LECTURE #13
M/G/1

204528

Queueing Theory and Applications in Networks

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Outline

- M/G/1
- Imbedded Markov Chain
  - Transition Probability
  - Mean Queue Length
- Mean Residual Service time

imbed [ɪm’bɛd] *vb* **-beds**, **-bedding**, **-bedded** a less common spelling of embed
http://www.thefreedictionary.com
fix (an object) firmly and deeply in a surrounding mass.

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M/G/1

- A single server
- Poisson arrivals
- Arbitrary service time

  → not necessary to be memoryless

M/M/1

- Let N(t) = the number of customers in the system
- If N(t) > 1
  - A customer is in service
  - The service time distribution is Memoryless
- So we can predict the future behavior of the system ➔ It's depends on the previous state
- The process N(t) is **a Markov Chain**

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M/G/1

- Let N(t) = the number of customers in the system
- If N(t) > 1
  - A customer is in service
  - The service time distribution is **NOT** **Memoryless**
- So, we need to know the time spent by customer to predict the future behavior of the system
- We want to summarize the complete past history of the system
- The process N(t) is **NOT** **a Markov Chain**

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M/G/1

- Let $X_0(t)$ = the service time already received by the customer in service at time t
- However, [N(t), $X_0(t)$] is a Markov Process
  - Summarize all past history
- But it is a two-dimensional state description

  ➔ can solved by the *method of supplementary variables*

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M/G/1

- We select the method of **Imbedded Markov Chain** (*Embedded Markov Chain*)
- We wish to simplify from two-dimensional description ➔ one-dimensional
- Must find some points in time that the number of the system will provide the future state of the system

  → Departure instance

  → For the departure point, $X_0(t)$ can be eliminated

Imbedded Markov Chain

- The departure of the customers is also called **Regeneration points**
- Let t_n = the time of departure of the n^{th} customer (where n =1,2,3,..)
- Let X_n = the number of customers in the system at time t_n

$$X_n = N(t_n)$$

- N(t) ➔ becomes the no. of customer **left behind** by a departure customer (not the same as previous N(t))

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For Poisson Arrival

- $P_k(t) = P[N(t) = k]$

  $= P[\text{arrival @ t finds k in system}]$

- This is always true

$$R_k(t) = \lim_{\Delta t \to \infty} P[N(t) = k]$$

$$P_k(t) = R_k(t)$$

For

$P_k(t)$ = Prob. that system is in state E_k

$R_k(t)$ = Prob. that an arriving customer finds the system is in state E_k

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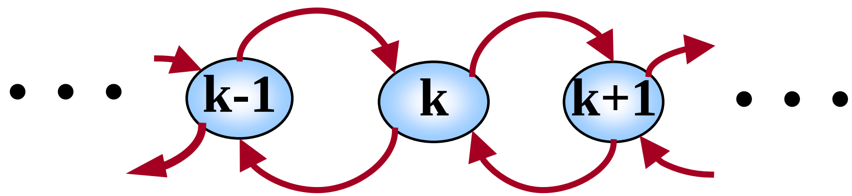
For Any System

- With the limiting distribution



$$\lim_{t \to \infty} P[N(t) = k] = \lim_{n \to \infty} P[X_n = k]$$

- P[arrival @ t finds k customers in the system]

  = P[departure @ t leaves k customers in the system]
- For M/G/1

  ➔ $r_k = d_k = p_k$

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Imbedded Markov Chain

- X_n = # customers in the system at time t_n
- Let Y_n = # customers arriving during the service time of the n^{th} customer

$$X_{n+1} = \begin{cases} X_n - 1 + Y_{n+1} & \text{if } X_n > 0 \\ Y_{n+1} & \text{if } X_n = 0 \end{cases}$$

Transition Probability

This notation is not related to the previous notation

- C_n = the n^{th} customer to enter the system
- τ_n = arrival time of C_n
- $t_n = \tau_n - \tau_{n-1}$ = interarrival time between C_{n-1} and C_n
- x_n = Service time of C_n
- $\mathbf{q_n}$ = number of customers left behind by departure of C_n
- v_n = number of customers arriving during the service of C_n

Transition Probability

- Want to find Distribution of q_n, $P[q_n = k]$
  - Time dependent
  - → limiting distribution $n \rightarrow \infty$
  - → $P[q_n = k] = p_k = P[N(t) = k]$

q_n = number of customers left behind by departure of C_n

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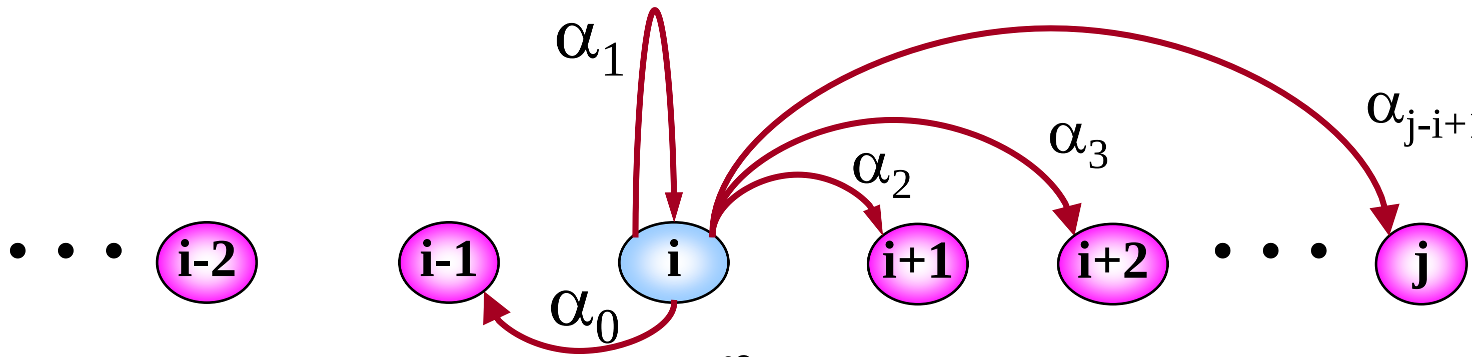
State transition diagram of M/G/1



$$\alpha_k = P[v_{n+1} = k] = \int_0^\infty P[\tilde{v} = k \mid \tilde{x} = x]\, b(x)\, dx$$

$$\alpha_k = \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x}\, b(x)\, dx$$

v_n = number of customers arriving during the service of C_n

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Transition Probability

- Define one step transition probability
- $P = p_{ij} = P[q_{n+1} = j \mid q_n = i]$

$$P = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \cdots \\ 0 & 0 & \alpha_0 & \alpha_1 & \cdots \\ 0 & 0 & 0 & \alpha_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

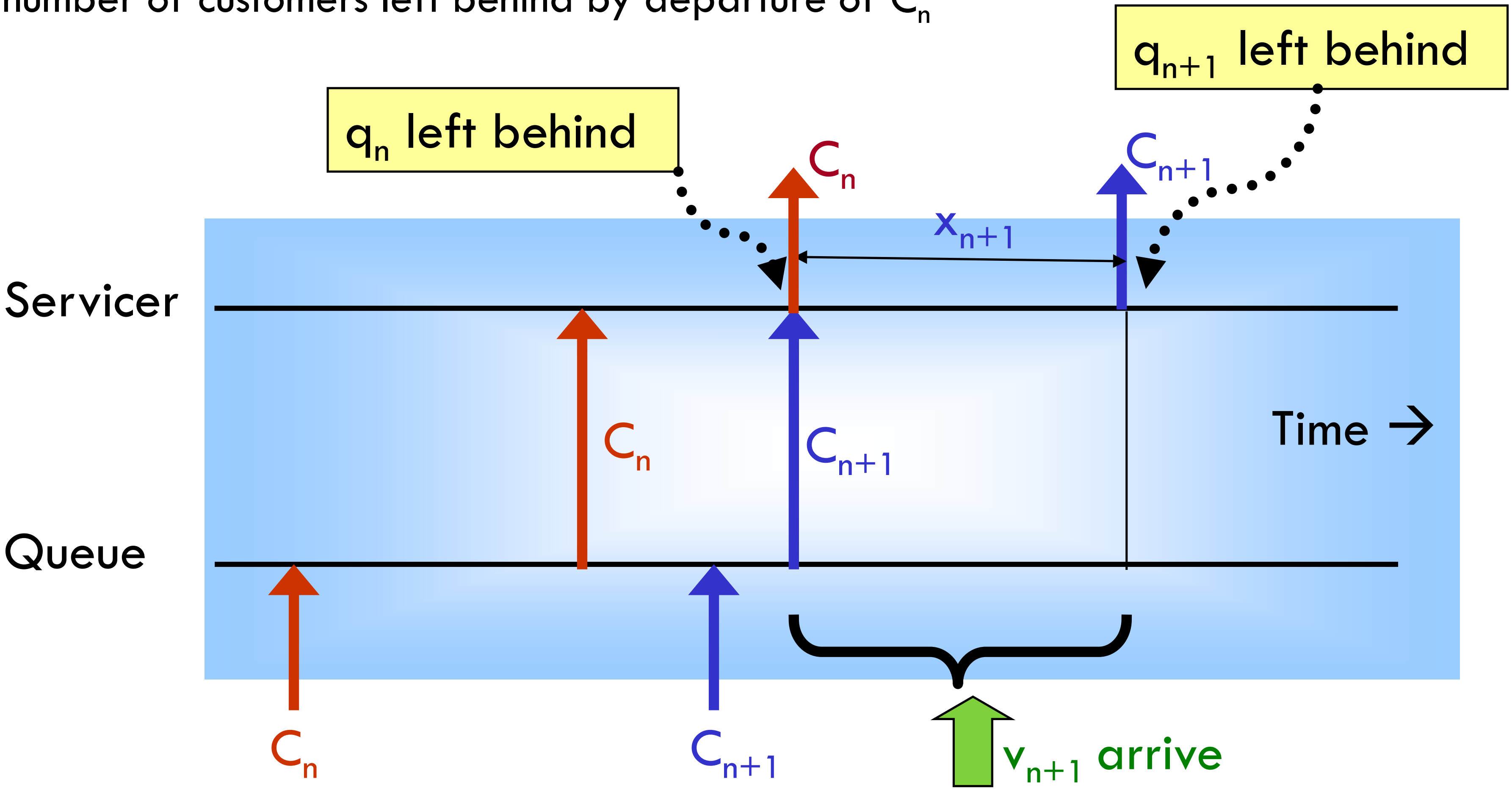
$$\alpha_k = P[v_{n+1} = k]$$

v_n = number of customers arriving during the service of C_n

q_n = number of customers left behind by departure of C_n

The Mean Queue Length ($q_n > 0$)

q_n = number of customers left behind by departure of C_n



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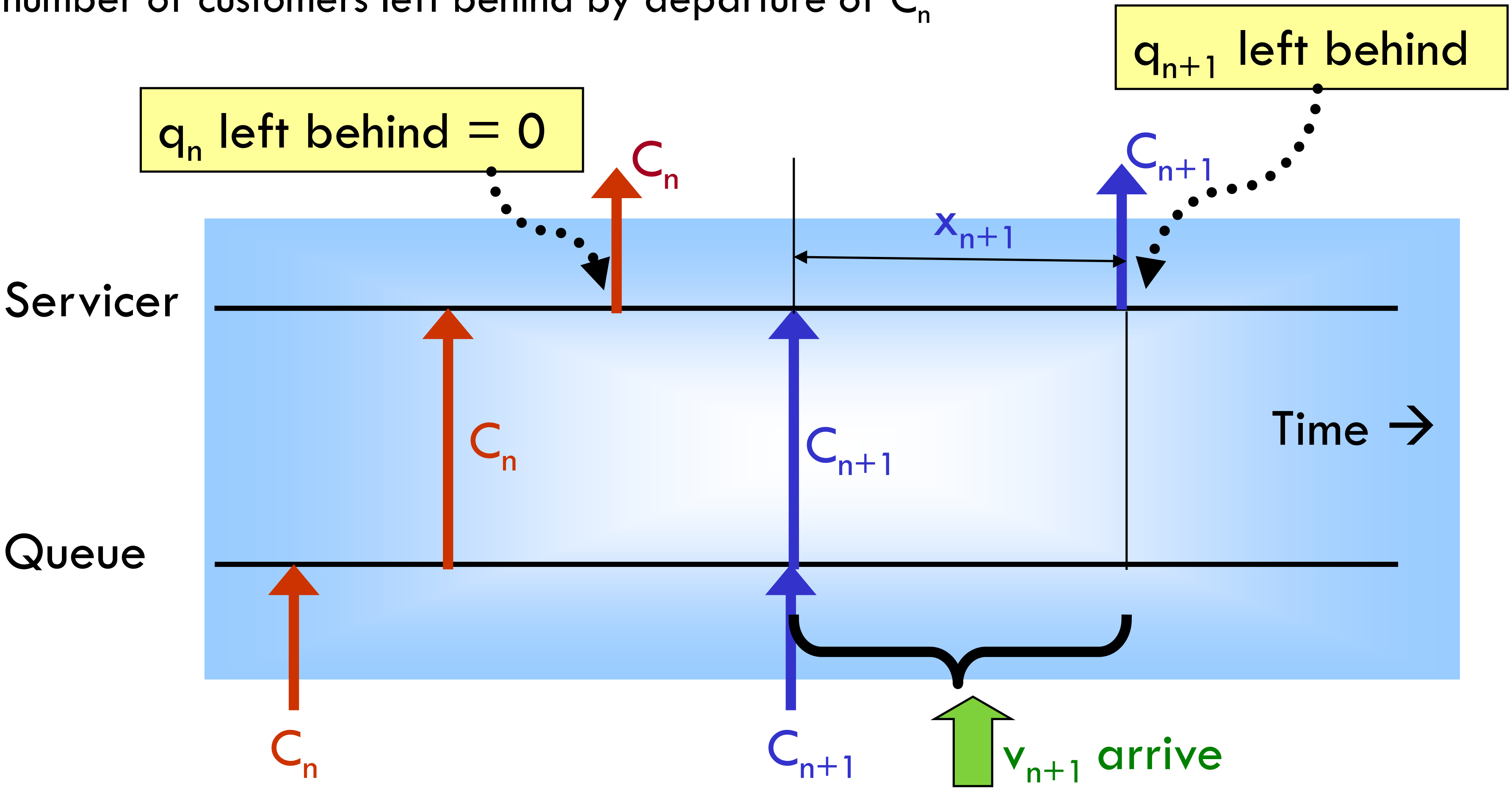
The Mean Queue Length ($q_n = 0$)

q_n = number of customers left behind by departure of C_n



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The Mean Queue Length, $E[q_{n+1}]$

From Slide: Cheng-Fu Chou, CMLab, CSIE, NTU

$$E[q_{n+1}] = E[q_n] - E[\Delta_{q_n}] + E[v_{n+1}]$$

- Take the limit as $n \to \infty$, $E[\tilde{q}] = E[\tilde{q}] - E[\Delta_{\tilde{q}}] + E[\tilde{v}]$
- We get,

$$E[\Delta_{\tilde{q}}] = E[\tilde{v}] = \text{average no. of arrivals in a service time}$$

- On the other hand,

$$\begin{aligned} E[\Delta_{\tilde{q}}] &= \sum_{k=0}^{\infty} \Delta_k P[\tilde{q} = k] \\ &= \Delta_0 P[\tilde{q} = 0] + \Delta_1 P[\tilde{q} = 1] + \cdots \\ &= P[\tilde{q} > 0] \end{aligned}$$

- Therefore $E[\Delta_{\tilde{q}}] = P[\tilde{q} > 0]$. Since we are dealing with single server, it is also equal to P[busy system]=ρ. Therefore,

$$E[\tilde{v}] = \rho$$

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The Mean Queue Length

Since we have

$$
\begin{aligned}
q_{n+1} &= q_n - \Delta_{q_n} + v_{n+1} \\
q_{n+1}^2 &= q_n^2 + \Delta_{q_n}^2 + v_{n+1}^2 - 2q_n\Delta_{q_n} + 2q_n v_{n+1} - 2\Delta_{q_n} v_{n+1}
\end{aligned}
$$

Note that : $(\Delta_{q_n})^2 = \Delta_{q_n}$ and $q_n\Delta_{q_n} = q_n$

$$
\begin{aligned}
\lim_{n\to\infty} E[q_{n+1}^2] &= \lim_{n\to\infty} \{E[q_n^2] + E[\Delta_{q_n}^2] + E[v_{n+1}^2] - \\
&\qquad 2E[q_n] + 2E[q_n v_{n+1}] - 2E[\Delta_{q_n} v_{n+1}]\} \\
0 &= E[\Delta_{\tilde{q}}] + E[\tilde{v}^2] - 2E[\tilde{q}] + 2E[\tilde{q}_n]E[\tilde{v}] - 2E[\Delta_{\tilde{q}}]E[\tilde{v}] \\
E[\tilde{q}] &= \rho + \frac{E[\tilde{v}^2] - E[\tilde{v}]}{2(1-\rho)}
\end{aligned}
$$

Now the remaining question is how to find $E[\tilde{v}^2]$

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The Mean Queue Length

- Let $V(Z) = \sum_{k=0}^{\infty} P[\tilde{v} = k]Z^k$

$$
\begin{aligned}
V(Z) &= \sum_{k=0}^{\infty} \int_0^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x} b(x) dx Z^k \\
&= \int_0^{\infty} e^{-\lambda x} \left(\sum_{k=0}^{\infty} \frac{(\lambda x Z)^k}{k!}\right) b(x) dx \\
&= \int_0^{\infty} e^{-\lambda x} e^{\lambda x Z} b(x) dx \\
&= \int_0^{\infty} e^{-(\lambda - \lambda Z)x} b(x) dx
\end{aligned}
$$

- Look at $B^*(s) = \int_0^{\infty} e^{-sx} b(x) dx$. Therefore,

$$V(Z) = B^*(\lambda - \lambda Z)$$

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The Mean Queue Length

- From this, we can get $E[\tilde{v}], E[\tilde{v}^2], \ldots$

$$\frac{dV(Z)}{dZ} = \frac{dB^*(\lambda - \lambda Z)}{dZ} = \frac{dB^*(\lambda - \lambda Z)}{d(\lambda - \lambda Z)} \bullet \frac{d(\lambda - \lambda Z)}{dZ}$$

$$= -\lambda \frac{dB^*(y)}{dy}$$

$$\frac{dV(Z)}{dZ}\bigg|_{Z=1} = -\lambda \frac{dB^*(y)}{dy}\bigg|_{y=0} = +\lambda \bar{x} = \rho$$

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The Mean Queue Length

$\frac{d^2V(Z)}{dZ^2} = \bar{v^2} - \bar{v}$, since $V(Z) = B^*(\lambda - \lambda Z)$

$$\frac{d^2V(Z)}{dZ^2} = \frac{d}{dZ}[-\lambda\frac{dB^*(y)}{dy}] = -\lambda\frac{d^2B^*(y)}{dy^2}\frac{dy}{dZ}$$

$$\frac{d^2V(Z)}{dZ^2}|_{Z=1} = \lambda^2\frac{dB^{2*}(y)}{dy^2}|_{y=0} = \lambda^2B^{*(2)}(0)$$

$$\bar{v^2} - \bar{v} = \lambda^2\bar{x^2} \quad \Rightarrow \quad \bar{v^2} = \bar{v} + \lambda^2\bar{x^2}$$

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The Mean Queue Length

Go back, since

$$E[\tilde{q}] = \rho + \frac{E[\tilde{v}^2] - E[\tilde{v}]}{2(1-\rho)}$$

$$E[\tilde{q}] = \rho + \frac{\lambda^2 \bar{x^2}}{2(1-\rho)}$$

$$= \rho + \rho^2 \frac{(1 + C_b^2)}{2(1-\rho)}$$

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The Mean Queue Length

$$q_{n+1} = \begin{cases} q_n - 1 + v_{n+1} & \text{if } q_n > 0 \\ v_{n+1} & \text{if } q_n = 0 \end{cases}$$

- For n → ∞

$$N = \rho + \frac{\lambda^2 \overline{x^2}}{2(1-\rho)}$$

Pollaczek-Khinchin (P-K) formula

Pollaczek-Khinchin (P-K) formula

- The formula was developed by ***Felix Pollaczek*** (Austrian-French) and ***Aleksandr Khinchin*** (Russian)
- A single server situation
  - Poisson arrival distribution
  - General service time distribution
- To determine
  - Mean waiting time in queue (queuing delay)
  - Mean end-to-end time through the system

Felix Pollaczek

Aleksandr Khinchin

http://en.wikipedia.org

Mean Residual Service time

- Let W_i = Waiting time in queue of i^{th} customer
- R_i = Residual service time
  - j being served when i arrives, R_i = remaining time until customer j's service time is completed
  - If no customer is in service, $R_i = 0$
- X_i = Service time of i^{th} customer
- N_i = # customers found waiting in queue

Mean Residual Service time

$$W_i = R_i + \sum_{j = i-N_i}^{i-1} X_j$$

$$E[W_i] = E[R_i] + E[\sum_{j = i-N_i}^{i-1} E[X_j | N_i]]$$

$$= E[R_i] + \overline{X} \; E[N_i]$$

$$W = R + \frac{1}{\mu} N_Q$$

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Mean Residual Service time

$$W = R + \frac{1}{\mu} N_Q$$

- By Little's Theorem

$$N_Q = \lambda W$$

$$\Rightarrow W = R + \rho W$$

$$\Rightarrow W = \frac{R}{1 - \rho}$$

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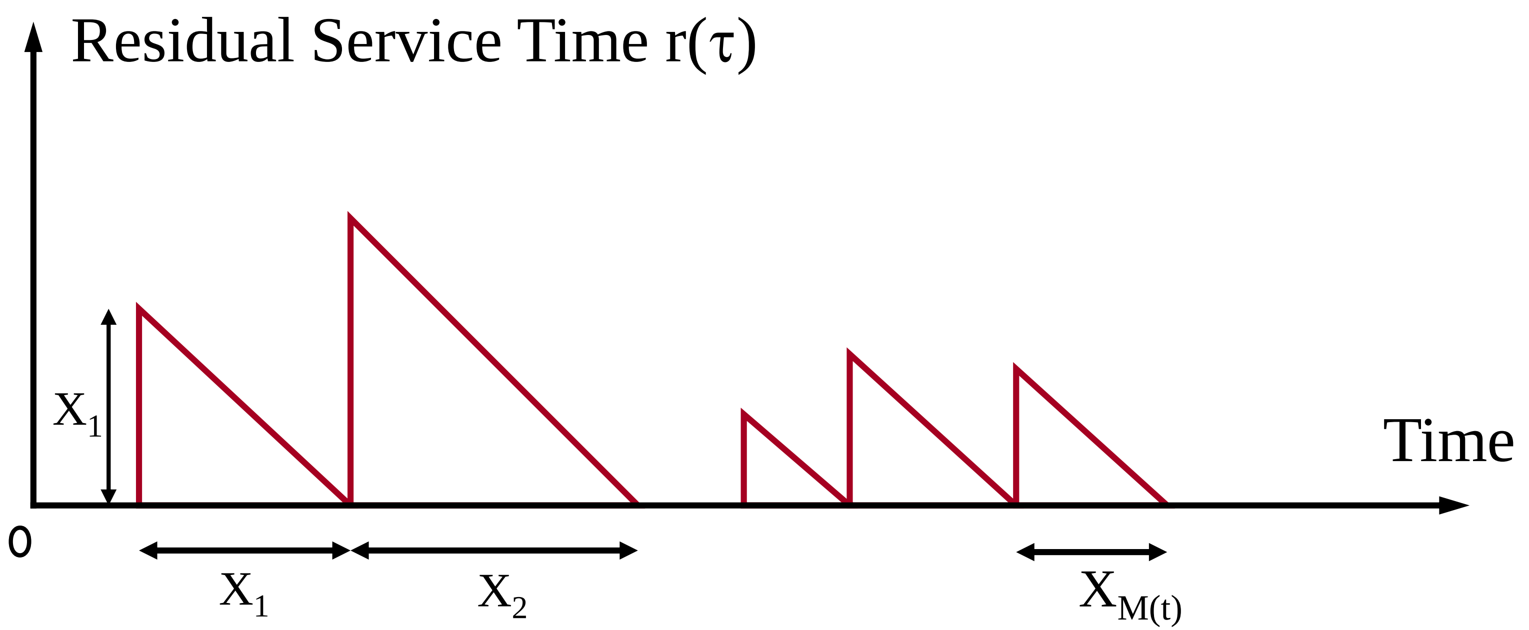
Mean Residual Service time

- Calculate R



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Mean Residual Service time

$$R = \frac{1}{t}\int_0^t r(\tau)\,d\tau = \frac{1}{t}\sum_{i=1}^{M(t)} \frac{1}{2} X_i^2$$

$$= \frac{1}{2}\,\frac{M(t)}{t}\,\frac{\sum_{i=1}^{M(t)} X_i^2}{M(t)}$$

$$R = \frac{1}{2}\lambda\overline{X^2} \qquad t \rightarrow \infty$$

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Mean Residual Service time

$$W = \frac{R}{1-\rho}$$

AND

$$R = \frac{1}{2}\lambda\overline{X^2}$$

$$W = \frac{\lambda\overline{X^2}}{2(1-\rho)}$$

$$T = \overline{X} + W \qquad = \overline{X} + \frac{\lambda\overline{X^2}}{2(1-\rho)}$$

$$N = \lambda T \quad\Rightarrow\quad N = \lambda\overline{X} + \frac{\lambda^2\overline{X^2}}{2(1-\rho)}$$

$$\Rightarrow\quad N = \rho + \frac{\lambda^2\overline{X^2}}{2(1-\rho)}$$

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P-K Formula: for M/M/1

- Service = exponential distribution

$$E[X] = \frac{1}{\mu}, \quad E[X^2] = \frac{2}{\mu^2}$$

$$E[W] = \frac{\lambda E[X^2]}{2(1-\rho)} = \frac{\rho}{\mu(1-\rho)}, \quad E[Q] = \frac{\lambda^2 E[X^2]}{2(1-\rho)} = \frac{\rho^2}{(1-\rho)}$$

$$E[T] = \frac{1}{\mu} + \frac{\lambda E[X^2]}{2(1-\rho)} = \frac{1}{\mu} + \frac{\rho}{\mu(1-\rho)} = \frac{1}{\mu-\lambda}, \quad E[N] = \lambda E[T] = \frac{\lambda}{\mu-\lambda}$$

P-K Formula: for M/M/1

$$N = \rho + \frac{\lambda^2 \overline{X^2}}{2(1-\rho)} \Rightarrow N = \frac{\rho}{(1-\rho)}$$

P-K Formula: for M/D/1

- Service = Deterministic service times all equal to 1/μ

$$E[X] = \frac{1}{\mu}, \quad E[X^2] = \frac{1}{\mu^2}$$

$$E[W] = \frac{\lambda E[X^2]}{2(1-\rho)} = \frac{\rho}{2\mu(1-\rho)}, \quad E[Q] = \frac{\lambda^2 E[X^2]}{2(1-\rho)} = \frac{\rho^2}{2(1-\rho)}$$

$$E[T] = \frac{1}{\mu} + \frac{\lambda E[X^2]}{2(1-\rho)} = \frac{1}{\mu} + \frac{\rho}{2\mu(1-\rho)} = \frac{2-\rho}{2\mu(1-\rho)}, \quad E[N] = \lambda E[T] = \frac{\rho(2-\rho)}{2(1-\rho)}$$