

LECTURE #2

PROBABILITY REVIEW (I)

204528

Queueing Theory and
Applications in Networks

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Outline

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- Probability meaning
- Random Variable
- Discrete RV
- Continuous RV
- Multiple RVs

What is Probability ?

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- Physical Property
 - Lottery
- Knowledge
 - Snow in Thailand
- Probability meaning
 - Situation cannot exactly replicate
 - But not chaotic (have a pattern)

Probability

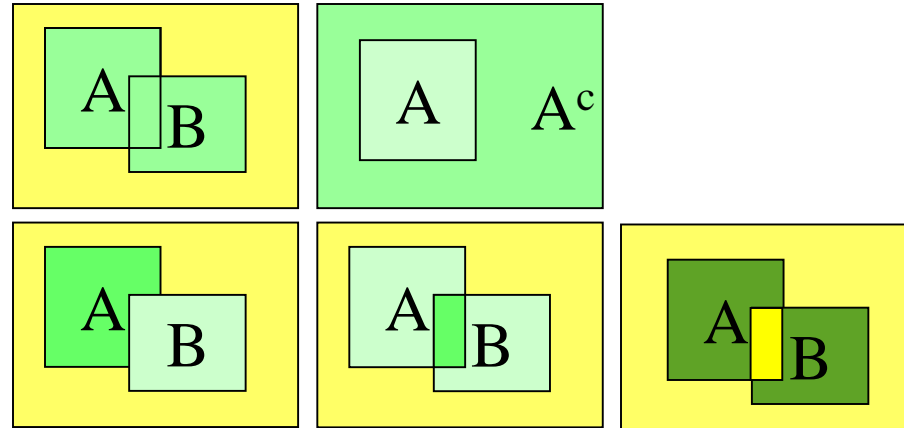
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- Definition
 - Logic of probability
- Axiom
 - Fact without proof
- Theorem
 - Derived from Definition, Axiom, or other Theorems

Probability Mathematics

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- Set Theory
 - Set operation
 - Set properties



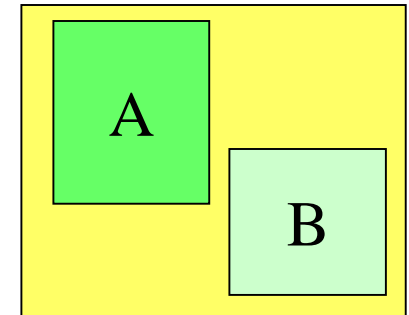
Important Set Properties

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1. Mutually Exclusive

$$A_i \cap A_j = \emptyset \quad \text{for } i \neq j$$

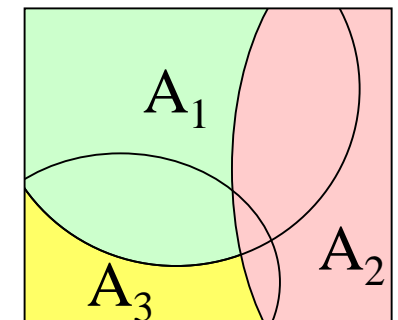
$A \cap B = \emptyset \rightarrow$ called **Disjoint** for only 2 sets



2. Collectively Exhaustive

$$A_1 \cup A_2 \cup \dots \cup A_n = S$$

$$\bigcup_{i=1}^n A_i = S$$



Experiment

What is an Experiment?

- Method for finding some facts/conclusions

Give an example?

- For movie “**Alice**”, is it fun?
- Stand in front of the theatre
- Ask audiences, fun or not?

Composition of an experiment

- Procedure
- Observation

Why experiment is needed?

- Uncertainty



Experiment

- Concern about movie “**Alice**” experiment
 - Should I ask man, women, or teenager?
 - Experience of the audiences
 - Knowledge of the audiences
- Complicated experiment → need **Model**
 - Real experiments: too complicate
 - Capture only the important part
 - Model Example:
 - Treat all audiences the same
 - Answer will only be like/dislike



Experiment

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**Same Procedure
but different Observations
→ Different Experiments**

Example:

1. Flip a coin 3 times, Observe the sequence of heads/tails
2. Flip a coin 3 times, Observe # of heads

Definition in Probability

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- **Outcome**
 - Any possible observation
- **Sample Space**
 - *Finest-grain*: each outcome is different
 - *Mutually exclusive*: if one outcome occurs, other will not occur
 - *Collectively exhaustive*: every outcome must be in the sample space
- **Event**
 - Set of outcomes (Must know all outcomes)
 - $\text{Event} \subset \text{Sample Space}$

Event Examples

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For an experiment:

Roll a dice, observe the shown numbers

Outcomes:

number = 1,2,3,4,5,6

Sample space:

$S = \{1,2,3,\dots,6\}$

Event examples:

$E_1 = \{\text{number} < 3\} = \{1,2\}$

$E_2 = \{\text{number is odd}\} = \{1,3,5\}$

Set VS. Probability

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Set Algebra	Probability
Set	Event
Universal set	Sample space
Element	Outcome

Probability of Event, $P[\text{😊}]$

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$P[\text{😊}]$

is a function that maps event
in the sample space to real number

From experiment: Roll a dice

Outcomes:

number = 1,2,3,4,5,6

Sample space:

$S = \{1,2,3,\dots,6\}$

Event examples:

$E_1 = \{\text{number} < 3\} = \{1,2\}$

$E_2 = \{\text{number is odd}\} = \{1,3,5\}$

$$P[E_1] = 2/6 = 1/3$$

$$P[E_2] = 3/6 = 1/2$$

Probability Axioms

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Axiom 1: For any event A , $P[A] \geq 0$

Axiom 2: $P[S] = 1$

Axiom 3: For events A_1, A_2, \dots, A_n of mutual exclusive events
 $P[A_1 \cup A_2 \cup \dots \cup A_n] = P[A_1] + P[A_2] + \dots + P[A_n]$

Example Theorems

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- **Theorem:** If A and B are disjoint, then

$$\mathbf{P[A \cup B] = P[A] + P[B]}$$

- **Theorem:** If $B = B_1 \cup B_2 \cup \dots \cup B_n$ and $B_i \cap B_j = \emptyset$ for $i \neq j$, then

$$\mathbf{P[B] = \sum_{i=1}^n P[B_i]}$$

Equally Likely

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Theorem:

For an experiment with sample space $S = \{s_1, \dots, s_n\}$
if each outcome is **equally likely**,

$$P[s_i] = 1/n \quad 1 \leq i \leq n$$

Consequences of Axioms

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Theorem:

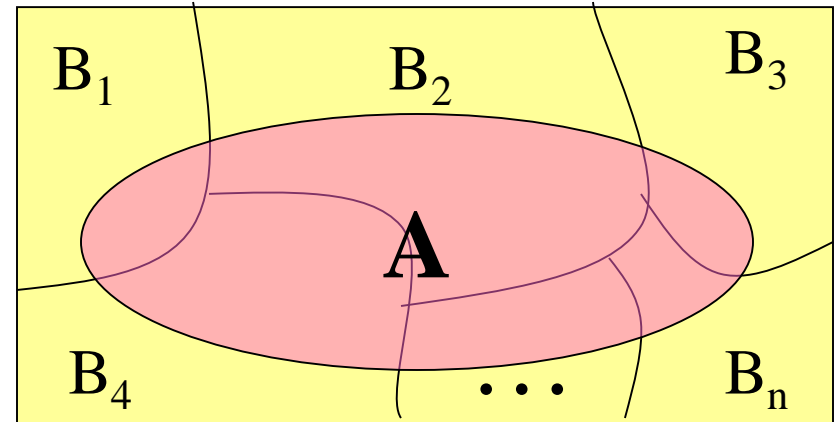
- $P[\emptyset] = 0$
- $P[A^c] = 1 - P[A]$
- For any A and B (not necessary disjoint)
$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$
- If $A \subset B$, then $P[A] \leq P[B]$

A Useful Theorem

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Let B_1, B_2, \dots, B_n be mutual exclusive events whose union equals sample space S

→ partition of S



For any event A

$$A = A \cap S = A \cap (B_1 \cup B_2 \cup \dots \cup B_n)$$

$$P[A] = P[A \cap B_1] + P[A \cap B_2] + \dots + P[A \cap B_n]$$

Theorem:

$$P[A] = \sum_{i=1}^n P[A \cap B_i]$$

Conditional Probability

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- In practice, it maybe impossible to find the precise outcome of an experiment
- However, if we know that Event B has occurred (the outcome of Event A is in set B)
 - Probability of A when B occurs can be described
 - Still don't know $P[A]$

Conditional Probability

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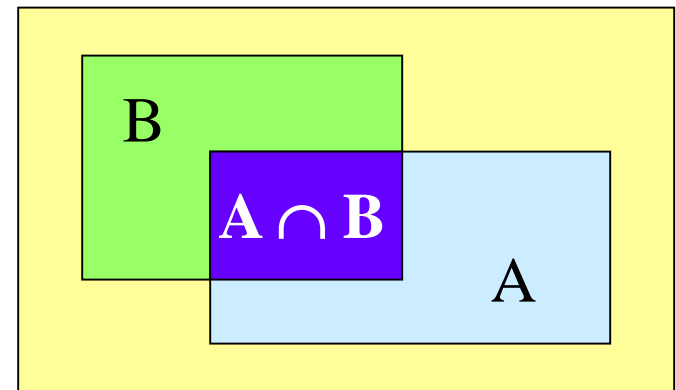
- **Notation:** $P[A|B]$
 - “Probability of A given B”
 - The condition probability of the event A given the occurrence of the event B

- **Definition:**

$$P[A|B] = \frac{P[AB]}{P[B]}$$

- **Example:**

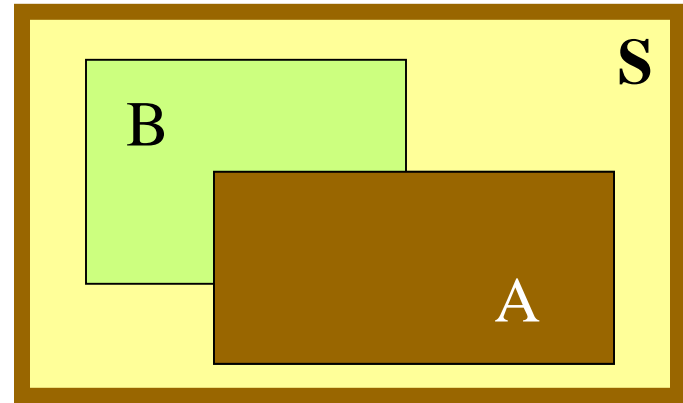
Tiger Woods hits Hole-in-one



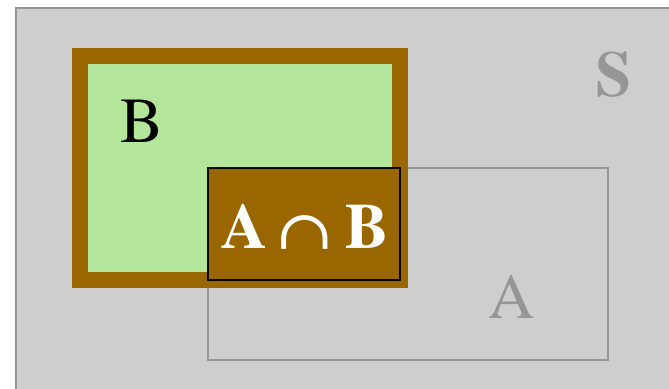
More Explanation

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$$\begin{aligned} P[A|S] &= \frac{P[AS]}{P[S]} \\ &= \frac{P[A]}{1} \\ &= P[A] \end{aligned}$$



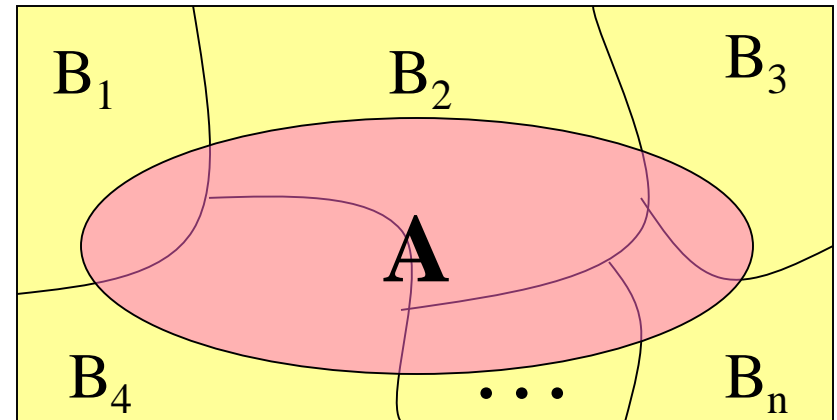
$$P[A|B] = \frac{P[AB]}{P[B]}$$



Law of Total Probability

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- Let B_1, B_2, \dots, B_n be mutual exclusive events whose union equals sample space S
- $P[B_i] > 0$



$$\text{Theorem: } P[A] = \sum_{i=1}^n P[A \cap B_i]$$
$$P[A] = P[A \cap B_1] + P[A \cap B_2] + \dots$$
$$P[A] = P[A|B_1]P[B_1] + P[A|B_2]P[B_2] + \dots$$

$$\text{Theorem: } P[A] = \sum_{i=1}^n P[A|B_i]P[B_i]$$

Bayes' Theorem

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$$\begin{aligned} P[B|A] &= \frac{P[BA]}{P[A]} \\ &= \frac{P[A|B]P[B]}{P[A]} \end{aligned}$$

$$P[A|B] = \frac{P[AB]}{P[B]}$$

Theorem:
$$P[B|A] = \frac{P[A|B]P[B]}{P[A]}$$

2 Independent Events

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Definition: Event A and B are independent iff

$$\mathbf{P[AB] = P[A]P[B]}$$

$$\begin{aligned} P[A|B] &= \frac{P[AB]}{P[B]} \\ &= \frac{P[A]P[B]}{P[B]} \end{aligned}$$

$$\mathbf{P[A|B] = P[A]}$$

$$\mathbf{P[B|A] = P[B]}$$

Independent Interpretation

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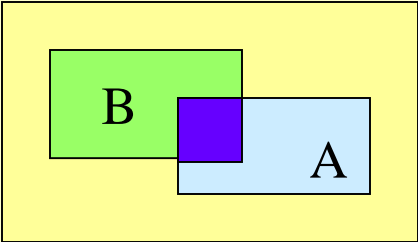
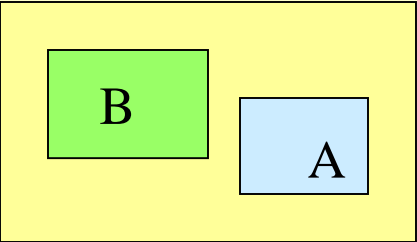
$$P[A] = 0.3$$

$$P[A|B] = 0.3$$

No matter event B occurs or not,
event A is not affected

Independent VS. Disjoint

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Independent	Disjoint
	
$P[AB] \neq 0$	$P[AB] = 0$
$P[A \cap B] = P[A] * P[B]$	$P[A \cup B] = P[A] + P[B]$

Note: Independent = Disjoint iff $P[A]=0$ or $P[B]=0$

3 Independent Events

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Definition: Event A_1, A_2 and A_3 are independent iff

- 1) A_1 and A_2 are independent
- 2) A_2 and A_3 are independent
- 3) A_1 and A_3 are independent
- 4) $P[A_1 \cap A_2 \cap A_3] = P[A_1] P[A_2] P[A_3]$

Why only number 4 is insufficient ?

Definition: Event A and B are independent iff

$$P[AB] = P[A]P[B]$$

Most Common Application

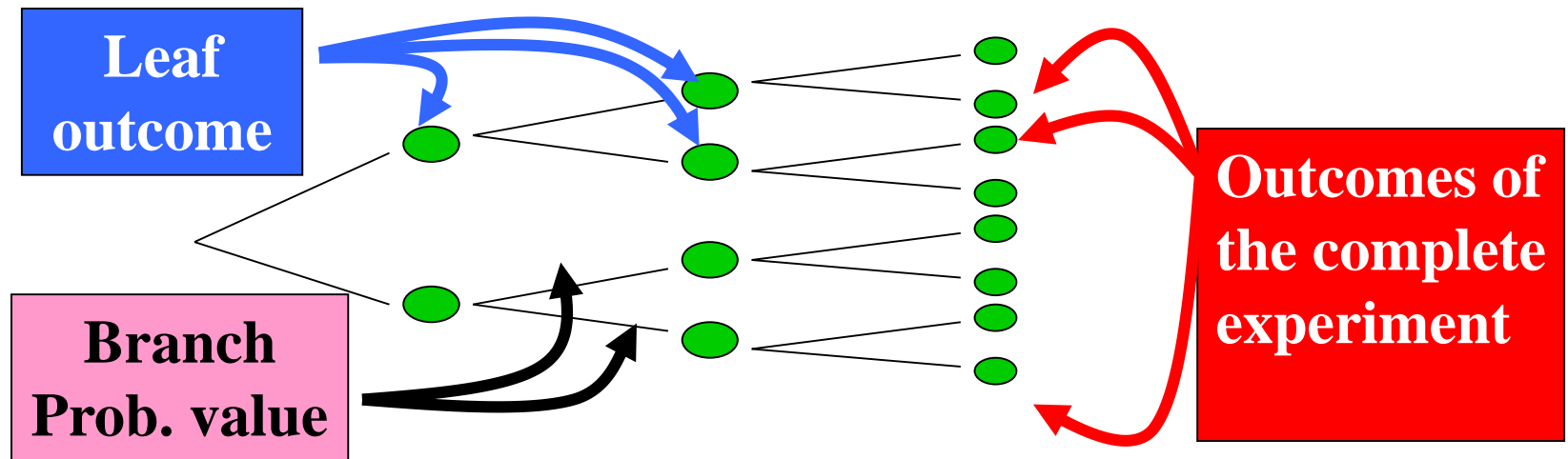
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- **Assume that the events of separate experiments are independent**
- Example:
 - Assume that outcome of a coin toss is independent of the outcomes of all prior and all subsequent coin tosses
 - $P[H] = P[T] = 1/2$
 - $P[HTH] = P[H]P[T]P[H] = 1/2 * 1/2 * 1/2 = 1/8$

Sequential Experiments

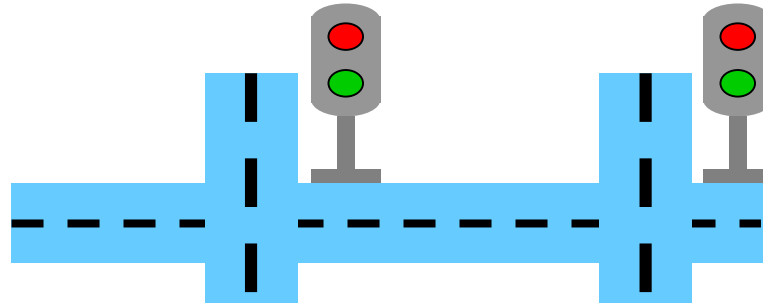
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- Experiment: in sequence
subexperiments \rightarrow subexperiments
- Each subexp. may depend on the previous one
- Represented by a **Tree Diagram**
- **Model Conditional Prob. \rightarrow Sequential Experiment**



Sequential Example

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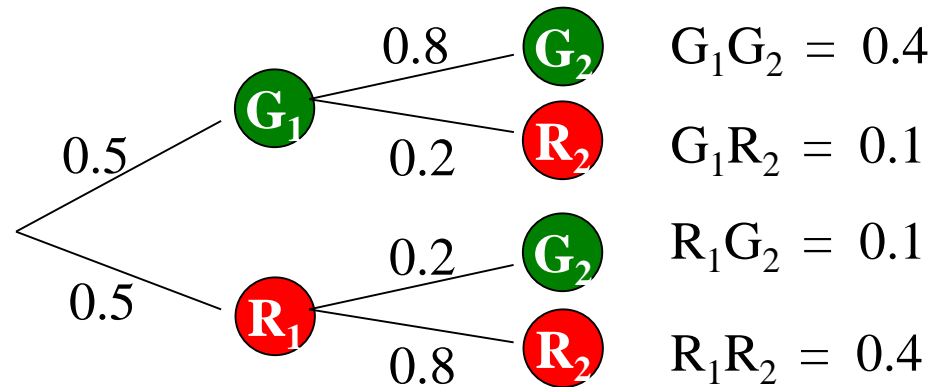
Timing coordination of 2 traffic lights

- $P[\text{the 2}^{\text{nd}} \text{ light is the same color as the 1}^{\text{st}}] = 0.8$
- Assume 1st light is equally likely to be green or red

Find $P[\text{The second light is green}]$?

Sequential Example

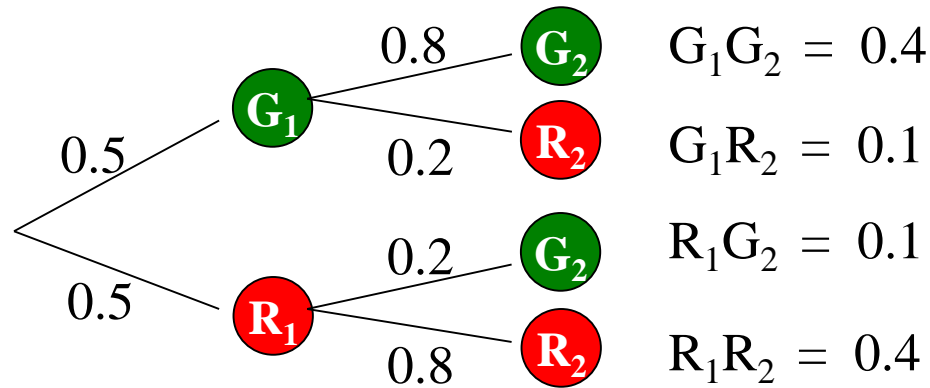
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- $P[G_1] = P[R_1] = 0.5$
- $P[G_2G_1] = P[G_2 | G_1]P[G_1] = (0.8)(0.5) = 0.4$

Sequential Example

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P[The second light is green] ?

$$P[G_2] = P[G_2G_1] + P[G_2R_1] = 0.4 + 0.1 = 0.5$$

$$P[G_2] = P[G_2|G_1]P[G_1] + P[G_2|R_1]P[R_1]$$



Counting Method

Principle of Counting Method

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If experiment A has **n** possible outcomes,
and experiment B has **k** possible outcomes,

→ Then there are **nk** possible outcomes
when you perform both experiments

k-permutations

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Theorem:

The number of **k**-permutations
(ordered sequence) of **n** distinguishable objects is

$$(n)_k = \frac{n!}{(n-k)!}$$

Choose with replacement

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Theorem: Given n distinguishable objects,
There are n^k ways to choose with replacement
a sample of k objects

k-combination

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Theorem:

The number of ways to choose **k** objects out of **n** distinguishable objects is

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}$$

Independent Trials

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- Perform repeated trials
- p = a success probability
- $(1-p)$ = a failure probability
- Each trial is independent
- $S_{k,n}$ = the event that k successes in n trials

$$P[S_{k,n}] = \binom{n}{k} p^k (1-p)^{n-k}$$

Independent Trials: Example

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- 3 trials with 2 successes
- 000 001 010 011 100 101 110 111
- How many way to choose 2 out of 3

$$= \binom{n}{k} = 3$$

- What is the probability of success for each way ?
- $p^2 * (1-p)$

$$P[S_{2,3}] = \binom{3}{2} p^2 (1-p)^{3-2}$$

Independent Trials

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Example: In the first round of a food contest, probability that a dish will pass the test is 0.8 .

From 10 candidates, what is the probability that x candidates will pass? $P[x = 8]$?

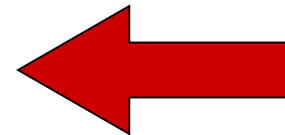
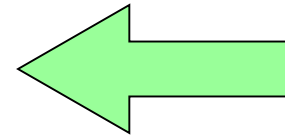
Solution:

$A = \{ \text{a dish passes the test} \}, \quad P[A] = 0.8$

Testing a dish is an independent trial

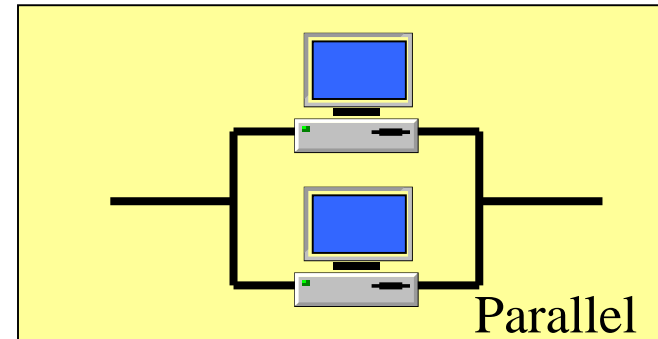
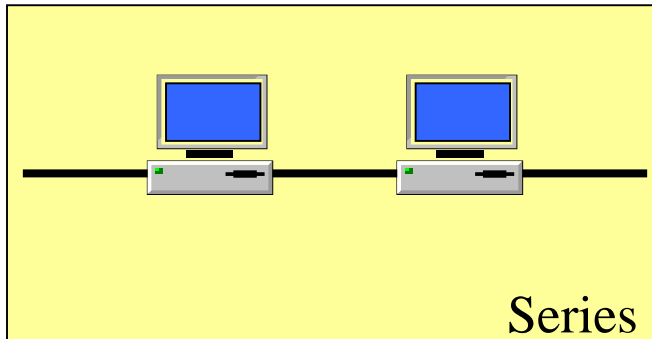
$$P[A_{x,10}] = \binom{10}{x} (0.8)^x (1-0.8)^{10-x}$$

$$P[A_{8,10}] = (45)(0.1678)(0.04) = 0.3$$



Independent Trials: Reliability

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Let probability that a computer works = p

Series: $P[A] = P[A_1 A_2] = p^2$

Parallel: $P[B] = ?$

$$\begin{aligned} P[B] &= 1 - P[B^c] \\ &= 1 - P[B_1^c B_2^c] \\ &= 1 - (1 - p)^2 \end{aligned}$$

Outline

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- Probability meaning
- Random Variable
- Discrete RV
- Continuous RV
- Multiple RVs



Random Variable

Random Variable

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Experiment (Physical Model)

- Compose of procedure & observation
- From observation, we get outcomes
- From all outcomes, we get a (mathematical) probability model called “Sample space”
- From the model, we get $P[A]$, $A \subset S$

Random Variable

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From a probability model

- Ex.: 2 traffic lights, observe the seq. of light

$$S = \{R_1R_2, R_1G_2, G_1R_2, G_1G_2\}$$

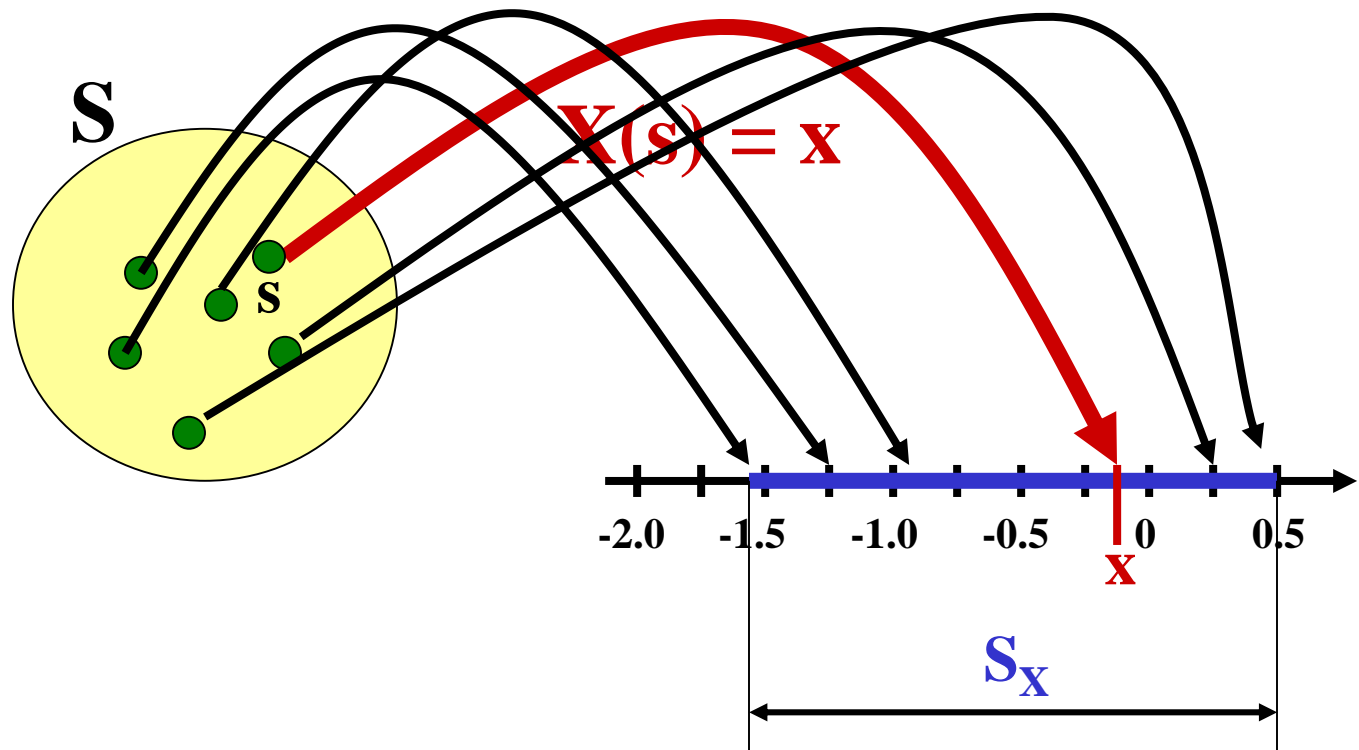
- If assign a number to each outcome in S , each number that we observe is called “**Random Variable**”
- Observe the number of red light

$$S_X = \{0, 1, 2\}$$

Random Variable

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X is a function that maps each outcome, s , in S to a real number $X(s)$, x



2 types of Random Variable

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- Discrete Random Variable

Example:

$X = \#$ of shuttle-cocks used in one badminton game

$Y = \#$ of people in a stand for a world cup soccer match

- Continuous Random Variable

Example:

$Z = \#$ of minutes for opening a web page

Discrete Random Variable

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Definition:

- **X** is a **discrete random variable** if the range of **X** is countable

$$S_x = \{x_1, x_2, \dots\}$$

- **X** is a **finite random variable** if all values with nonzero probability are in the finite set

$$S_x = \{x_1, x_2, \dots, x_n\}$$

Why do we need a RV?

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- For a probability model (experiment), the outcome in S can be in arbitrary form
- If we implement a Random Variable, we can calculate the average !
- In Probability, the average is called “**expected value**” of a random variable

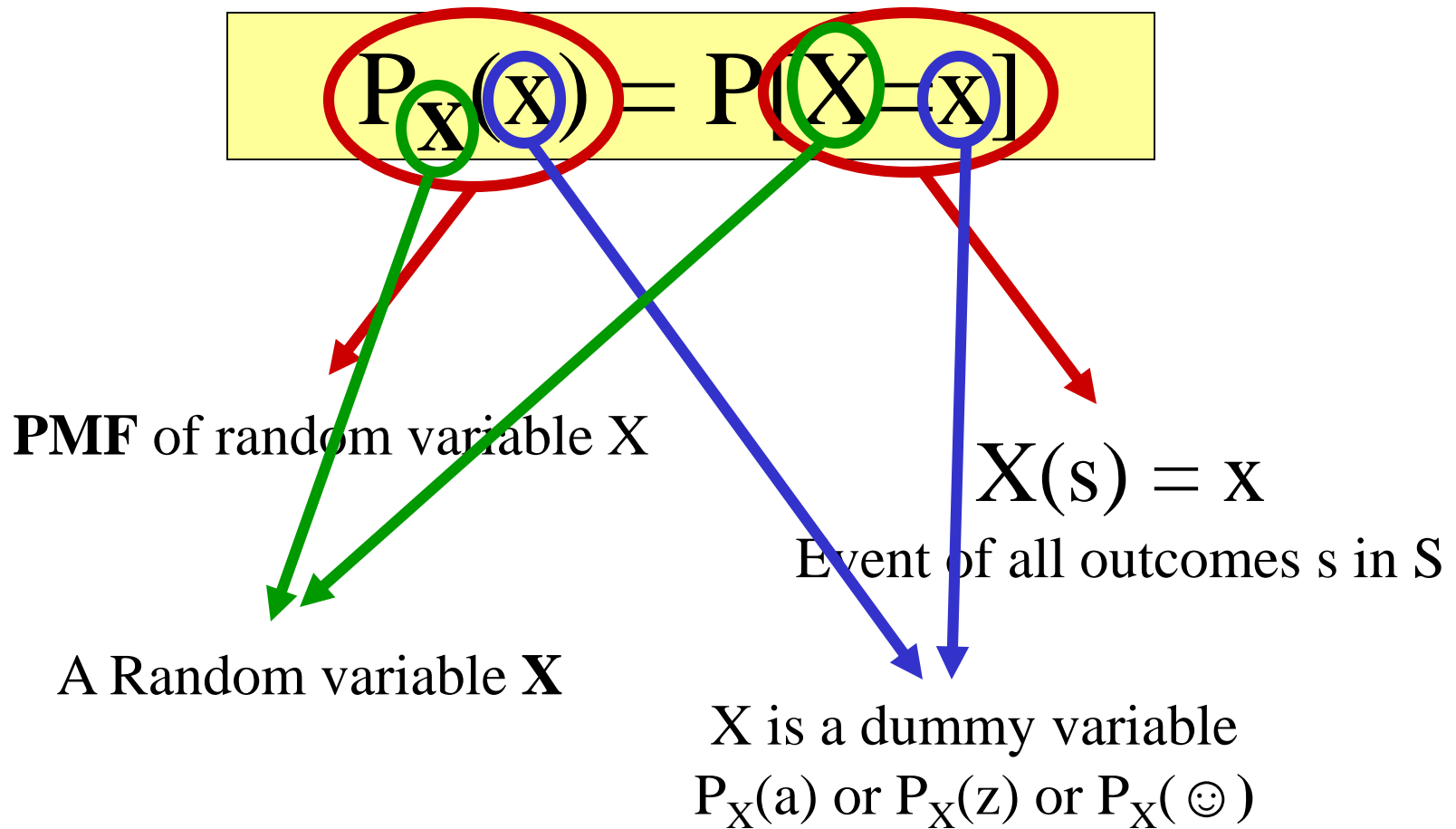
Probability Mass Function

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- For a (discrete) probability model, $\mathbf{P[A] = [0,1]}$
- For a discrete random variable, the probability model is called a “**Probability Mass Function (PMF)**”

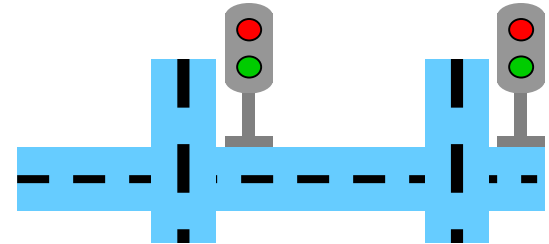
Probability Mass Function

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PMF Example

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Example:

- 2 traffic lights, observe the seq. of light
 $S = \{ R_1R_2, R_1G_2, G_1R_2, G_1G_2 \}$
- **Find PMF of T, the number of red light**

PMF Example

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- T is a random variable of # of red light

→ Find $P_T(t)$

→ $P_T(t) = P[T = t]$

→ First, find probability for each t

→ Each outcome is equally likely → 1/4

$$P[T=0] = P[\{G_1G_2\}] = 1/4$$

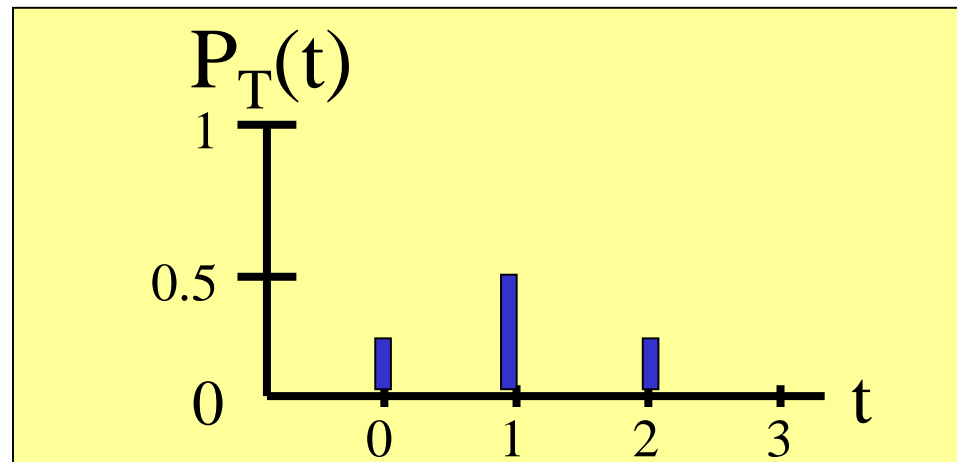
$$P[T=1] = P[\{R_1G_2, G_1R_2\}] = 2/4 = 1/2$$

$$P[T=2] = P[\{R_1R_2\}] = 1/4$$

PMF Example

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$$P_T(t) = \begin{cases} 1/2 & t = 1 \\ 1/4 & t = 0, 2 \\ 0 & \text{Otherwise} \end{cases}$$



PMF Theorem

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Theorem: For a discrete random variable X with PMF $P_X(x)$ and Range S_X :

- 1) For any x , $P_X(x) \geq 0$
- 2) $\sum_{x \in S_X} P_X(x) = 1$
- 3) For event $B \subset S_X$, $P[B]$, the probability that X is in the set B is

$$P[B] = \sum_{x \in B} P_X(x)$$

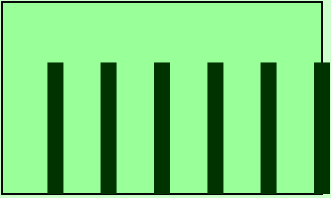
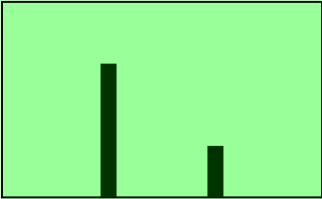
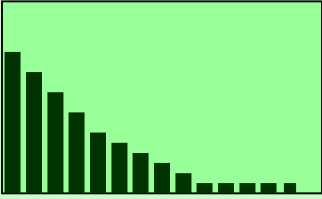
Useful Discrete RV

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- Discrete Uniform Random Variable
- Bernoulli Random Variable
- Geometric Random Variable
- Binomial Random Variable
- Pascal Random Variable
- Poisson Random Variable

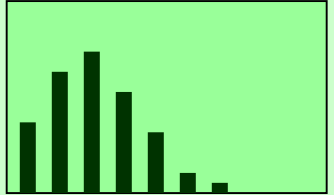
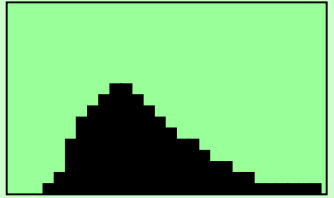
Useful Discrete RV

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<p><u>Uniform</u> Equiprobable outcomes</p>	$\begin{cases} 1/(j-k+1) & x = k, k+1, k+2, \dots, j \\ 0 & \textit{Otherwise} \end{cases}$	
<p><u>Bernoulli</u> Pass/Fail</p>	$\begin{cases} 1 - p & x = 0 \\ p & x = 1 \\ 0 & \textit{Otherwise} \end{cases}$	
<p><u>Geometric</u> # tests until fail</p>	$\begin{cases} p(1 - p)^{x-1} & x = 1, 2, 3, \dots \\ 0 & \textit{Otherwise} \end{cases}$	

Useful Discrete RV

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<p><u>Binomial</u></p> <p># fails in n tests</p>	$\begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x=1,2,\dots,n \\ 0 & \text{Otherwise} \end{cases}$	
<p><u>Pascal</u></p> <p># tests until k fails</p>	$\begin{cases} \binom{x-1}{k-1} p^k (1-p)^{x-k} & x = k, k+1, \dots \\ 0 & \text{Otherwise} \end{cases}$	
<p><u>Poisson</u></p> <p>occurrence in a period</p>	$\begin{cases} \frac{(\lambda T)^x e^{-(\lambda T)}}{x!} & x = 0, 1, 2, \dots \\ 0 & \text{Otherwise} \end{cases}$	