

# LECTURE #8

## EQUILIBRIUM BEHAVIOR OF BIRTH-DEATH QUEUEING SYSTEM

204528

Queueing Theory and  
Applications in Networks

**Assoc. Prof. Anan Phonphoem, Ph.D.** (รศ.ดร. อนันต์ พลเพิ่ม)  
**Computer Engineering Department, Kasetsart University**

# Outline

2

- M/M/1
- Discouraged Arrivals
- Responsive Servers (M/M/ $\infty$ )
- Finite Storage (M/M/1/K)

3

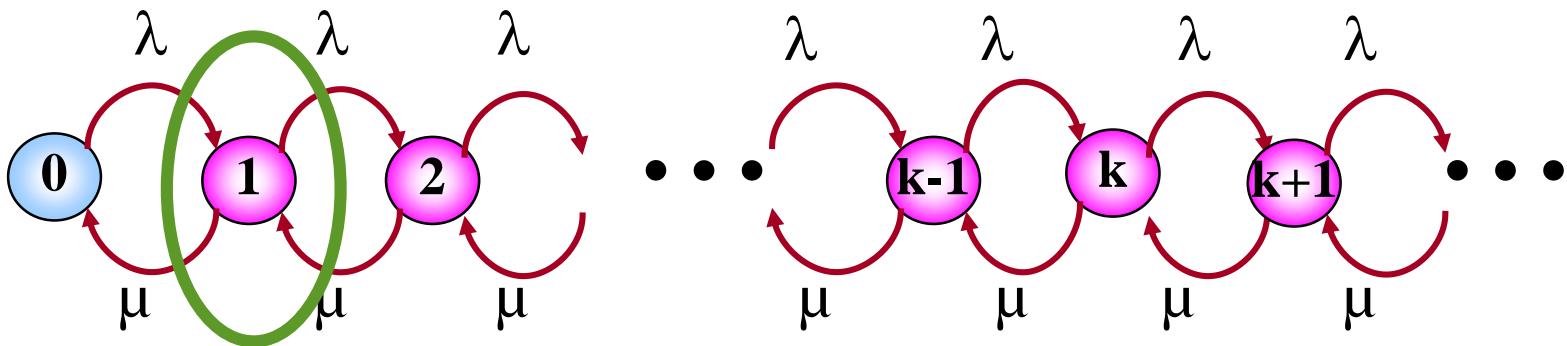
**M/M/1**

# A Birth-Death Process : M/M/1

4

- Assumption

- $\lambda_k = \lambda$  for  $k \geq 0$
- $\mu_k = \mu$  for  $k \geq 1$
- The system begins at time  $t_0$  with 0 member



# A Birth-Death Process : M/M/1

5

- A Birth-Death Process
  - Constant coefficients  $\lambda$  and  $\mu$
- Interarrival Time / Service Time / #Servers
- Memoryless / Memoryless / 1 Server
- M/M/1 = A single-server queue with a Poisson arrival and an exponential distribution for service time

# A Birth-Death Process : M/M/1

6

- $A(t)$  = CDF of the arrival time
  - $\lambda_k = \lambda$  for  $k \geq 0$
  - $A(t) = 1 - e^{-\lambda t}$
- $B(x)$  = CDF of the service time
  - $\mu_k = \mu$  for  $k \geq 1$
  - $B(x) = 1 - e^{-\mu x}$

# A Birth-Death Process : M/M/1

7

- $dp_0(t)/dt = -\lambda p_0(t) + \mu p_1(t)$
- $dp_k(t)/dt = -(\lambda + \mu)p_k(t) + \lambda p_{k-1}(t) + \mu p_{k+1}(t)$
- Now we try to find the solution!
  - Hint: Solved by using  $z$ -transforms

# A Birth-Death Process : M/M/1

8

$$p_k(t) = e^{-(\lambda+\mu)t} \left( \begin{array}{l} \rho^{(k-i)/2} I_{k-i}(at) \\ + \rho^{(k-i-1)/2} I_{k+i+1}(at) \\ + (1-\rho)\rho^k \sum_{j=k+i+2}^{\infty} \rho^{-j/2} I_j(at) \end{array} \right)$$

- Where  $\rho = \lambda/\mu$  and  $a = 2\mu\rho^{1/2}$
- $I_k(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{k+2m}}{(k+m)!m!} \quad k \geq -1$
- Time dependent behavior of the state prob.



# Equilibrium Solution

9

- The time-dependent solution is unmanageable
- $p_k(t) \rightarrow$  not too useful  $\rightarrow$  transient
- Let  $p_k \equiv$  limiting probability (system = k members)  
$$= \lim_{t \rightarrow \infty} p_k(t) = \text{System in state } E_k$$
- $p_k$  is not time-dependent

# Equilibrium Solution

10

- From

$$dp_0(t)/dt = -\lambda_0 p_0(t) + \mu_1 p_1(t)$$

$$dp_k(t)/dt = -(\lambda_k + \mu_k)p_k(t) + \lambda_{k-1}p_{k-1}(t) + \mu_{k+1}p_{k+1}(t)$$

- If set  $\lim_{t \rightarrow \infty} dp_k(t)/dt = 0$

- Obtain the result

$$0 = -\lambda_0 p_0 + \mu_1 p_1 \quad k = 0$$

$$0 = -(\lambda_k + \mu_k)p_k + \lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1} \quad k \geq 1$$

# Equilibrium Solution

11

- From conservation relation  $\sum_{k=0}^{\infty} p_k(t) = 1$
- Find the answer for  $p_0$  and  $p_i$

# Equilibrium Solution

12

- From

$$0 = -\lambda_0 p_0 + \mu_1 p_1 \quad k = 0$$

$$0 = -(\lambda_k + \mu_k) p_k + \lambda_{k-1} p_{k-1} + \mu_{k+1} p_{k+1} \quad k \geq 1$$

- Yield

$$\lambda_0 p_0 = \mu_1 p_1$$

$$p_1 = (\lambda_0 / \mu_1) p_0$$

# Equilibrium Solution

13

- And for  $k = 1$

$$(\lambda_k + \mu_k)p_k = \lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1}$$

$$(\lambda_1 + \mu_1)p_1 = \lambda_0 p_0 + \mu_2 p_2$$

$$(\lambda_1 + \mu_1)(\lambda_0 / \mu_1)p_0 = \lambda_0 p_0 + \mu_2 p_2$$

$$(\lambda_1 \lambda_0 / \mu_1)p_0 + \lambda_0 p_0 = \lambda_0 p_0 + \mu_2 p_2$$

$$(\lambda_1 \lambda_0 / \mu_1)p_0 = \mu_2 p_2$$

$$p_2 = \frac{\lambda_1 \lambda_0}{\mu_1 \mu_2} p_0$$

# Equilibrium Solution

14

$$p_i = p_0 \left( \prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}} \right) \quad \forall i \geq 1$$

$$1 = \left( p_0 + \sum_{i=1}^{\infty} p_i \right)$$

$$p_0 = \left( 1 + \sum_{i=1}^{\infty} \prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}} \right)^{-1}$$

# M/M/1 Equilibrium Solution

15

$$p_i = p_0 \left( \prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}} \right) \quad \forall i \geq 1$$

- From  $\lambda_k = \lambda$  for  $k \geq 0$  and  $\mu_k = \mu$  for  $k \geq 1$

$$p_i = p_0 \left( \prod_{k=0}^{i-1} \frac{\lambda}{\mu} \right) \quad \forall i \geq 1$$

$$p_i = p_0 \left( \frac{\lambda}{\mu} \right)^i \quad \forall i \geq 0$$

# M/M/1 Equilibrium Solution

16

$$p_0 = \left( 1 + \sum_{i=1}^{\infty} \prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}} \right)^{-1}$$

- From  $\lambda_k = \lambda$  for  $k \geq 0$  and  $\mu_k = \mu$  for  $k \geq 1$

$$p_0 = \left( 1 + \sum_{i=1}^{\infty} \left( \frac{\lambda}{\mu} \right)^i \right)^{-1}$$



# Birth-Death Process Classification

17

Define:  $S_1 \triangleq \sum_{i=0}^{\infty} \prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}}$  .....From  $p_0$

$$S_2 \triangleq \sum_{i=0}^{\infty} \left( \lambda_i \prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}} \right)^{-1}$$

Ergodic:  $S_1 < \infty$  and  $S_2 = \infty$

Recurrent Null:  $S_1 = \infty$  and  $S_2 = \infty$

Transient:  $S_1 = \infty$  and  $S_2 < \infty$

# Ergodicity

18

- $E_j = \text{Ergodic}$  if
  - $E_j = \text{Aperiodic}$  and *Recurrent Nonnull*
- $f_j = 1$ ,  $M_j < \infty$ , and  $\beta = 1$
- A Markov Chain is **ergodic**
  - If **all** states of a Markov Chain are **ergodic**
  - If number of states is **finite** and **all** states of a Markov Chain are **aperiodic**, and **irreducible**

$f_j = P[\text{the process returns to state } j \text{ after leaving state } j]$

Anan Phonphoem

Dept. of Computer Engineering, Kasetsart University, Thailand

# Stability Condition

19

- $p_0 > 0$
- The sufficient condition for ergodicity in M/M/1 is  $\lambda < \mu$

$$p_0 = \left( 1 + \sum_{i=1}^{\infty} \left( \frac{\lambda}{\mu} \right)^i \right)^{-1}$$
$$= \frac{1}{1 + \frac{\lambda/\mu}{1 - \lambda/\mu}} = 1 - \frac{\lambda}{\mu}$$

# M/M/1

20

- $\rho = \lambda / \mu$
- For stability conditions  $0 \leq \rho < 1$

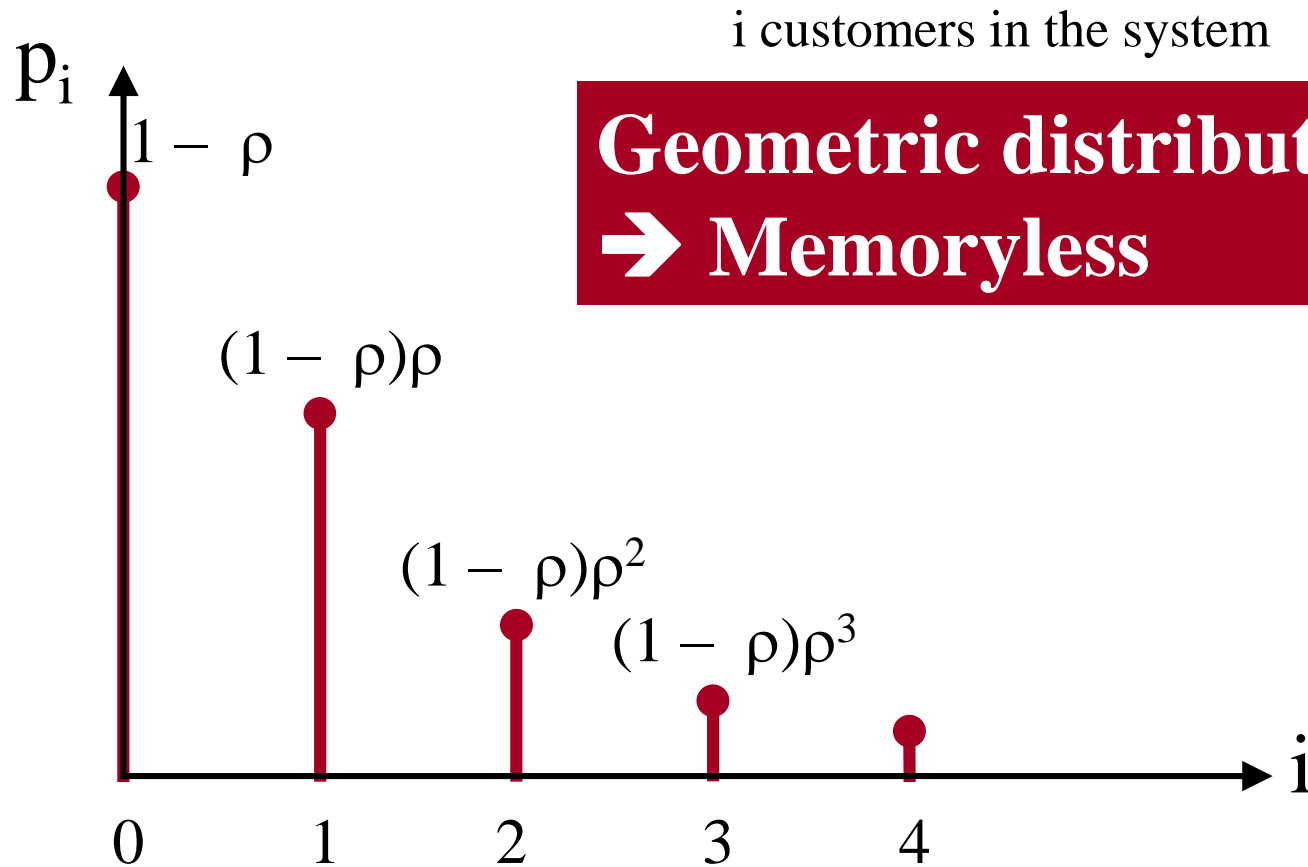
$$p_0 = 1 - \rho$$

$$p_i = p_0 \left[ \frac{\lambda}{\mu} \right]^i$$

$$p_i = (1 - \rho)\rho^i$$

# The Solution of $p_i$

21



# The average # of customers

22

$$\begin{aligned}\bar{N} &= \sum_{i=0}^{\infty} i p_i &= (1 - \rho) \sum_{i=0}^{\infty} i \rho^i \\ & &= (1 - \rho) \rho \frac{\partial}{\partial \rho} \sum_{i=0}^{\infty} \rho^i \\ & &= (1 - \rho) \rho \frac{\partial}{\partial \rho} \frac{1}{(1 - \rho)} \\ & &= (1 - \rho) \rho \frac{1}{(1 - \rho)^2} \\ & &= \frac{\rho}{(1 - \rho)}\end{aligned}$$

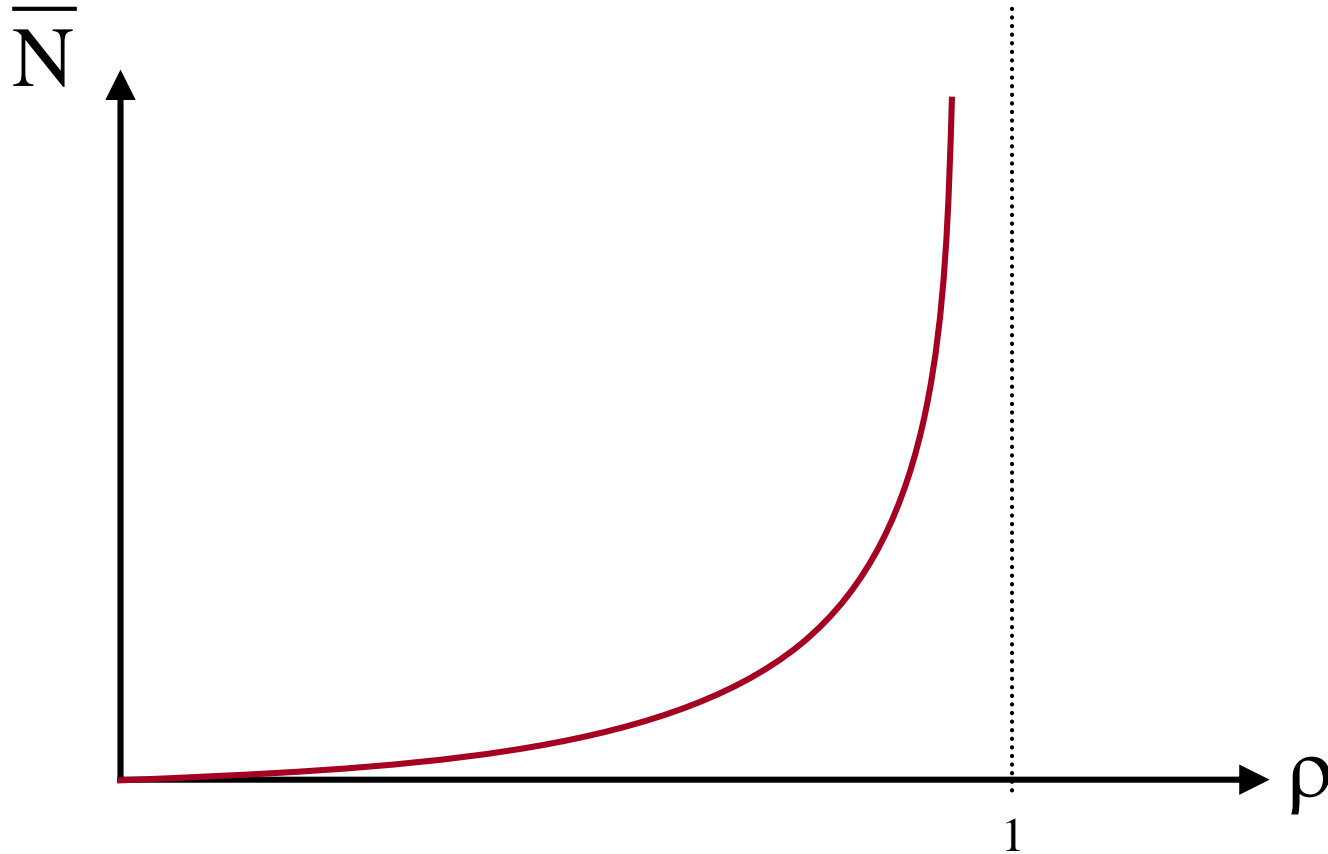
# The average # of customers

23

$$\begin{aligned}\bar{N} &= \frac{\rho}{(1 - \rho)} \\ &= \frac{\lambda/\mu}{(1 - \lambda/\mu)} \\ &= \frac{\lambda}{(\mu - \lambda)}\end{aligned}$$

# The average # of customers

24





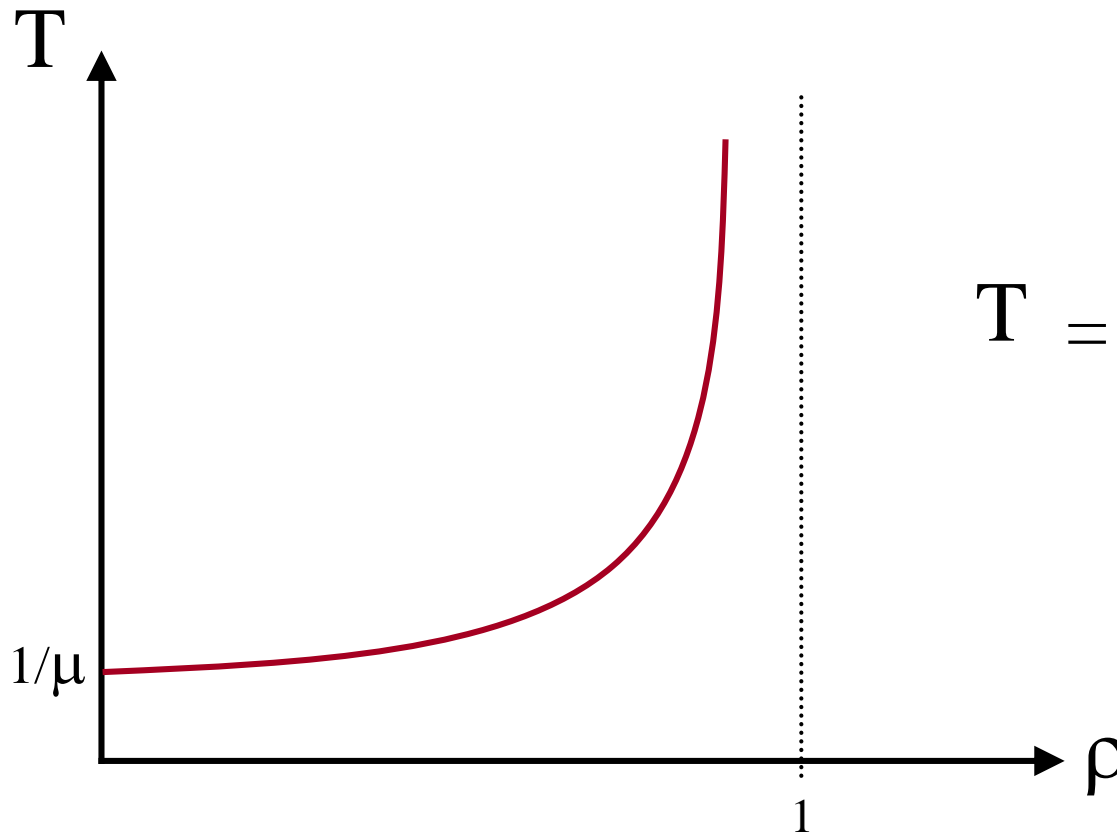
# The average delay

25

$$\begin{aligned} T &= \frac{\bar{N}}{\lambda} \\ &= \frac{1}{\lambda} \left( \frac{\rho}{(1-\rho)} \right) \\ &= \frac{1/\mu}{(1-\rho)} \\ &= \frac{1}{(\mu - \lambda)} \end{aligned}$$

# The average delay

26

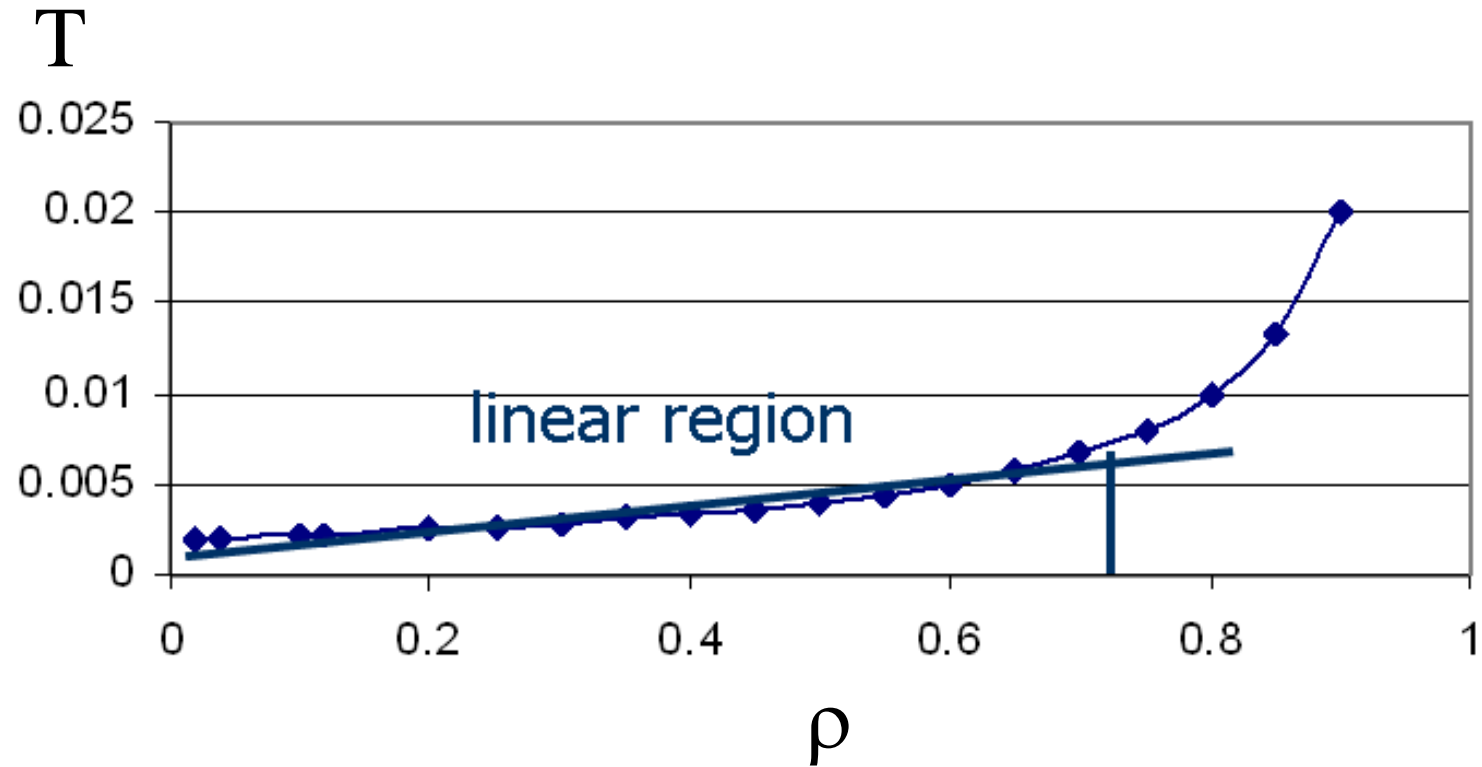


$$T = \frac{1 / \mu}{(1 - \rho)}$$

$\rho = 0 \rightarrow$  only service time

# Stable Region

27



# The average # of customers in Q

28

- $W = \text{Total Delay} - \text{Service time} = T - S$

$$\overline{W} = \frac{1 / \mu}{(1 - \rho)} - (1 / \mu)$$

$$= \frac{\rho / \mu}{(1 - \rho)}$$

$$\overline{N}_Q = \lambda \overline{W} = \frac{\rho^2}{(1 - \rho)}$$

# Example 1

29

- A wireless channel of IEEE 802.11b = 11Mbps
- For a video file needed to be sent at 2 Mbps
- $\lambda = 2$  Mbps
- $\mu = 11$  Mbps
- $\rho = \lambda/\mu = 2/11 = 0.182$

# Example 1

30

- The avg. Tx time  $= \frac{1 / \mu}{(1 - \rho)}$   
 $= (1/11)/(1 - 0.182)$   
 $= 111.14 \text{ ms}$
- The avg.# of waiting bits  $= 0.182^2/(1 - 0.182)$   
 $= 0.04049$

# Example 2

- On a network gateway, measurements show that the packets arrive at a mean rate of 125 packets per second (pps)
- Gateway takes about 2 milliseconds to forward them
- Assuming an M/M/1 model
  - What is the probability of buffer overflow if the gateway had only 13 buffers ?
  - How many buffers are needed to keep packet loss below one packet per million?

# Example 2

32

- Measurement of a network gateway:
  - mean arrival rate : 125 Packets/s
  - mean response time : 2 ms
- Assuming exponential arrivals:
  - What is the gateway's utilization?
  - What is the probability of n packets in the gateway?
  - mean number of packets in the gateway?
  - The number of buffers so  $P(\text{overflow})$  is  $<10^{-6}$ ?



# Example 2

33

- Arrival rate  $\lambda =$
- Service rate  $\mu =$
- Gateway utilization  $\rho = \lambda/\mu =$
- Prob. of  $n$  packets in gateway =
- Mean number of packets in gateway =

# Example 2

34

- Arrival rate  $\lambda = 125$  pps
- Service rate  $\mu = 1/0.002 = 500$  pps
- Gateway utilization  $\rho = \lambda/\mu = 0.25$
- Prob. of  $n$  packets in gateway

$$(1 - \rho)\rho^n = 0.75(0.25)^n$$

- Mean number of packets in gateway

$$\frac{\rho}{1 - \rho} = \frac{0.25}{0.75} = 0.33$$

# Example 2

35

- Probability of buffer overflow:  
= P(more than 13 packets in gateway)  
=  $\rho^{13} = 0.25^{13} = 1.49 \times 10^{-8}$   
= 15 packets per billion packets
- To limit the probability of loss to less than  $10^{-6}$ :  
$$\rho^n \leq 10^{-6}$$
$$n > \log(10^{-6}) / \log(0.25)$$
$$= 9.96$$

# HW: M/M/1

36

- A Bangkok gas station named “Smile Pump”
- Normally, they provide all kinds of Diesel and Gasohal (91, 95).
- Due to the Fuel crisis, the station installs a NGV gas pump which is the only one NGV in the area.

# HW: M/M/1

37

- Car (mostly Taxi) arrives at Poisson with rate 6 per hour. Each car take 5 min. exponentially distributed time for filling up.
  - What is the system utilization ?
  - How long does each car spend at the gas station ?
  - How big is the waiting area that Gas station needs to prepare ?
  - How long does each car have to wait in the Queue?
  - How many car in the queue ?

# HW: M/M/1

38

- During a Bangkok Music Festival, a lot of people coming to Bangkok. The arrival rate of cars increase to 10 per hour.
  - What is the system utilization ?
  - How long does each car spend at the gas station ?
  - How long does each car have to wait in the queue?
  - How many car in the queue ?

# HW: M/M/1

39

- During a Songkran Festival, people leaving Bangkok. The arrival rate of cars drop to 3 per hour.
  - What is the system utilization ?
  - How long does each car spend at the gas station ?
  - How long does each car have to wait in the queue?
  - How many car in the queue ?

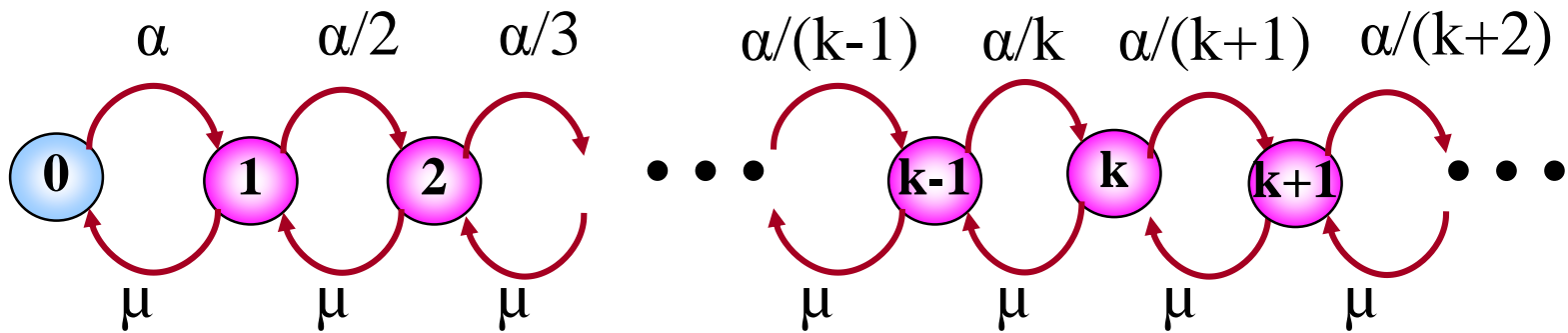
# DISCOURAGED ARRIVALS





# Discouraged Arrivals

41



- Assumption

- $\lambda_k = \alpha / (k+1)$  for  $k \geq 0$
- $\mu_k = \mu$  for  $k \geq 1$

# Discouraged Arrivals

42

$$\begin{aligned} p_k &= p_0 \left( \prod_{i=0}^{k-1} \frac{\alpha/(i+1)}{\mu} \right) \\ &= p_0 \left( \frac{\alpha}{\mu} \right)^k \left( \frac{1}{k!} \right) \\ p_0 &= \left( 1 + \sum_{k=1}^{\infty} \left( \frac{\alpha}{\mu} \right)^k \left( \frac{1}{k!} \right) \right)^{-1} \\ &= e^{-\alpha/\mu} \end{aligned}$$

# Discouraged Arrivals

43

$$\rho = 1 - e^{-\alpha/\mu}$$

- The ergodic condition is  $\alpha/\mu < \infty$

$$\begin{aligned} p_k &= p_0 \frac{(\alpha/\mu)^k}{k!} \\ &= \frac{(\alpha/\mu)^k}{k!} e^{-\alpha/\mu} \end{aligned}$$

Poisson Distribution

# Discouraged Arrivals

44

$$\bar{N} = \frac{\alpha}{\mu}$$

$$\lambda = \mu\rho = \mu(1 - e^{-\alpha/\mu})$$

- From Little's result

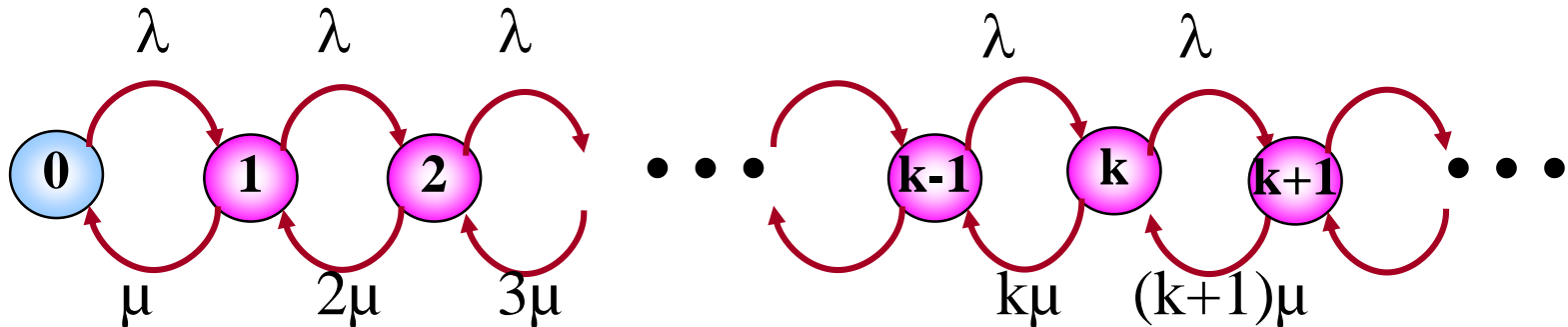
$$T = \frac{\alpha}{\mu^2 (1 - e^{-\alpha/\mu})}$$

# RESPONSIVE SERVERS (M/M/∞)



# Responsive Servers (M/M/∞)

46



- Assumption
  - $\lambda_k = \lambda$  for  $k \geq 0$
  - $\mu_k = k\mu$  for  $k \geq 1$
- Infinite number of servers

# Responsive Servers (M/M/∞)

47

$$\begin{aligned} p_k &= p_0 \left( \prod_{i=0}^{k-1} \frac{\lambda}{(i+1)\mu} \right) \\ &= p_0 \left( \frac{\lambda}{\mu} \right)^k \left( \frac{1}{k!} \right) \\ p_0 &= \left( 1 + \sum_{k=1}^{\infty} \left( \frac{\lambda}{\mu} \right)^k \left( \frac{1}{k!} \right) \right)^{-1} \\ &= e^{-\lambda/\mu} \end{aligned}$$

# Responsive Servers (M/M/∞)

48

$$\rho = 1 - e^{-\lambda/\mu}$$

- The ergodic condition is  $\lambda/\mu < \infty$

$$\begin{aligned} p_k &= p_0 \frac{(\lambda/\mu)^k}{k!} \\ &= \frac{(\lambda/\mu)^k}{k!} e^{-\lambda/\mu} \end{aligned}$$

Poisson Distribution



# Responsive Servers (M/M/∞)

49

$$\bar{N} = \frac{\lambda}{\mu}$$

- From Little's result

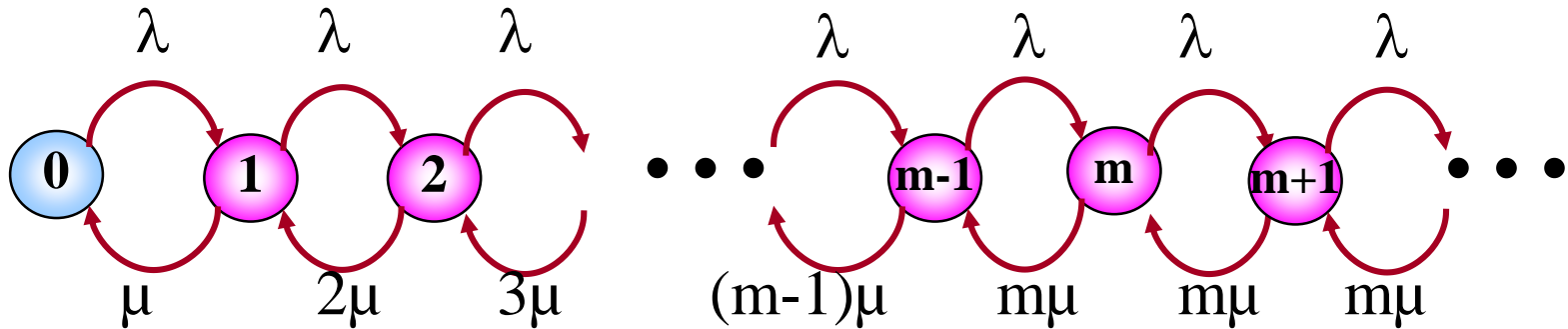
$$T = \frac{\bar{N}}{\lambda} = 1 / \mu \quad \leftarrow \text{Always get serve}$$

# M-SERVER (M/M/M)



# m-Server (M/M/m)

51



- Assumption

- $\lambda_k = \lambda$  for  $k \geq 0$

- $\mu_k = \min [ k\mu, m\mu ]$

$$= \begin{cases} k\mu & 0 \leq k \leq m \\ m\mu & m \leq k \end{cases}$$

# m-Server (M/M/m)

52

- For  $k \leq m$

$$p_k = p_0 \left( \prod_{i=0}^{k-1} \frac{\lambda}{(i+1)\mu} \right)$$

$$= p_0 \left( \frac{\lambda}{\mu} \right)^k \left( \frac{1}{k!} \right)$$

# m-Server (M/M/m)

53

- For  $k \geq m$

$$\begin{aligned} p_k &= p_0 \prod_{i=0}^{m-1} \frac{\lambda}{(i+1)\mu} \prod_{j=m}^{k-1} \frac{\lambda}{m\mu} \\ &= p_0 \left[ \frac{\lambda}{\mu} \right]^k \frac{1}{m! m^{k-m}} \end{aligned}$$

# m-Server (M/M/m)

54

$$p_k = \begin{cases} p_0 \frac{(m\rho)^k}{k!} & k \leq m \\ p_0 \frac{(\rho)^k m^m}{m!} & k \geq m \end{cases}$$

$$\rho = \frac{\lambda}{m\mu} < 1$$

# m-Server (M/M/m)

55

$$p_0 = \left( 1 + \sum_{k=1}^{m-1} \frac{(m\rho)^k}{k!} + \sum_{k=m}^{\infty} \frac{(m\rho)^k}{m!} \frac{1}{m^{k-m}} \right)^{-1}$$

$$p_0 = \left( \sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \left( \frac{(m\rho)^m}{m!} \right) \left( \frac{1}{1-\rho} \right) \right)^{-1}$$

# m-Server (M/M/m)

56

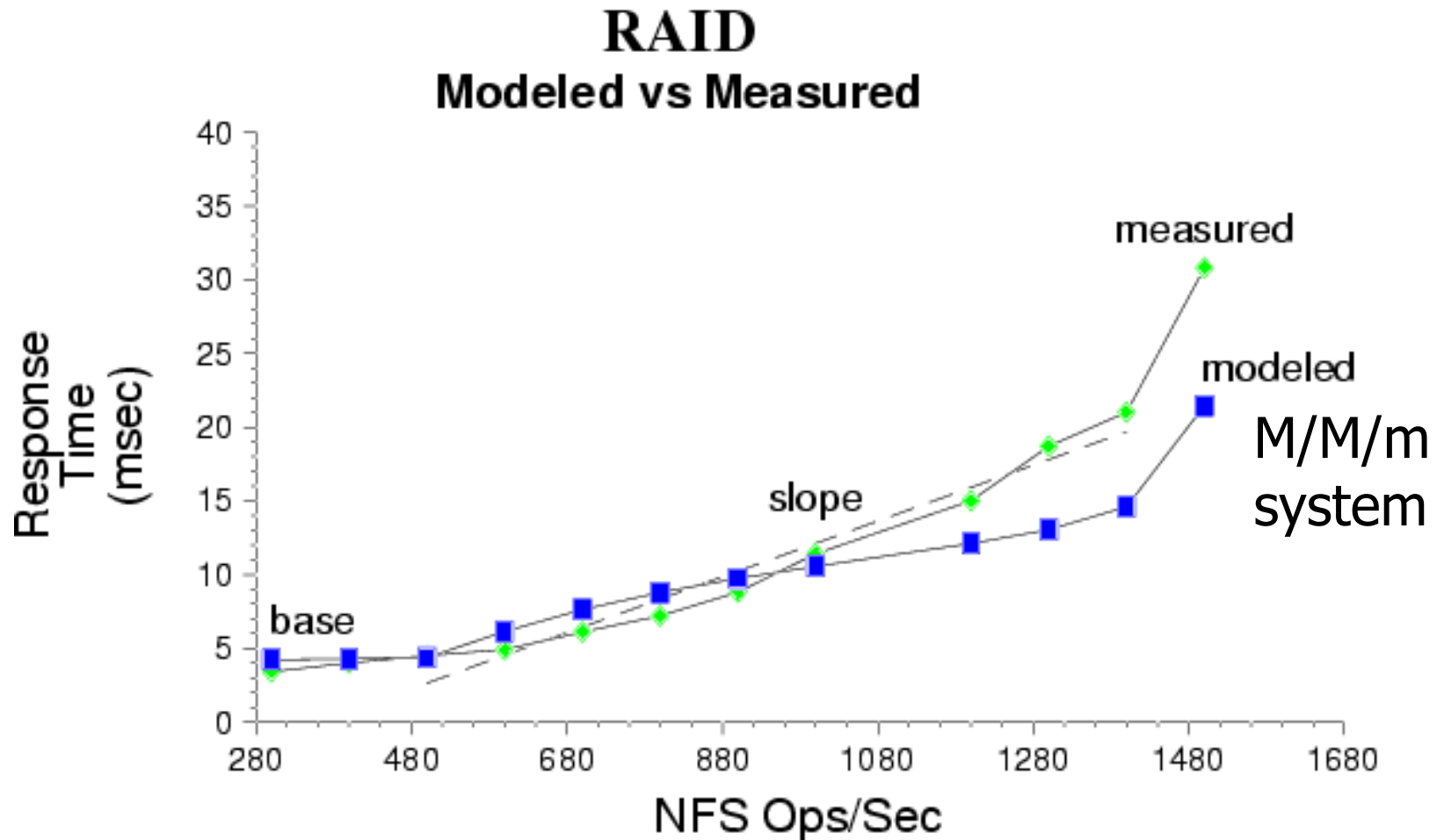
$$\begin{aligned} P[\text{queueing}] &= \sum_{k=m}^{\infty} p_k \\ &= \sum_{k=m}^{\infty} p_0 \frac{(m\rho)^k}{m!} \frac{1}{m^{k-m}} \end{aligned}$$

$$P[\text{queueing}] = \frac{\left( \frac{(m\rho)^m}{m!} \right) \left( \frac{1}{1-\rho} \right)}{\left( \sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \left( \frac{(m\rho)^m}{m!} \right) \left( \frac{1}{1-\rho} \right) \right)}$$

Erlang's C formula  
 $C(m, \lambda/\mu)$



# Empirical Example

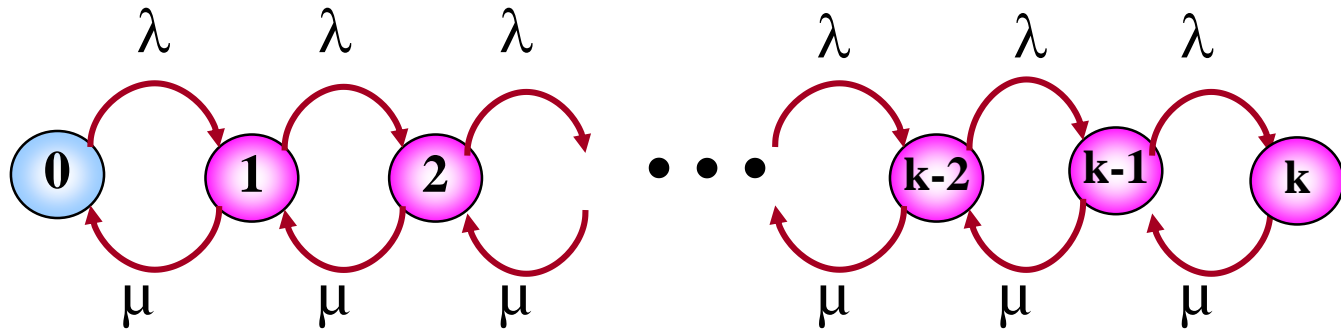


# FINITE STORAGE (M/M/1/K)



# Finite Storage (M/M/1/K)

59



- Assumption

- $\mu_k = \mu$

$$1 \leq k \leq K$$

- $\lambda_k = \begin{cases} \lambda \\ 0 \end{cases}$

$$k < K$$

$$k \geq K$$

# Finite Storage (M/M/1/K)

60

- For  $k \leq K$

$$p_k = p_0 \prod_{i=0}^{k-1} \frac{\lambda}{\mu}$$

$$= p_0 \left( \frac{\lambda}{\mu} \right)^k$$

- For  $k > K$

$$p_k = 0$$

# Finite Storage (M/M/1/K)

61

$$\begin{aligned} p_0 &= \left( 1 + \sum_{k=1}^K \left( \frac{\lambda}{\mu} \right)^k \right)^{-1} \\ &= \left( 1 + \frac{(\lambda/\mu)(1 - (\lambda/\mu)^K)}{1 - (\lambda/\mu)} \right)^{-1} \\ &= \frac{1 - (\lambda/\mu)}{1 - (\lambda/\mu)^{K+1}} \end{aligned}$$

# Finite Storage (M/M/1/K)

62

$$p_k = \begin{cases} \frac{1 - (\lambda/\mu)}{1 - (\lambda/\mu)^{K+1}} \left(\frac{\lambda}{\mu}\right)^k & 0 \leq k \leq K \\ 0 & \text{Otherwise} \end{cases}$$

# Finite Storage (M/M/1/K)

63

- For  $K = 1$  (Blocked call cleared)

$$p_k = \begin{cases} \frac{1}{1 + (\lambda/\mu)} & k = 0 \\ \frac{\lambda/\mu}{1 + (\lambda/\mu)} & k = 1 = K \\ 0 & \text{Otherwise} \end{cases}$$

# Note on M/M/1/K

64

- $\rho > 1$  is possible