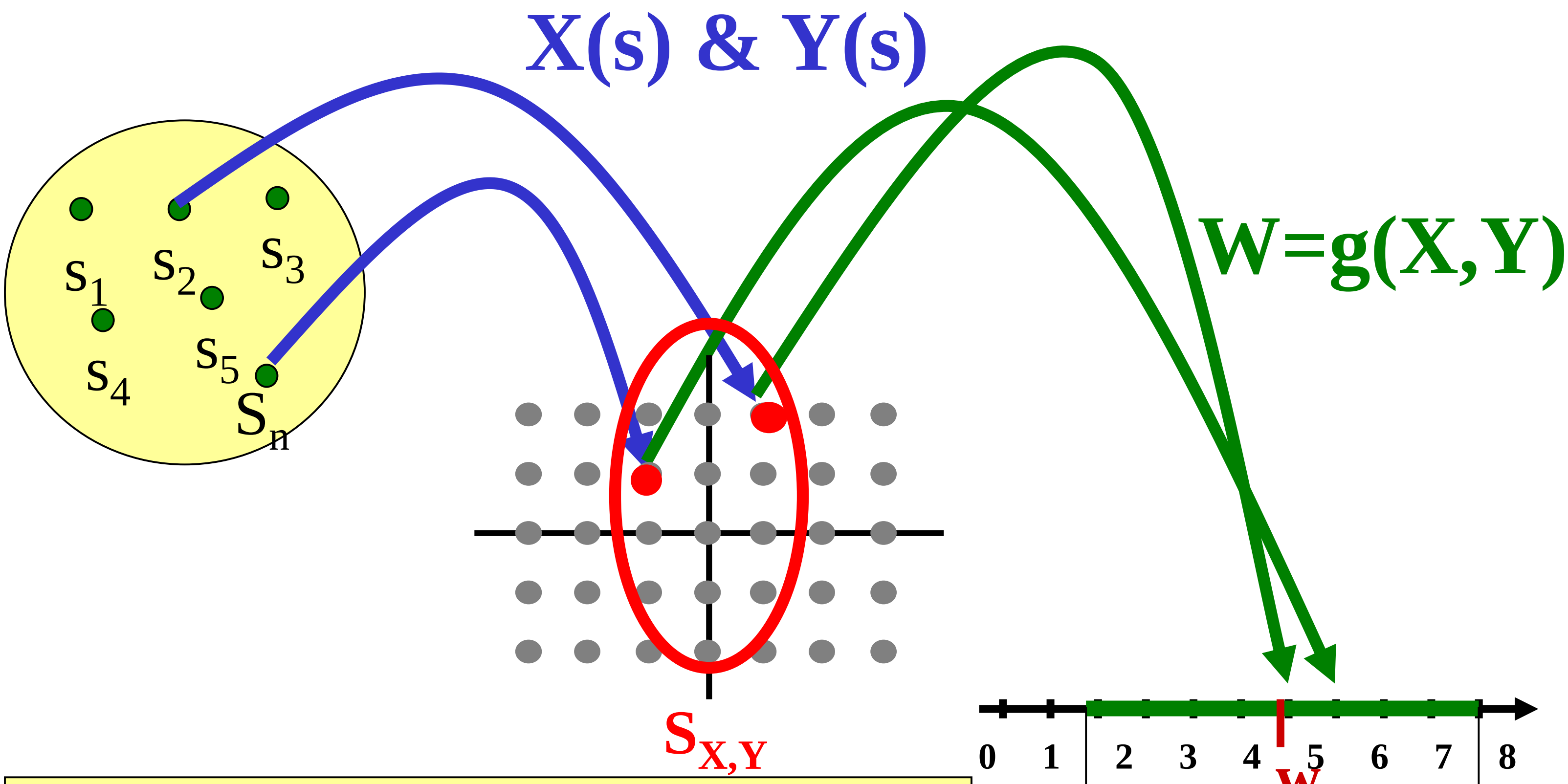
Outline

- Derived Random Variable (Functions of 2 RVs)
- Covariance
- Correlation
- Conditional Joint PMF by an Event

Derived Random Variable
Functions of 2 RVs



$$P_W(w) = \sum_{(x,y):g(x,y)=w} P_{X,Y}(x,y)$$

Example

- E-Document System
- Scanning (text: 40 sec & graphics: 60 sec)

$P_{S,T}(s,t)$	t = 40	t = 60
s = 1 sheet	0.15	0.1
s = 2 sheets	0.3	0.2
s = 3 sheets	0.15	0.1

Let D = duration for sending one Job

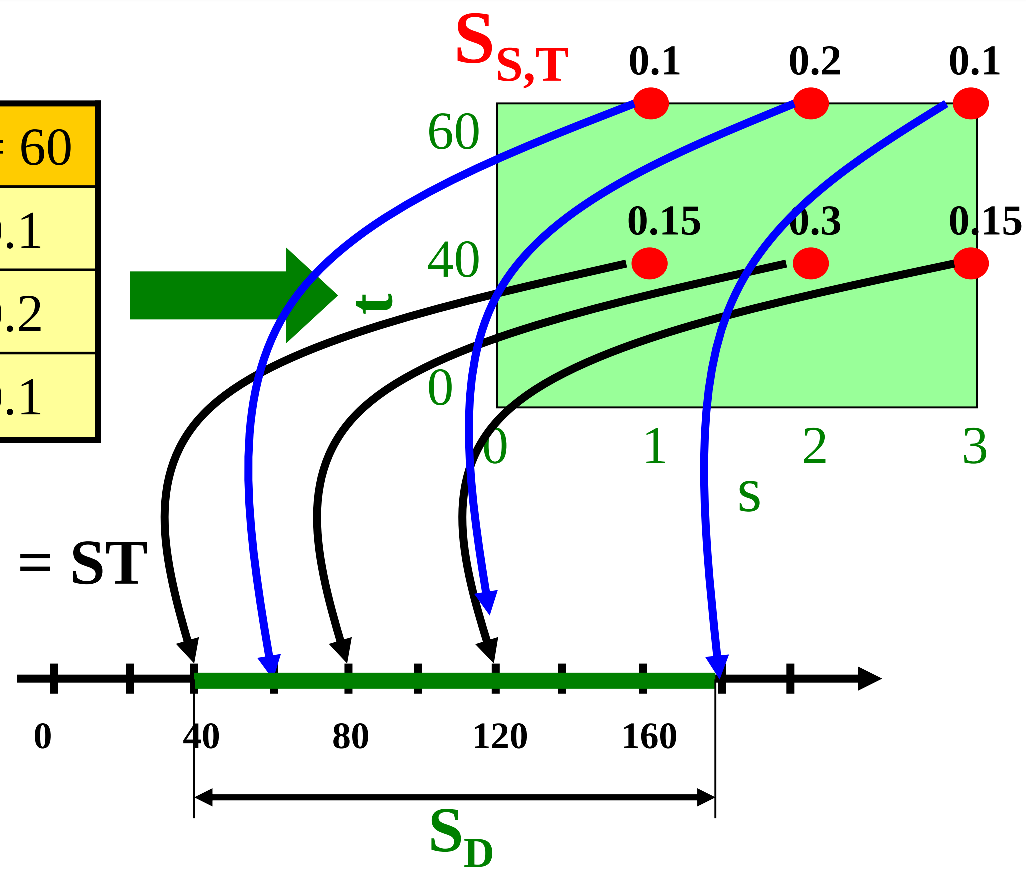
= g(S,T)

= ST

Find $P_D(d)$, S_D, and E[D]

Example

$P_{S,T}(s,t)$	t = 40	t = 60
s = 1 sheet	0.15	0.1
s = 2 sheets	0.3	0.2
s = 3 sheets	0.15	0.1



$D = g(S,T) = ST$

$S_D = \{40, 60, 80, 120, 180\}$

Example

$$S_D = \{40, 60, 80, 120, 180\}$$

$$P_D(d) = \sum_{(s,t):g(s,t)=d} P_{S,T}(s,t)$$

$$P_D(d) = \begin{cases} 0.15 & d = 40 \\ 0.1 & d = 60 \\ 0.3 & d = 80 \\ 0.15 + 0.2 & d = 120 \\ 0.1 & d = 180 \\ 0 & \text{Otherwise} \end{cases}$$

$$E[D] = \sum_{d \in S_D} d P_D(d)$$

$$= 96 \text{ sec}$$

Expectations

- E[W]   for W = g(X,Y)
- E[X+Y]
- Var[X+Y]   (Variance)
- Cov[X,Y]   (Covariance)
- $r_{X,Y}$ (Correlation)
- $\rho_{X,Y}$ (Correlation Coefficient)

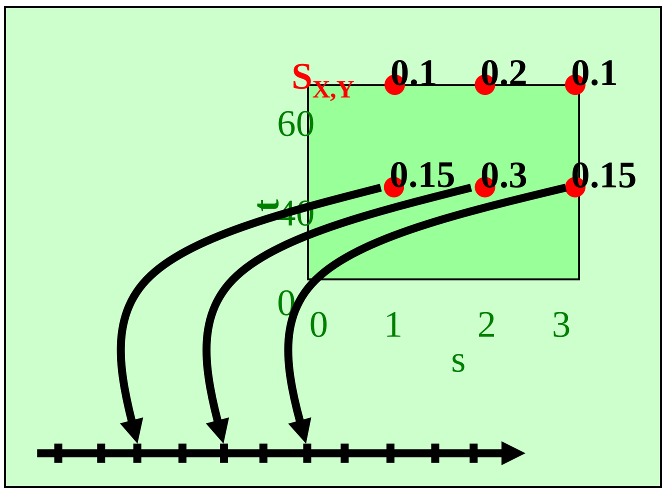
Expected Value of g(X,Y)

Theorem: for W = g(X,Y)

$$E[W] = \sum_{x \in S_X} \sum_{y \in S_Y} g(X,Y)\ P_{X,Y}(x,y)$$

From Last Example

To find the E[D], D = g(S,T) = ST



$$P_D(d) = \begin{cases} 0.15 & d = 40 \\ 0.1 & d = 60 \\ 0.3 & d = 80 \\ 0.15 + 0.2 & d = 120 \\ 0.1 & d = 180 \\ 0 & \text{Otherwise} \end{cases}$$

$$E[D] = \sum_{d \in S_D} d P_D(d)$$

Map g(S,T) → D

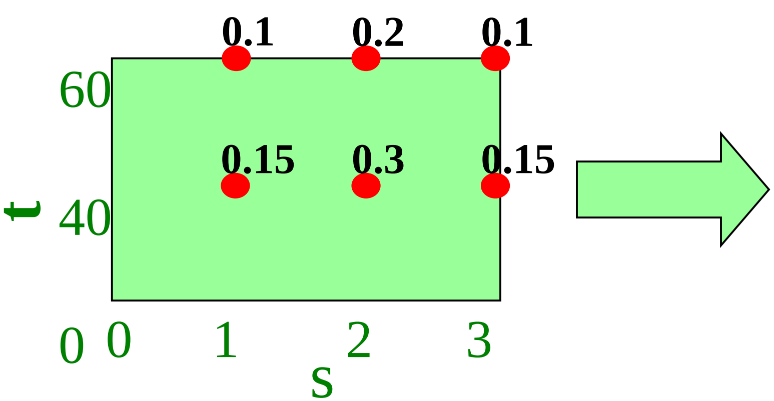
Find $P_D(d)$

Find E[D]

From Last Example

With the theorem, we can directly find E[D]

$$E[D] = \sum_{s=1}^{3} \sum_{t=40,60} st \; P_{S,T}(s,t)$$



E[D] = 1*40*0.15 + 1*60*0.1 +
2*40*0.3 + 2*60*0.2 +
3*40*0.15 + 3*60*0.1
= 96 sec

For any 2 RVs

Theorem:

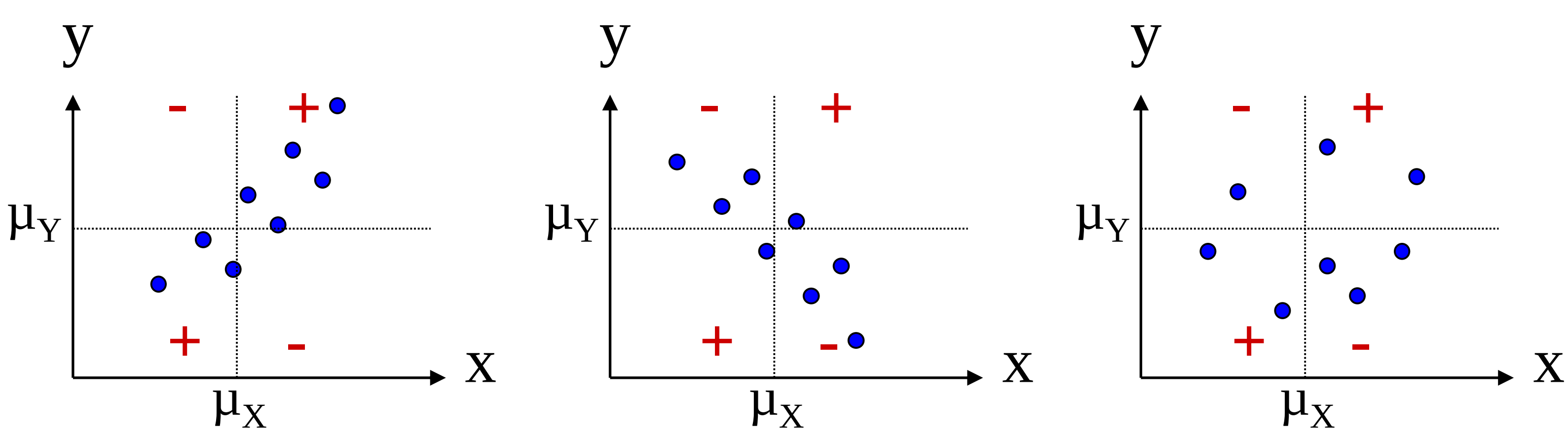
$$E[X + Y] = E[X] + E[Y]$$

- Find E[X] and E[Y]

  $\Rightarrow$ Marginal PMF

Covariance of X and Y

- If X and Y are not independent,
- Interesting question:
  - How X and Y are related ?



Strong Positive Relationship

Strong Negative Relationship

Not Strongly Related

Var[X+Y]

Definition: $Var[X] = E[(X - \mu_X)^2]$

$$
\begin{aligned}
Var[X+Y] &= E[((X+Y) - \mu_{X+Y})^2] \\
&= E[((X+Y) - (\mu_X + \mu_Y))^2] \\
&= E[((X-\mu_X) + (Y-\mu_Y))^2] \\
&= E[(X-\mu_X)^2 + 2(X-\mu_X)(Y-\mu_Y) + (Y-\mu_Y)^2] \\
&= E[(X-\mu_X)^2] + 2E[(X-\mu_X)(Y-\mu_Y)] + E[(Y-\mu_Y)^2]
\end{aligned}
$$

Theorem:

$$Var[X+Y] = Var[X] + Var[Y] + 2\,E[(X-\mu_X)(Y-\mu_Y)]$$

Covariance

Covariance of X and Y

Definition: $Cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$

$$
\begin{aligned}
Cov[X,Y] &= E[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y] \\
&= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X \mu_Y \\
&= E[XY] - \mu_X \mu_Y - \mu_X \mu_Y + \mu_X \mu_Y
\end{aligned}
$$

Theorem: $Cov[X,Y] = E[XY] - \mu_X \mu_Y$

Example (1)

$P_{S,T}(s,t)$	t = 40	t = 60	$P_S(s)$
s = 1 sheet	0.15	0.1	0.25
s = 2 sheets	0.3	0.2	0.5
s = 3 sheets	0.15	0.1	0.25
$P_T(t)$	0.6	0.4	1

$$\mu_S = \sum s\, P_S(s)$$
$$= (0.25*1)+(0.5*2)+(0.25*3)$$
$$= 2$$

$$\mu_T = \sum t\, P_T(t)$$
$$= (0.6*40)+(0.4*60)$$
$$= 48$$

Example (1)

$P_{S,T}(s,t)$	t = 40	t = 60	$P_S(s)$
s = 1 sheet	0.15	0.1	0.25
s = 2 sheets	0.3	0.2	0.5
s = 3 sheets	0.15	0.1	0.25
$P_T(t)$	0.6	0.4	1

$\mu_S = 2$, $\mu_T = 48$

Definition: $Cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$

$$Cov(S,T) = \sum_{(s,t)}\sum (s-2)(t-48)\, P_{S,T}(s,t)$$

$$\begin{aligned} = \ & (1-2)(40-48) *0.15 \\ & + (1-2)(60-48) *0.1 \\ & + (2-2)(40-48) *0.3 \\ & + (2-2)(60-48) *0.2 \\ & + (3-2)(40-48) *0.15 \\ & + (3-2)(60-48) *0.1 \end{aligned}$$

$$\begin{aligned} = \ & +1.2 \\ & -1.2 \\ & +0 \\ & +0 \\ & -1.2 \\ & +1.2 \end{aligned}$$

= 0

Example (1)

$P_{S,T}(s,t)$	t = 40	t = 60	$P_S(s)$
s = 1 sheet	0.15	0.1	0.25
s = 2 sheets	0.3	0.2	0.5
s = 3 sheets	0.15	0.1	0.25
$P_T(t)$	0.6	0.4	1

$\mu_S = 2$, $\mu_T = 48$

Theorem: $Cov[X,Y] = E[XY] - \mu_X \mu_Y$

$Cov(S,T) = E[ST] - \mu_S\mu_T$

$= ((1*40)*0.15 + (1*60)*0.1 + (2*40)*0.3 + (2*60)*0.2 + (3*40)*0.15 + (3*60)*0.1) – (2*48)$

$= (+6 +6 +24 +24 +18 +18) – 96$

$= (96) – 96$

$= 0$ ← Same as previous method

S and T are
Not Strongly Related

Example (2)

- Insurance Company
  - Automobile policy ($100, $250)
  - Homeowner’s policy ($0, $100, $200)
- Customer purchase both gets deductible (in dollars)
  - X = deductible amount on Auto Policy
  - Y = deductible amount on Homeowner’s Policy
- $P_{X,Y}(x,y)$
- Cov(X,Y)

Example (2)

$P_{X,Y}(x,y)$	y=0	y=100	y=200
x=100	0.20	0.10	0.20
x=250	0.05	0.15	0.30

X = deduct. Auto Policy
Y = deduct. Home Policy

$$\mu_X = \sum x\, P_X(x)$$
$$= (0.5*100)+(0.5*250)$$
$$= 175$$

$$\mu_Y = \sum y\, P_Y(y)$$
$$= (0.25*0)+(0.25*100)+(0.5*200)$$
$$= 125$$

Example (2)

$P_{X,Y}(x,y)$	y=0	y=100	y=200	$P_X(x)$
x=100	0.20	0.10	0.20	0.50
x=250	0.05	0.15	0.30	0.50
$P_Y(y)$	0.25	0.25	0.5	1

Definition: $Cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$

$$Cov(X,Y) = \sum_{(x,y)}\sum (x - 175)(y - 125)\, P_{X,Y}(x,y)$$

$$\begin{aligned} = \; & (100 - 175)(0 - 125) *0.2 \\ & + (100 - 175)(100 - 125) *0.1 \\ & + (100 - 175)(200 - 125) *0.2 \\ & + (250 - 175)(0 - 125) *0.05 \\ & + (250 - 175)(100 - 125) *0.15 \\ & + (250 - 175)(200 - 125) *0.3 \end{aligned}$$

$= 1875$

X and Y are
Strong Positive Relationship

Covariance ⇒ **Linear Dependence** between two RVs

Example (2)

- If the deductible in cents (not in dollars)
  - $X' \Rightarrow 100X, \quad Y' \Rightarrow 100Y$
  - $Cov(X',Y') \Rightarrow Cov(100X, 100Y) \Rightarrow 100*100*Cov(X,Y)$
- From 1,875 (in \$) ⇒ 18,750,000 (in cents)
- From 1,875 (in \$) ⇒ 0.1875 (in hundreds of \$)

Defect of Covariance:
Covariance critically **depends on Unit** measurement !!!
⇒We need Dimensionless measurement

⇒ **Correlation Coefficient**

Covariance of X and Y

Theorem: $Cov[X,Y] = E[XY] - \mu_x \mu_y$

Correlation

If $X = Y \Rightarrow$ $Cov[X,X] = E[XX] - \mu_x \mu_x$

$$= E[X^2] - \mu_x^2$$

$$= E[X^2 - 2\mu_x^2 + \mu_x^2]$$

$$= E[X^2 - 2\mu_x X + \mu_x^2]$$

$$= E[(X - \mu_x)^2]$$

$$= Var[X]$$

If μ_X or $\mu_Y = 0 \Rightarrow Cov[X,Y] = E[XY]$

Correlation

Definition: The correlation of X and Y is $r_{X,Y}$

$$r_{X,Y} = E[XY]$$

Theorem: $Cov[X,Y] = r_{X,Y} - \mu_x \mu_y$

More Definition

Definition 1:

X and Y are **Orthogonal** if $r_{X,Y} = 0$; $E[XY]=0$

Definition 2:

X and Y are **Uncorrelated** if $Cov[X,Y] = 0$

Definition 3:

Correlation Coefficient of X and Y is

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}} = \frac{Cov[X,Y]}{\sigma_X \sigma_Y} = [-1, 1]$$

Correlation Coefficient

- $\rho_{X,Y}$
  - Describes the info about X by observing Y
- $\rho_{X,Y} > 0$
  - If X $\Uparrow$ (relative to mean) $\Rightarrow$ Y $\Uparrow$
  - If X $\Downarrow$ (relative to mean) $\Rightarrow$ Y $\Downarrow$
- $\rho_{X,Y} < 0$
  - If X $\Uparrow$ (relative to mean) $\Rightarrow$ Y $\Downarrow$
  - If X $\Downarrow$ (relative to mean) $\Rightarrow$ Y $\Uparrow$
- Example:
  - X = student's height, Y= student's weight $\rho_{X,Y} > 0$
  - X = cell phone distance, Y= received signal Strength $\rho_{X,Y} < 0$

Example (2) Revisit

$P_{X,Y}(x,y)$	y=0	y=100	y=200	$P_X(x)$
x=100	0.20	0.10	0.20	0.50
x=250	0.05	0.15	0.30	0.50
$P_Y(y)$	0.25	0.25	0.5	1

$\mu_X = 175$, $\mu_Y = 125$

$$E[X^2] = \sum x^2 P_X(x)$$
$$= (100^2 * 0.5) + (250^2 * 0.5)$$
$$= 5{,}000 + 31{,}250$$
$$= 36{,}250$$

$$\sigma^2_X = E[X^2] - \mu^2_X$$
$$= 36{,}250 - (175)^2$$
$$\sigma_X = 75$$

$$E[Y^2] = \sum y^2 P_Y(y)$$
$$= (0^2 * 0.25) + (100^2 * 0.25) + (200^2 * 0.5)$$
$$= 22{,}500$$

$$\sigma^2_Y = E[Y^2] - \mu^2_Y$$
$$= 22{,}500 - (125)^2$$
$$\sigma_Y = 82.92$$

Example (2) Revisit

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sigma_X \sigma_Y} = \frac{1,875}{75 * 82.92}$$

$$= 0.301$$

From $\rho_{X,Y} > 0$

If X $\Uparrow$ (relative to mean) $\Rightarrow$ Y $\Uparrow$

If X $\Downarrow$ (relative to mean) $\Rightarrow$ Y $\Downarrow$

From **Cov(X,Y)**

X and Y are

Strong Positive Relationship

Uncorrelated

If X and Y are **Independent**, then

$\Rightarrow Cov[X,Y] = 0 \Rightarrow \rho_{X,Y} = 0$

$\Rightarrow$ X and Y are **Uncorrelated**

Note:

If X and Y are **Uncorrelated**,

$\Rightarrow$ X and Y **may or may not Independent**

Conditional Joint PMF by an Event

Conditional Joint PMF by an Event

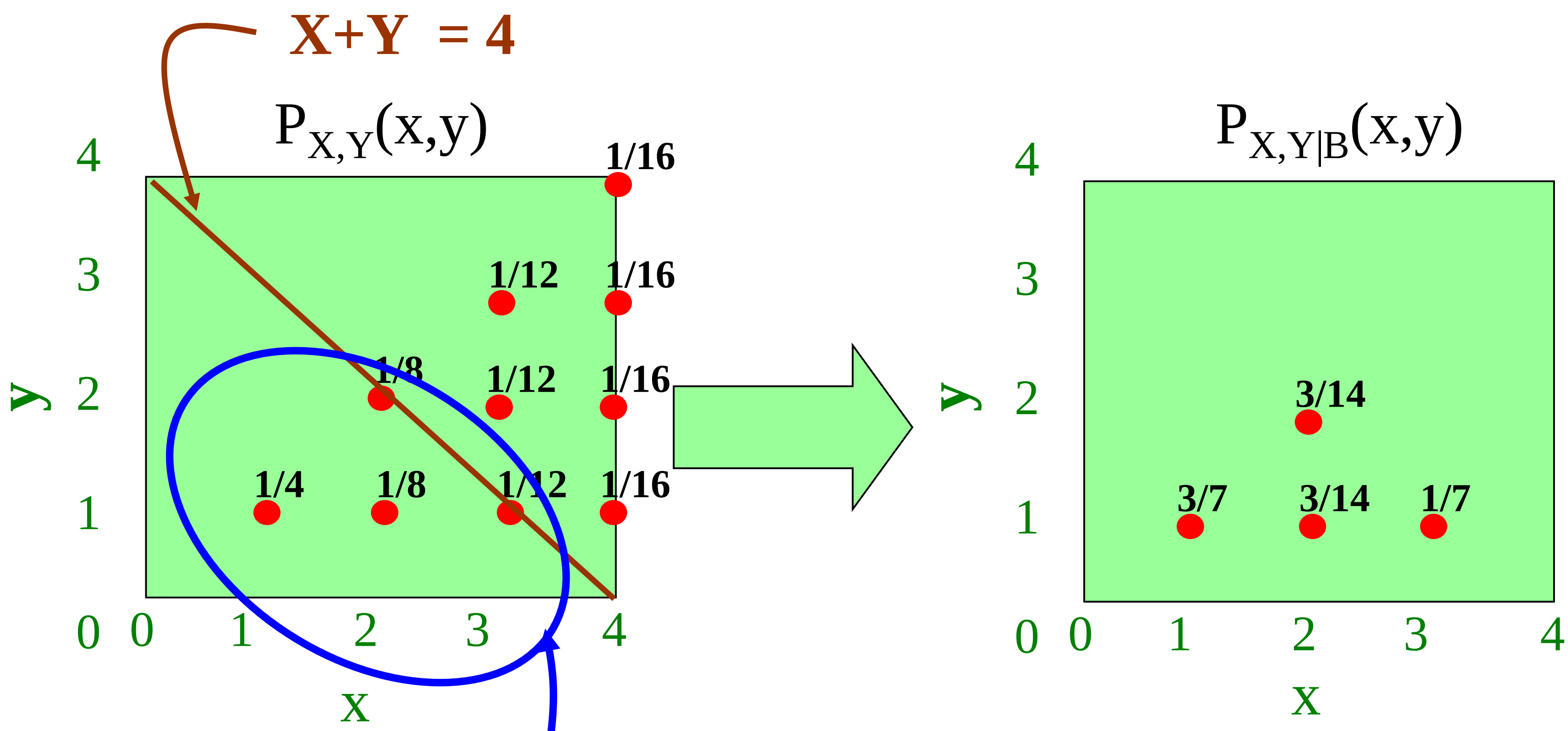
$$P_{X,Y|B}(x,y) = \frac{P[(X=x, Y=y) \cap B]}{P[B]}$$

If $(X=x, Y=y) \in B \Rightarrow (X=x, Y=y) \cap B = (X=x, Y=y)$

$$P_{X,Y|B}(x,y) = \begin{cases} \dfrac{P[(X=x, Y=y)]}{P[B]} & (x,y) \in B \\ 0 & \text{Otherwise} \end{cases}$$

Example $P_{X,Y|B}(x,y)$

Let $B = \{X+Y \leq 4\}$ Find $P_{X,Y|B}(x,y)$



$B = \{X+Y \leq 4\} \Rightarrow P[B] = 7/12$

Conditional PMF

- Special case of Conditional Joint PMF by an Event

  $\Rightarrow$ the Event is X=x or Y=y
- $P_{X,Y|B}(x,y)$ when B = {Y=y}

  $\Rightarrow P_{X,Y|Y=y}(x,y) = P_{X|Y}(x|y)$

Definition: $P_{X|Y}(x|y) = P[X=x \mid Y=y]$

Conditional PMF

$$P_{X|Y}(x|y) = P[X=x \mid Y=y]$$

$$= \frac{P[X=x, Y=y]}{P[Y=y]}$$

$$= \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

Theorem:

$$P_{X,Y}(x,y) = P_{X|Y}(x|y)P_Y(y) = P_{Y|X}(y|x)P_X(x)$$

Independent RVs

- From the independent definition
  - A and B are independent iff P[AB] = P[A]P[B]
- X and Y are independent RVs iff
  - {X=x} and {Y=y} are independent for all x, y in $S_{X,Y}$

Definition: $P_{X,Y}(x,y) = P_X(x)P_Y(y)$

Independent RVs

Theorem:

(a) $r_{X,Y} = E[XY] = E[X]E[Y]$

(b) $E[X|Y = y] = E[X]$ for all $y \in S_Y$

(c) $E[Y|X = x] = E[Y]$ for all $x \in S_X$

(d) $Var[X+Y] = Var[X] + Var[Y]$

(e) $Cov[X,Y] = \rho_{X,Y} = 0$

More than 2 RVs

Definition: Joint PMF of discrete RV $X_1,\ldots X_N$ is

$$P_{X_1,\ldots,X_N}(x_1,\ldots,x_N) = P[X_1=x_1, \ldots, X_N=x_N]$$

Summary

- Joint PMF
- Marginal PMF
- Covariance
- Correlation Coefficient
- Conditional Joint PMF by an Event