LECTURE #14
M/G/1

204528

Queueing Theory and Applications in Networks

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Outline

- More on M/G/1
- Busy period and its duration
- M/G/1 with Vacations

M/G/1

$$W_q = \frac{\lambda \overline{X^2}}{2(1-\rho)}$$

$$N_q = \frac{\lambda^2 \overline{x^2}}{2(1-\rho)}$$

$$W_T = \overline{X} + \frac{\lambda \overline{X^2}}{2(1-\rho)}$$

$$N_T = \rho + \frac{\lambda^2 \overline{x^2}}{2(1-\rho)}$$

The coefficient of variation (C_V)

- A normalized measure of dispersion of a probability distribution
- Defined as the ratio of the standard deviation to the mean

$$C_v = \frac{\sigma}{\mu}$$

- Only defined for *non-zero* mean
- C_V should only be computed for data measured on a **ratio scale**

dispersion = การกระจาย

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The coefficient of variation (C_V)

- Example (from wikipedia)
  - For a group of temperatures
  - An object changes its temperature by 1 K also changes its temperature by 1 C
  - The standard deviation does not depend on whether the Kelvin or Celsius scale
  - However, the mean temp would differ in each scale by 273
  - So, the coefficient of variation would differ
- Investment Dictionary (http://www.answers.com/topic/coefficient-of-variation)
  - The lower the ratio, the better your risk-return tradeoff
  - The higher the ratio, the higher the risk

The coefficient of variation (C_V)

http://en.wikipedia.org/wiki/Coefficient_of_variation

- In queueing theory
  - **Exponential Dist.** is often more important than the **Normal Dist.**
- For $C_V = 1$
  - E.g. Exponential distribution ➔ ($\sigma = \mu$)
- For $C_V < 1$ ➔ low-variance
  - E.g. Erlang distribution [r-stage Erlangian server (E_r)]
- For $C_V > 1$ ➔ high-variance
  - E.g. Hyper-Exponential distribution [r-Stage Parallel Servers (H_r)]
- Some formulas are expressed using **squared coefficient of variation** (SCV)

More on P-K Formula

From: J. Virtamo, 38.3143 Queueing Theory / The *M/G/1* queue

Squared coefficient of variation C_v^2

$$C_v^2 = \frac{\mathrm{Var}[X]}{\left(\overline{X}\right)^2}$$

$$\overline{X^2} = \mathrm{Var}[X] + \left(\overline{X}\right)^2$$

$$= (1 + C_v^2) \bullet \left(\overline{X}\right)^2$$

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More on P-K Formula

$$W_q = \frac{\lambda \overline{X^2}}{2(1-\rho)} = \frac{1 + C_v^2}{2} \cdot \frac{\rho}{1-\rho} \cdot \overline{X}$$

$$W_T = \overline{X} + \frac{\lambda \overline{X^2}}{2(1-\rho)} = \left(1 + \frac{1 + C_v^2}{2} \cdot \frac{\rho}{1-\rho}\right) \cdot \overline{X}$$

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More on P-K Formula

$$N_q = \frac{\lambda^2 \overline{x^2}}{2(1-\rho)} = \frac{1 + C_v^2}{2} \bullet \frac{\rho^2}{1-\rho}$$

$$N_T = \rho + \frac{\lambda^2 \overline{x^2}}{2(1-\rho)} = \rho + \frac{1 + C_v^2}{2} \bullet \frac{\rho^2}{1-\rho}$$

More on P-K Formula

- Mean values depend only on the expectation E[X] and variance Var[X] of the service time distribution but not on higher moments.
- Mean values increase linearly with the variance.
- Randomness, ‘disarray’, leads to an increased waiting time and queue length.
- The formula are similar to those of the *M/M/1* queue;
  - the only difference is the extra factor $\dfrac{1 + C_v^2}{2}$

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M/G/1 Steady-State

From www.cse.msu.edu/~cse807/notes/slides/queueing.ppt

$$W_q = \frac{\lambda(\overline{X^2} + \sigma^2)}{2(1-\rho)}$$

$$N_q = \frac{\lambda^2(\overline{X^2} + \sigma^2)}{2(1-\rho)}$$

$$W_T = \overline{X} + \frac{\lambda(\overline{X^2} + \sigma^2)}{2(1-\rho)}$$

$$N_T = \rho + \frac{\lambda^2(\overline{X^2} + \sigma^2)}{2(1-\rho)}$$

$$p_0 = 1 - \rho$$

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M/G/1 Example

- There are two workers competing for a job.
- Pual claims that an average service time is **faster** than Jar's
- But Jar claims to be **more consistent**, if not as fast.
- The arrivals is a Poisson process at a rate of $\lambda = 2$ per hour. (1/30 per minute).
- Paul's service statistics are an average service time of 24 minutes with a standard deviation of 20 minutes.
- Jar's service statistics are an average service time of 25 minutes, but a standard deviation of only 2 minutes.

M/G/1 Example

- If the <u>average length of the queue</u> is the criterion for hiring, which worker should be hired (Paul or Jar)?

M/G/1 Example

- For Paul,

$\lambda = 1/30$ (per min)

$1/\mu = 24$ min.

$\rho = \lambda/\mu = 24/30 = 4/5$

$\sigma^2 = 20^2 = 400$ min^2

$$N_q = \frac{\lambda^2(\overline{X}^2 + \sigma^2)}{2(1-\rho)} = \frac{(1/30)^2\,(24^2 + 400)}{2(1 - 4/5)}$$

$= 2.711$ customers

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M/G/1 Example

- For Jar,

$\lambda = 1/30$ (per min)

$1/\mu = 25$ min.

$\rho = \lambda/\mu = 25/30 = 5/6$

$\sigma^2 = 2^2 = 4 \text{ min}^2$

$$N_q = \frac{\lambda^2(\overline{X^2} + \sigma^2)}{2(1-\rho)} = \frac{(1/30)^2\,(25^2 + 4)}{2(1 - 5/6)}$$

= 2.097 customers

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M/G/1 Example

- Although working faster on the average,
  - Paul's greater service variability results in an average queue length about 30% greater than Jar's.
- On the other hand, the proportion of arrivals who would find Paul idle and thus experience no delay is $p_0 = 1 - \rho = 1/5 = 20\%$
- While the proportion who would find Jar idle and thus experience no delay is $p_0 = 1 - \rho = 1/6 = 16.7\%$.
- On the basis of average queue length, Nq,
  - Jar wins.

Busy and Idle period

- To derive the distribution for the M/G/1 queue
  - the length of the Idle period
  - the length of the busy period

Busy and Idle period

- C_n = the n^{th} customer to enter the system
- τ_n = arrival time of C_n
- $t_n = \tau_n - \tau_{n-1}$ = interarrival time between C_{n-1} and C_n
- x_n = Service time of C_n

Busy and Idle period

- U(t) = Unfinished work in the system
  = Virtual waiting time at time t
- Y_n = Busy period
- I_n = Idle period

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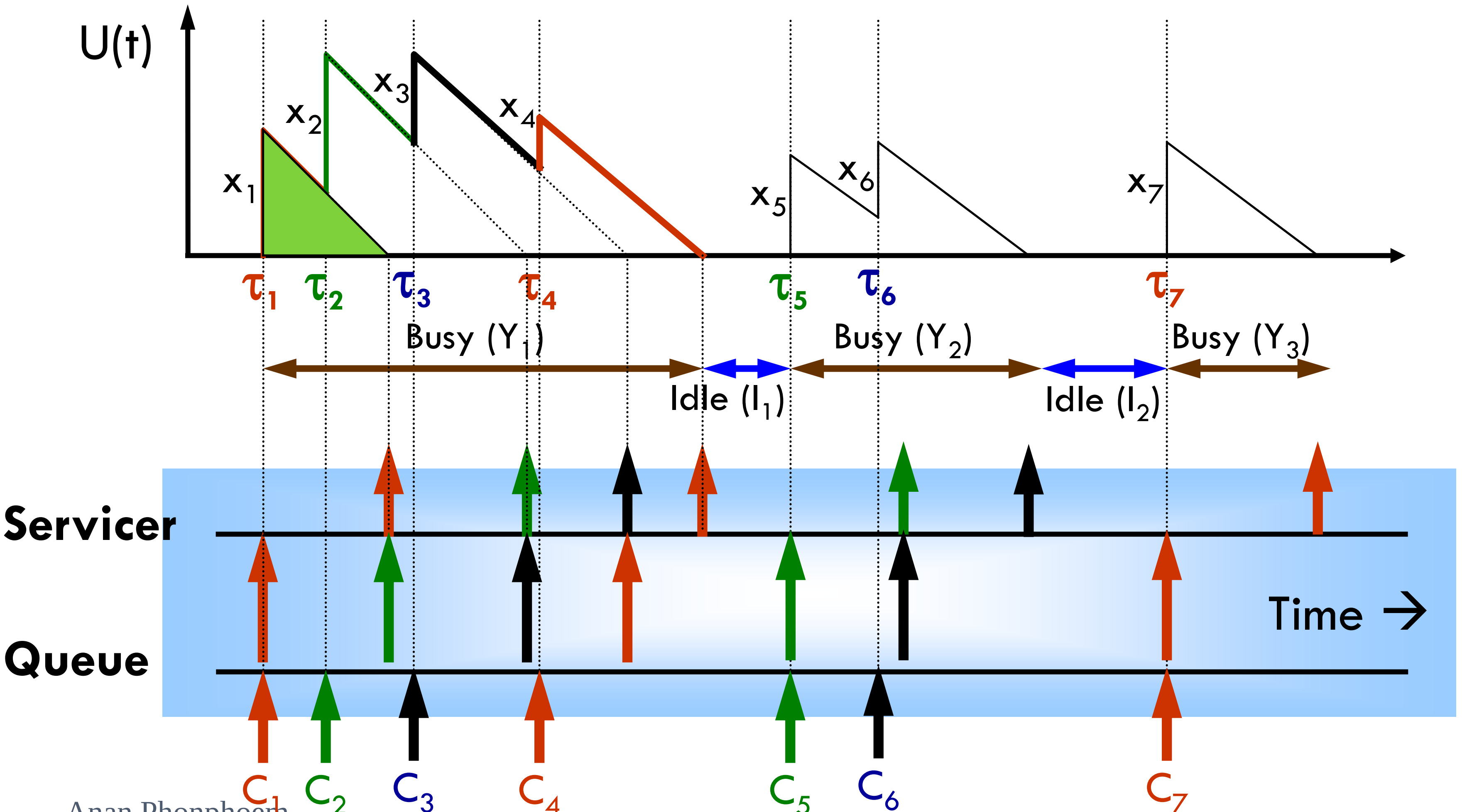
The Unfinished Work (FCFS)



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M/G/1 (FCFS)

- $A(t) = P[t_n \leq t] = 1 - e^{-\lambda t}$ $\quad t \geq 0$
- $B(x) = P[x_n \leq x]$
- A(t) and B(x) are independent on n
- F(y) = Idle period distribution

  $= P[I_n \leq y]$
- G(y) = Busy period distribution

  $= P[Y_n \leq y]$

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M/G/1 (FCFS)

- For Idle period F(y)
  - After busy period ➔ start the idle period
  - A new idle period will stop immediately when the new customer arrives
  - Therefore, from the memoryless distribution, Idle period distribution F(y)

$$F(y) = 1 - e^{-\lambda y} \qquad y \geq 0$$

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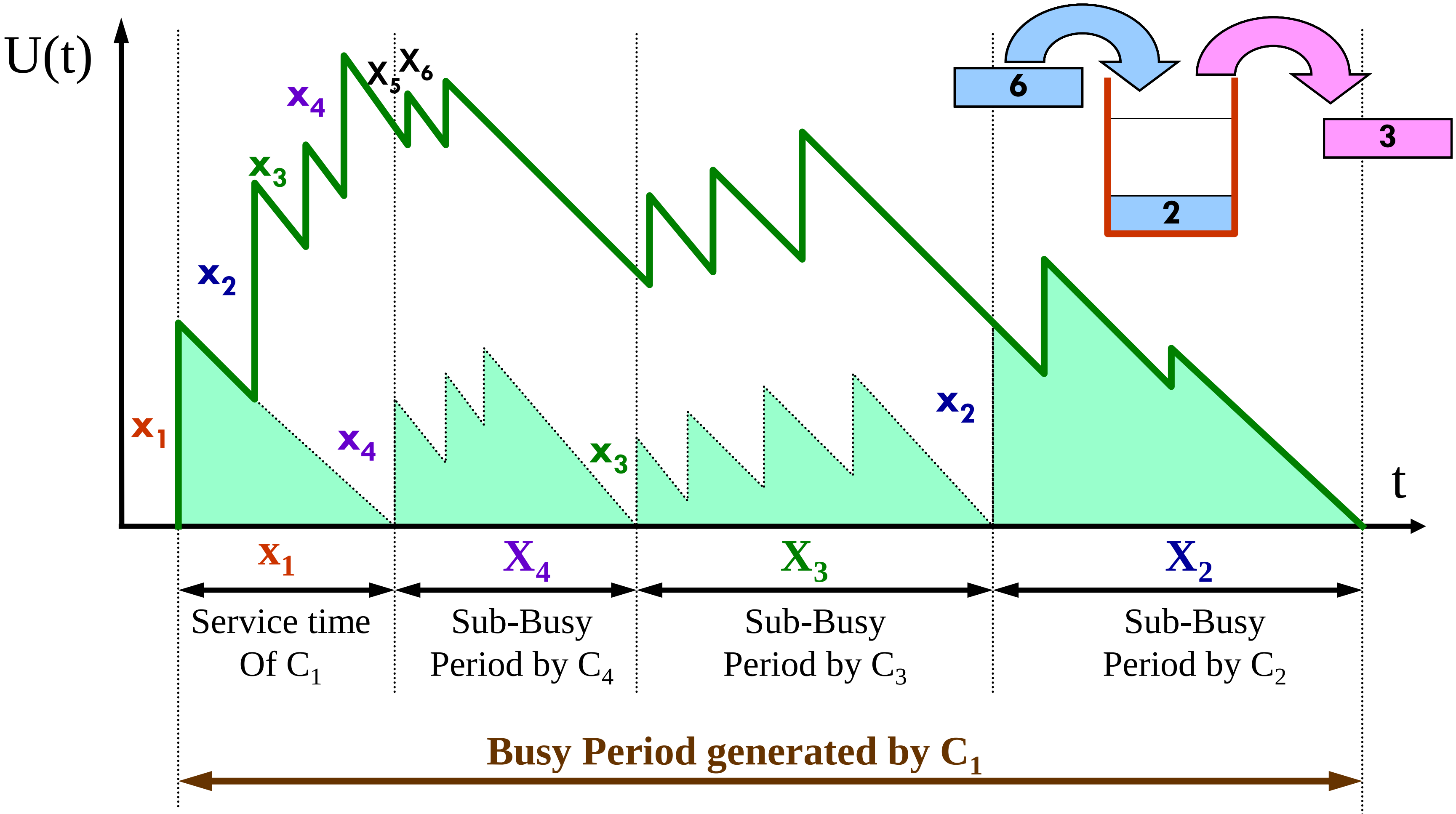
The Unfinished Work (LCFS)



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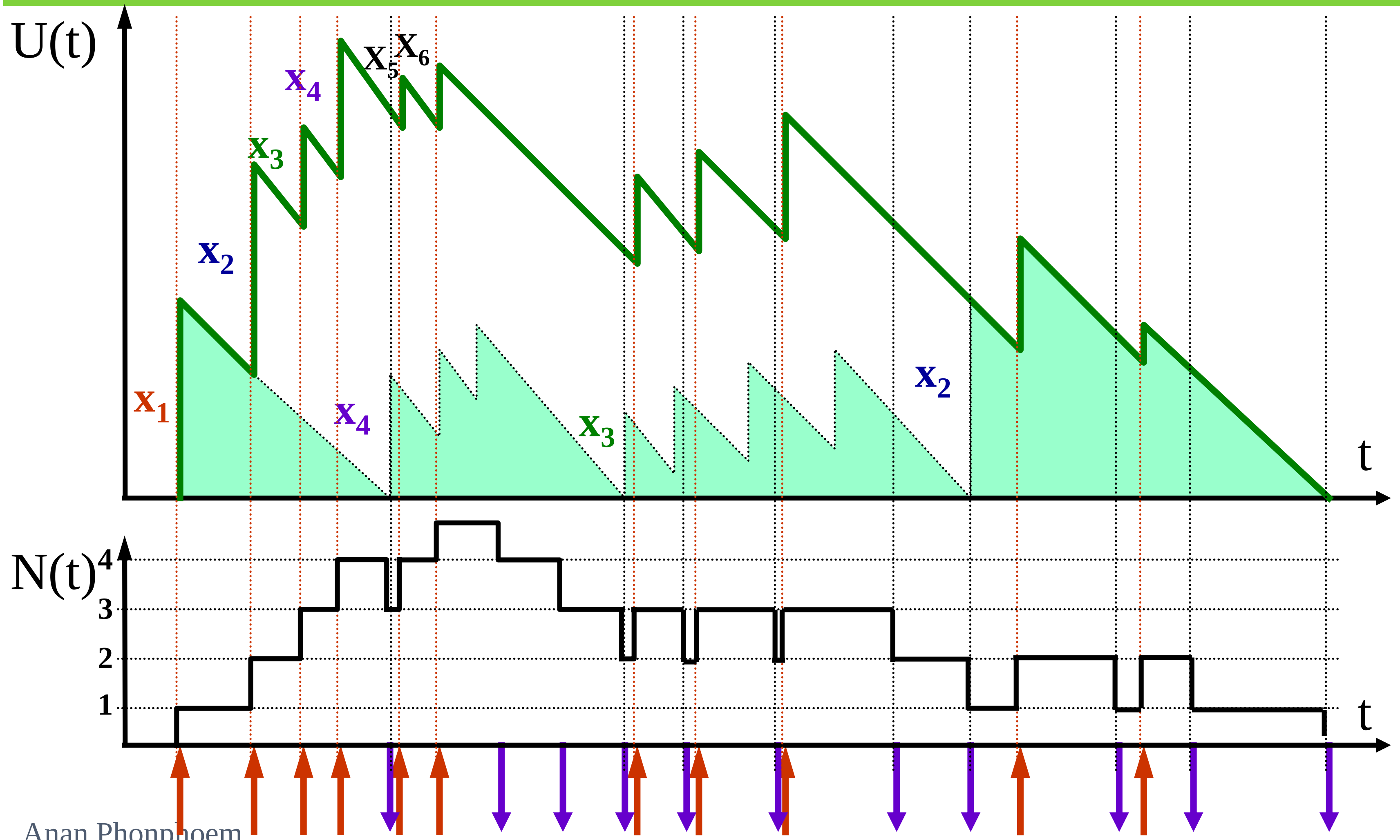
Number of Users (LCFS)



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M/G/1 (LCFS)

- Each sub-busy period behaves statistically the same as the major busy period
- The duration of busy period Y

$$Y = X_1 + X_{v+1} + X_{v+2} + \ldots + X_3 + X_2$$

- X_v = sub-busy period
- v = an RV = # of customer arrives during C_1 service interval

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M/G/1 (LCFS)

- For Busy period G(y)

  $G(y) = P[Y_n \leq y] \qquad\qquad y \geq 0$

- Transform of M/G/1 busy-period distribution

  $G^*(s) = B^*[s + \lambda - \lambda G^*(s)]$

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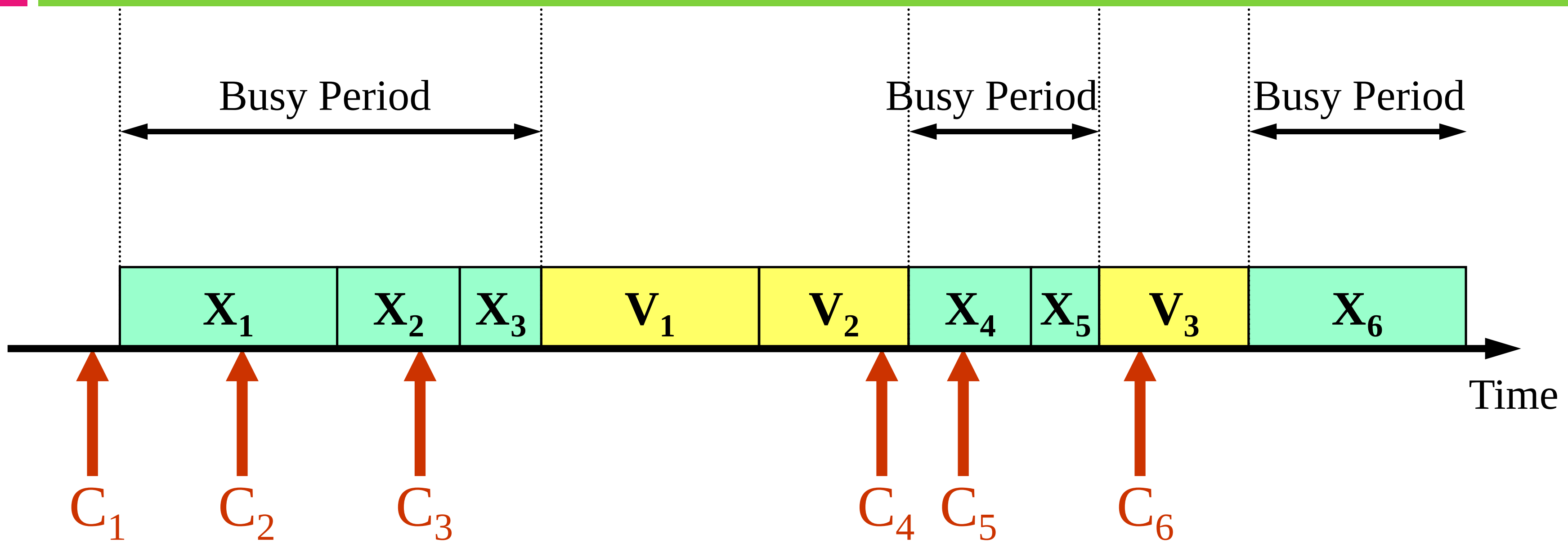
M/G/1 with Vacations

- At the end of busy period
  - The server goes on “vacation”
  - The vacation period = random interval of time
  - A new arrive during vacation has to wait until the end of vacation period
  - If the system is idle after vacation, a new vacation starts right away

M/G/1 with Vacations



- V_n = Vacation period with $\overline{V}$ and $\overline{V^2}$
  = IID random variable and independent of customer interarrival and service time
- X_n = Service period

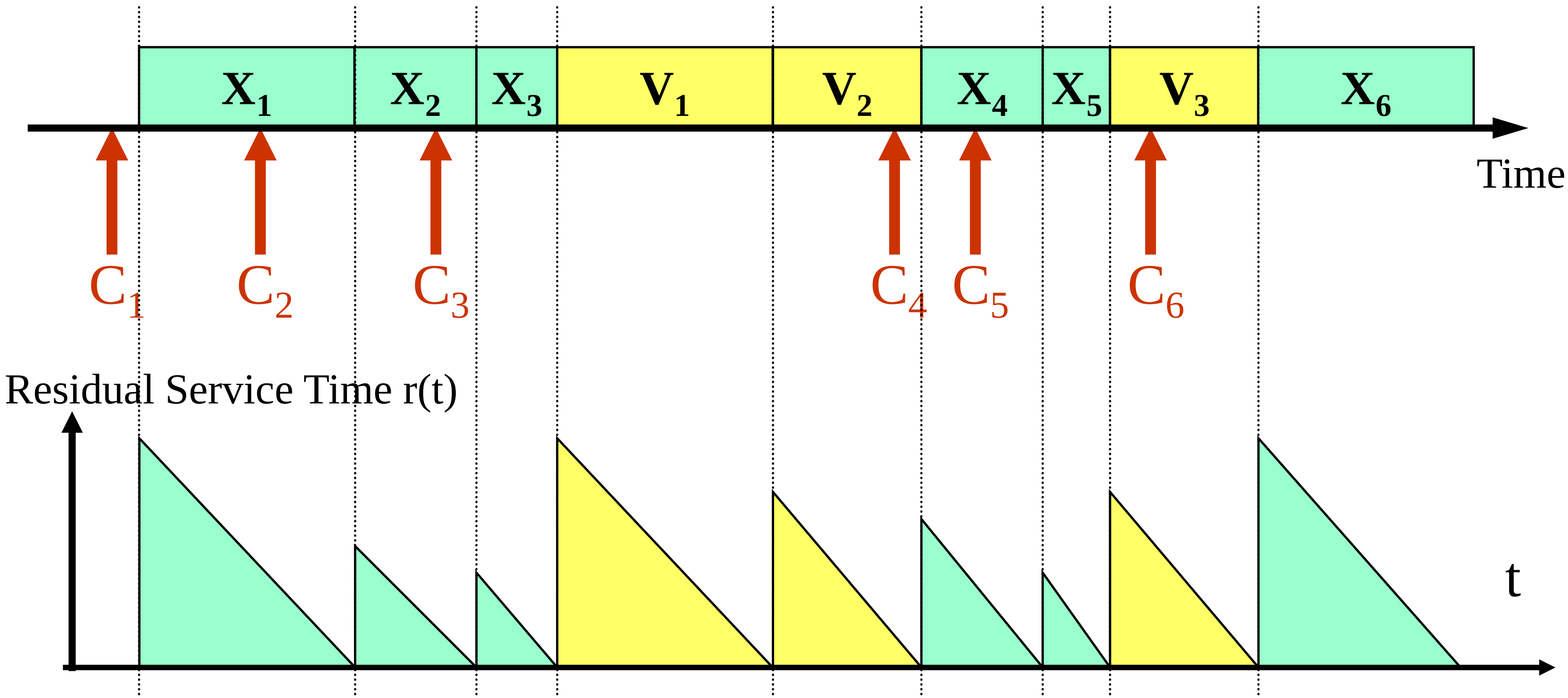
M/G/1 with Vacations

- A new customer is Poisson arrival and service time is general distribution
- The waiting time for customer is W

$$W = \frac{R}{1 - \rho}$$

- R = Residual Time

M/G/1 with Vacations



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M/G/1 with Vacations

$$R = \frac{1}{t}\int_0^t r(\tau)\,d\tau = \frac{1}{t}\sum_{i=1}^{M(t)} \frac{1}{2}X_i^2 + \frac{1}{t}\sum_{i=1}^{L(t)} \frac{1}{2}V_i^2$$

$$= \frac{M(t)}{t}\,\frac{\sum_{i=1}^{M(t)} \frac{1}{2}X_i^2}{M(t)} + \frac{L(t)}{t}\,\frac{\sum_{i=1}^{L(t)} \frac{1}{2}V_i^2}{L(t)}$$

M/G/1 with Vacations

$$t \to \infty, \qquad \frac{M(t)}{t} = \lambda \quad and \quad \frac{L(t)}{t} = \frac{(1-\rho)}{\overline{V}}$$

$$R = \frac{1}{2}\lambda\overline{X^2} + \frac{(1-\rho)\overline{V^2}}{2\overline{V}}$$

$$W = \frac{\lambda\overline{X^2}}{2(1-\rho)} + \frac{\overline{V^2}}{2\overline{V}}$$

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Examples

- Slotted M/D/1
- FDM
- Slotted FDM
- TDM

Slotted M/D/1

Modified from "Eytan Modiano", MIT

- Packet transmission starts only @beginning of the slot
- Each slot, only one packet will be transmitted with rate $1/\mu$
- If no packet is waiting in queue, the slot is idle (Server is on vacation)

Slotted M/D/1

$$W = \frac{\lambda \overline{X^2}}{2(1-\rho)} + \frac{\overline{V^2}}{2\overline{V}}$$

- $\overline{X} = \overline{V} = 1/\mu$
- $\overline{X^2} = \overline{V^2} = 1/\mu^2$

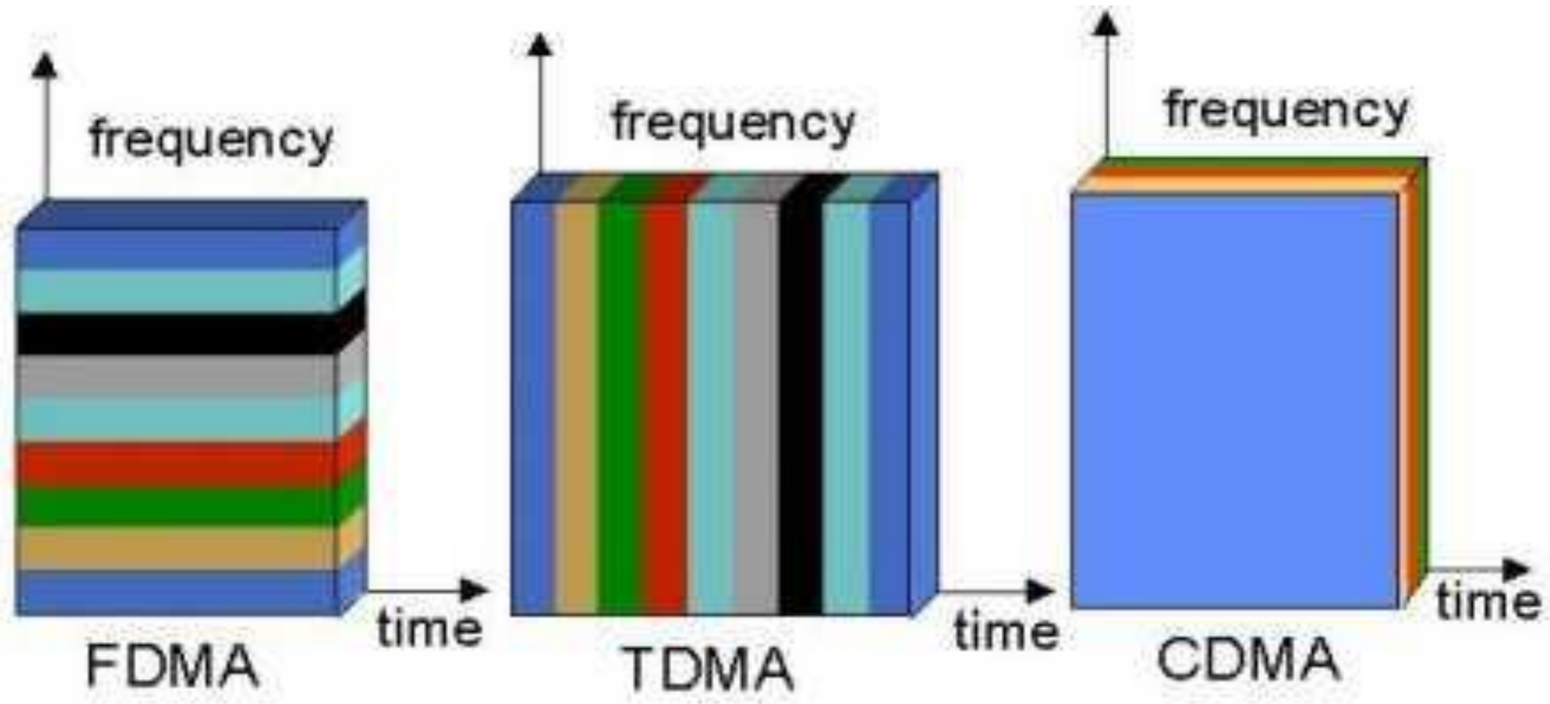
$$W = \frac{\lambda/\mu^2}{2(1-\lambda/\mu)} + \frac{1/\mu^2}{2/\mu}$$

$W_{M/D/1}$

$$\frac{1/\mu}{2} = \frac{\overline{X}}{2}$$

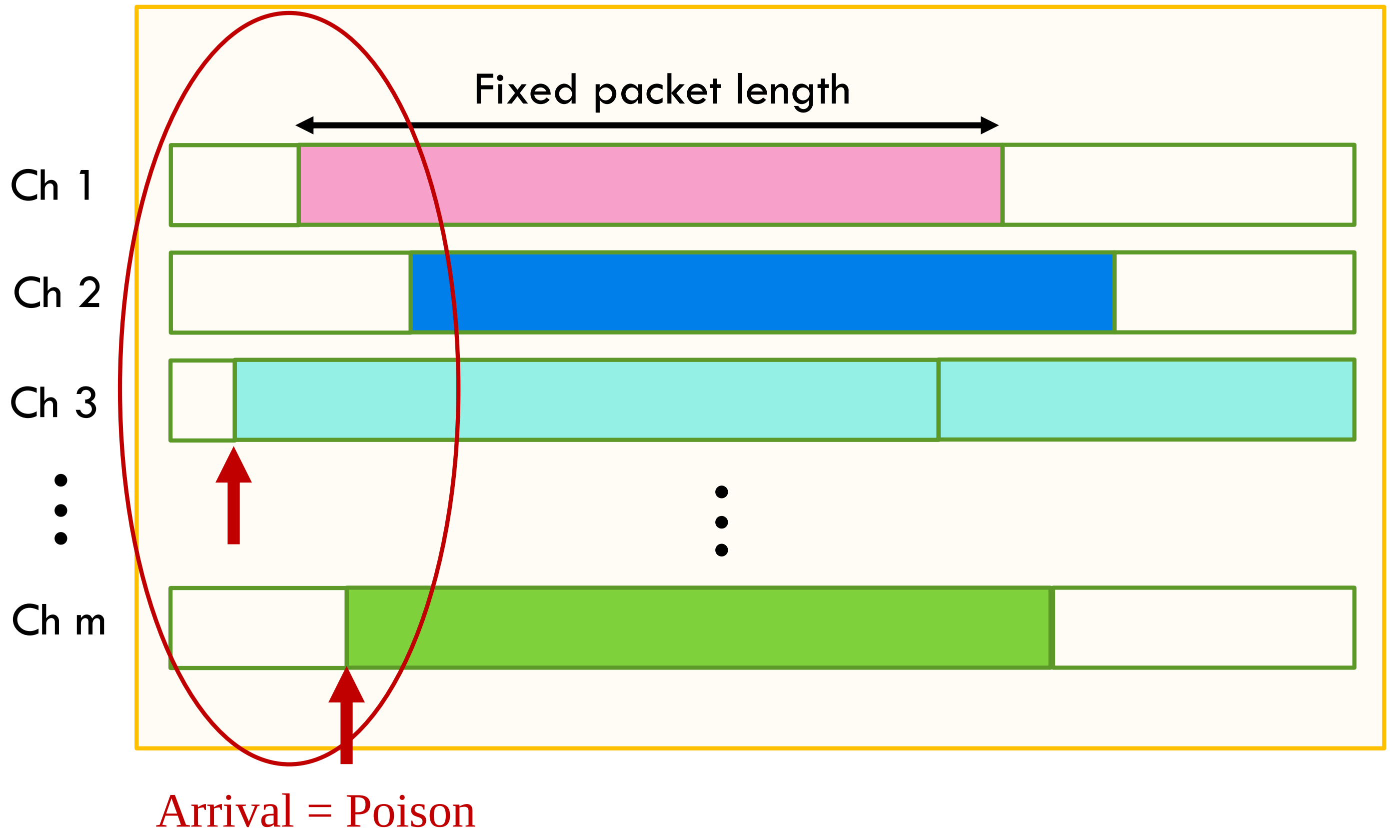
Wait for half slot

FDM, TDM, and CDMA



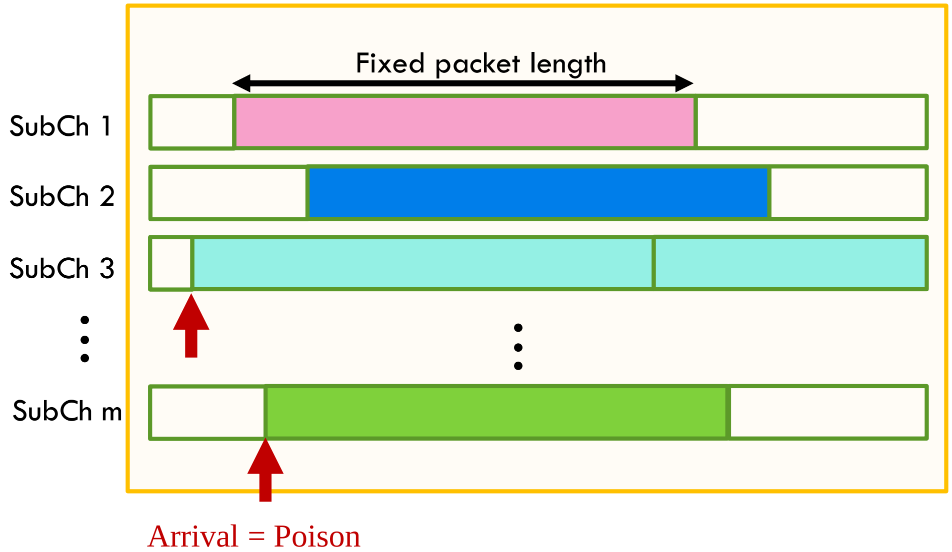
http://www.cs.ru.nl/~ths/a3/html/h2/h2.html

FDM



FDM

Modified from “Eytan Modiano”, MIT



- m Poisson streams of fixed length packets
- Arrival rate λ/m each multiplexed by FDM on m subchannels
- Total traffic = λ
- Each slot, only one packet will be transmitted with rate $\mu = 1/m$
- The total system load: $\rho = \lambda$

FDM

M/G/1 $$W = \frac{\lambda \, \overline{X^2}}{2(1-\rho)}$$

- $\overline{X} = 1/\mu = m$
- $\overline{X^2} = 1/\mu^2 = m^2$
- $\rho = \lambda$

$$W_{FDM} = \frac{(\lambda/m)\, m^2}{2\,(1-\rho)} = W_{M/D/1} = \frac{\rho\, m}{2\,(1-\rho)}$$

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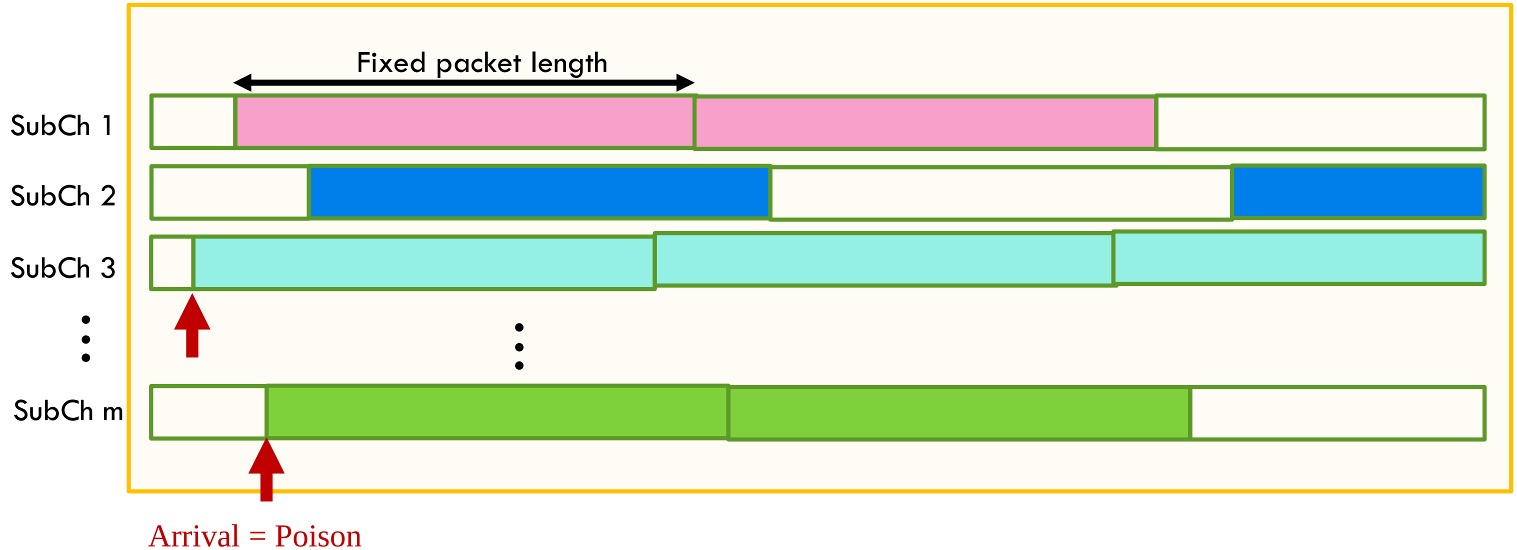
Slotted FDM

Modified from "Eytan Modiano", MIT



- Packet transmission starts only @beginning of the slot
- Each slot, only one packet will be transmitted with rate $\mu=1/m$
- m Poisson streams of fixed length packets
- Arrival rate λ/m each multiplexed by FDM on m subchannels

Slotted FDM

$$W = \frac{\lambda \overline{X^2}}{2(1-\rho)} + \frac{\overline{V^2}}{2\overline{V}}$$

- $\overline{X} = \overline{V} = 1/\mu = m$
- $\overline{X^2} = \overline{V^2} = 1/\mu^2 = m^2$

$$W = \frac{\rho m}{2(1-\rho)} + \frac{m}{2}$$

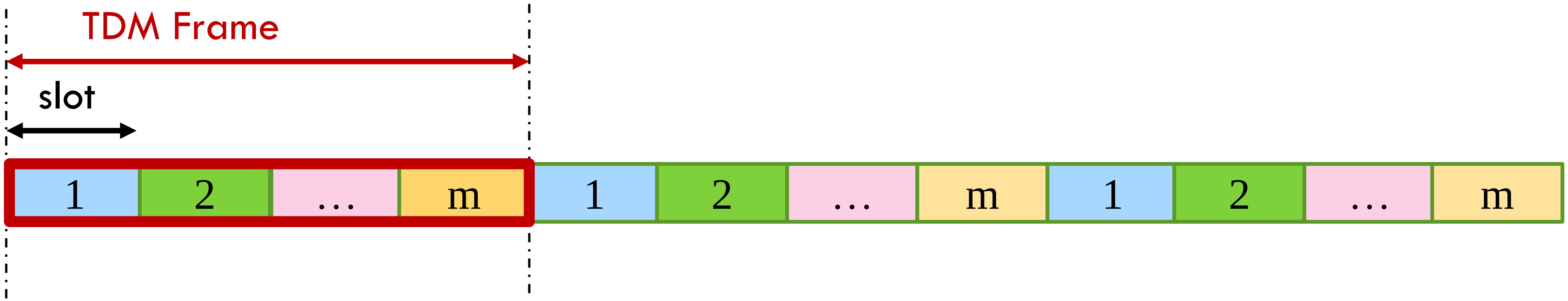
W_{FDM}

$$\frac{m^2}{2m} = \frac{\overline{X}}{2}$$

Wait for half slot

TDM

Modified from “Eytan Modiano”, MIT



- Packet transmission starts only @beginning of the slot
- Each slot, only one packet will be transmitted with rate $\mu=1/m$
- A stream(session) has to wait for its his own slot
- m Poisson streams of fixed length packets
- Arrival rate λ/m each multiplexed by TDM on m subchannels