

01204312

Probability Theory and Random Processes

Department of Computer Engineering, Faculty of Engineering,
Kasetsart University, THAILAND

1st Semester 2015
(Aug – Dec)

Lecture #6

Multiple Discrete Random Variable

Assoc. Prof. Anan Phonphoem, Ph.D.

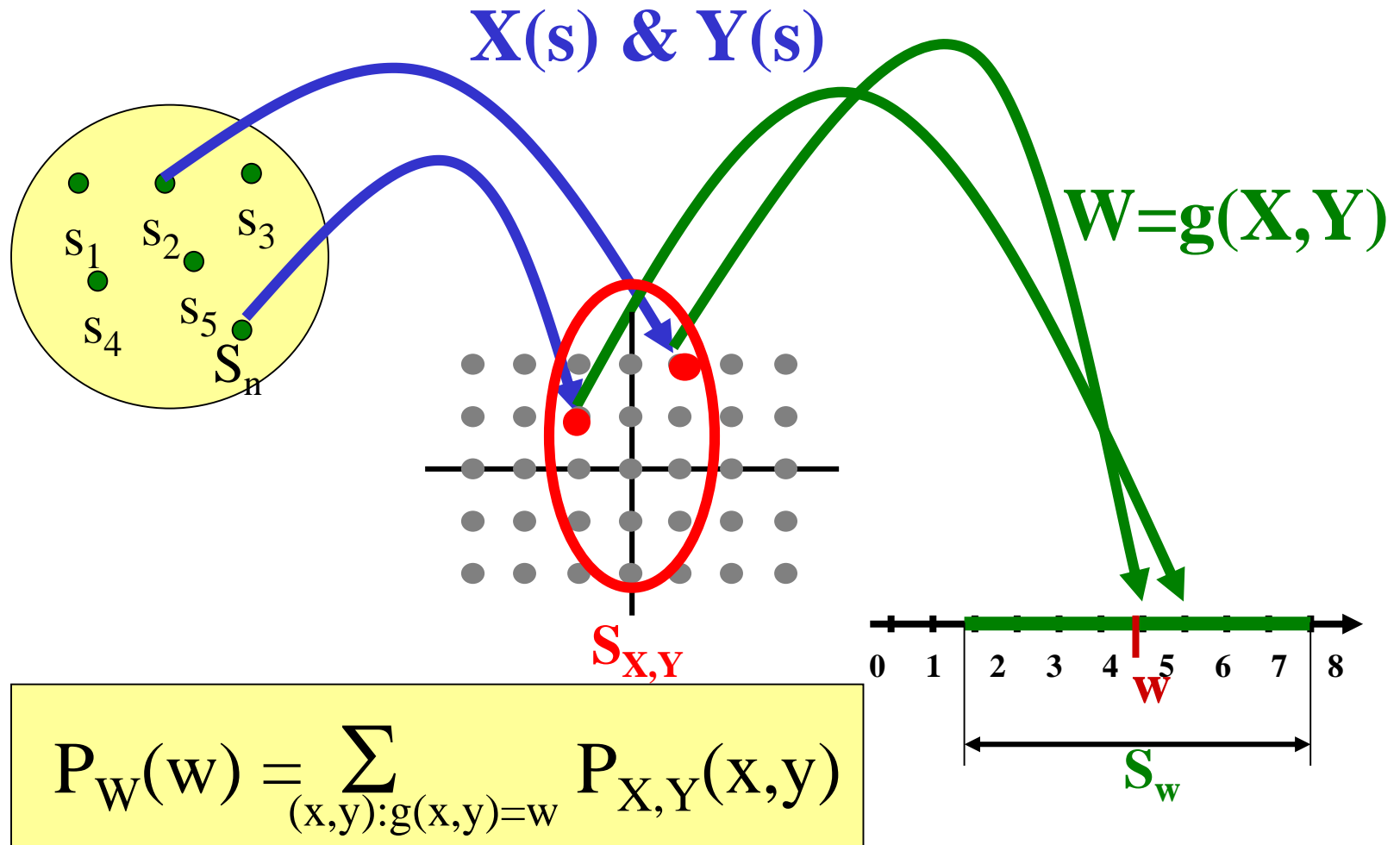
<http://www.cpe.ku.ac.th/~anan>

Department of Computer Engineering, Faculty of Engineering,
Kasetsart University, THAILAND

Outline

- Derived Random Variable
(Functions of 2 RVs)
- Covariance
- Correlation
- Conditional Joint PMF by an Event

Derived Random Variable Functions of 2 RVs



Example

- E-Document System
- Scanning (text: 40 sec & graphics: 60 sec)

$P_{S,T}(s,t)$	$t = 40$	$t = 60$
$s = 1$ sheet	0.15	0.1
$s = 2$ sheets	0.3	0.2
$s = 3$ sheets	0.15	0.1

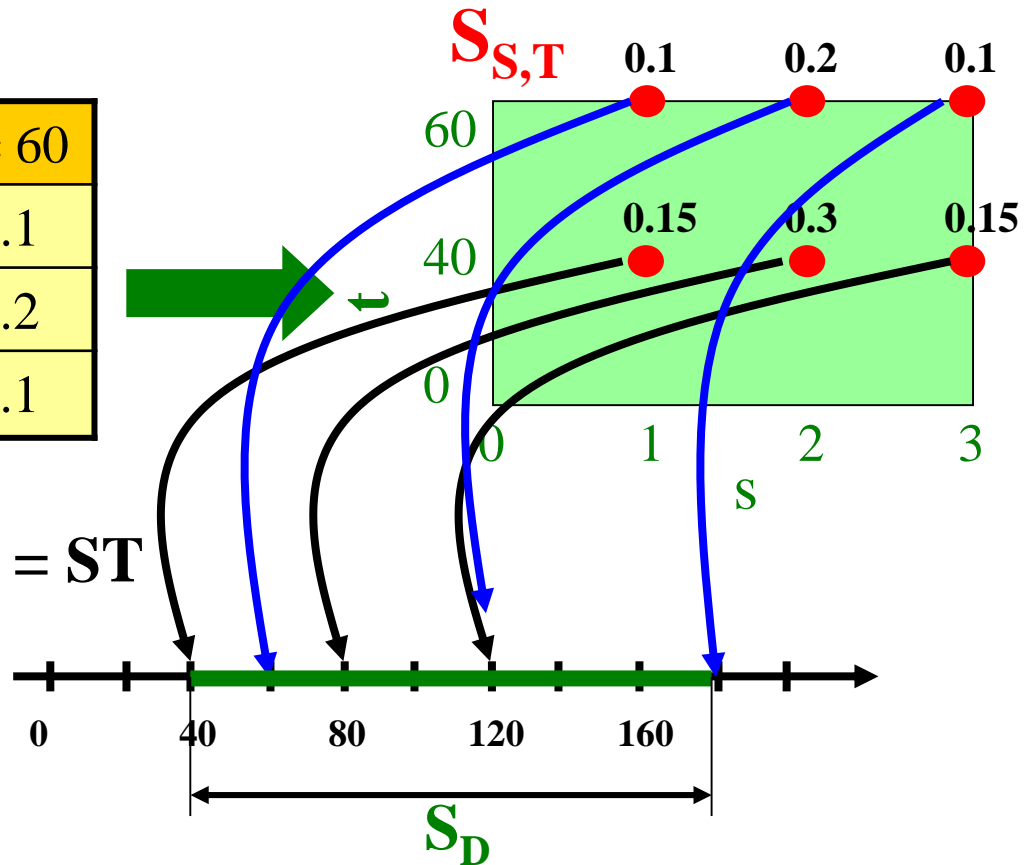
Let D = duration for sending one Job
= $g(S,T)$
= ST

Find $P_D(d)$, S_D , and $E[D]$

Example

$P_{S,T}(s,t)$	$t = 40$	$t = 60$
$s = 1$ sheet	0.15	0.1
$s = 2$ sheets	0.3	0.2
$s = 3$ sheets	0.15	0.1

$$D = g(S,T) = ST$$



$$S_D = \{40, 60, 80, 120, 180\}$$

Example

$$S_D = \{40, 60, 80, 120, 180\}$$

$$P_D(d) = \sum_{(s,t):g(s,t)=d} P_{S,T}(s,t)$$

$$P_D(d) = \begin{cases} 0.15 & d = 40 \\ 0.1 & d = 60 \\ 0.3 & d = 80 \\ 0.15 + 0.2 & d = 120 \\ 0.1 & d = 180 \\ 0 & \text{Otherwise} \end{cases}$$

$$\begin{aligned} E[D] &= \sum_{d \in S_D} dP_D(d) \\ &= 96 \text{ sec} \end{aligned}$$

Expectations

- $E[W]$ for $W = g(X, Y)$
- $E[X+Y]$
- $\text{Var}[X+Y]$ (Variance)
- $\text{Cov}[X, Y]$ (Covariance)
- $r_{X, Y}$ (Correlation)
- $\rho_{X, Y}$ (Correlation Coefficient)

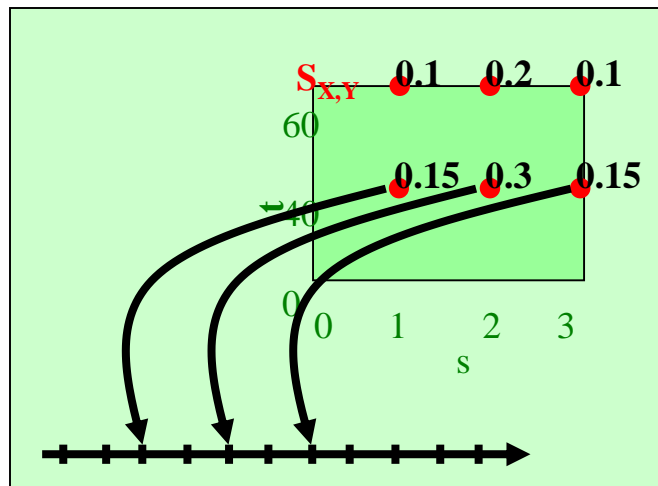
Expected Value of $g(X, Y)$

Theorem: for $W = g(X, Y)$

$$E[W] = \sum_{x \in S_X} \sum_{y \in S_Y} g(X, Y) P_{X, Y}(x, y)$$

From Last Example

To find the $E[D]$, $D = g(S,T) = ST$



Map $g(S,T) \rightarrow D$

$P_D(d) =$	0.15	$d = 40$
	0.1	$d = 60$
	0.3	$d = 80$
	$0.15 + 0.2$	$d = 120$
	0.1	$d = 180$
	0	Otherwise

Find $P_D(d)$

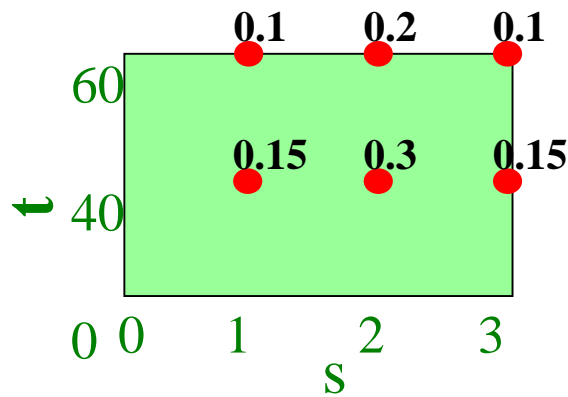
$$E[D] = \sum_{d \in S_D} dP_D(d)$$

Find $E[D]$

From Last Example

With the theorem, we can directly find $E[D]$

$$E[D] = \sum_{s=1}^3 \sum_{t=40,60} st P_{S,T}(s,t)$$



$$\begin{aligned} E[D] &= 1*40*0.15 + 1*60*0.1 + \\ &\quad 2*40*0.3 + 2*60*0.2 + \\ &\quad 3*40*0.15 + 3*60*0.1 \\ &= 96 \text{ sec} \end{aligned}$$

For any 2 RVs

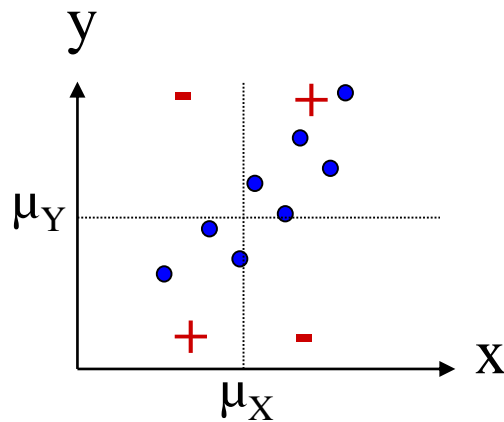
Theorem:

$$E[X + Y] = E[X] + E[Y]$$

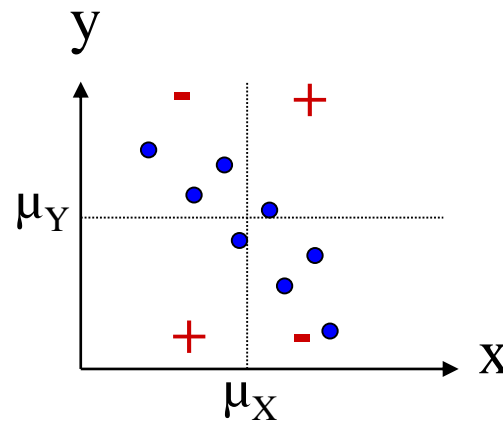
- Find $E[X]$ and $E[Y]$
 \Rightarrow Marginal PMF

Covariance of X and Y

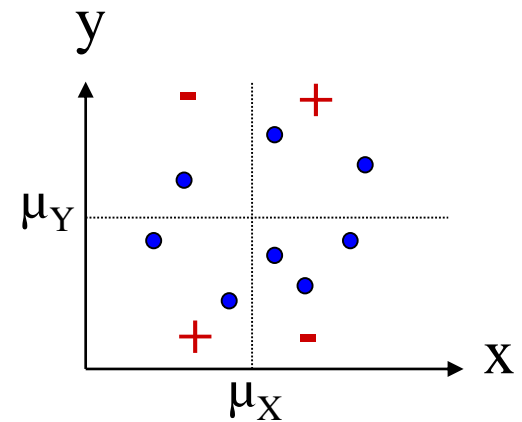
- If X and Y are not independent,
- Interesting question:
 - How X and Y are related ?



Strong Positive Relationship



Strong Negative Relationship



Not Strongly Related

Var[X+Y]

Definition: $\text{Var}[X] = E[(X - \mu_X)^2]$

$$\begin{aligned}\text{Var}[X+Y] &= E[((X+Y) - \mu_{X+Y})^2] \\ &= E[((X+Y) - (\mu_X + \mu_Y))^2] \\ &= E[((X - \mu_X) + (Y - \mu_Y))^2] \\ &= E[(X - \mu_X)^2 + 2(X - \mu_X)(Y - \mu_Y) + (Y - \mu_Y)^2] \\ &= E[(X - \mu_X)^2] + 2E[(X - \mu_X)(Y - \mu_Y)] + E[(Y - \mu_Y)^2]\end{aligned}$$

Theorem:

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2E[(X - \mu_X)(Y - \mu_Y)]$$

Covariance

Covariance of X and Y

Definition: $\text{Cov}[X, Y] = E[(X - \mu_X)(Y - \mu_Y)]$

$$\begin{aligned}\text{Cov}[X, Y] &= E[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y] \\ &= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X \mu_Y \\ &= E[XY] - \mu_X \mu_Y - \mu_X \mu_Y + \mu_X \mu_Y\end{aligned}$$

Theorem: $\text{Cov}[X, Y] = E[XY] - \mu_X \mu_Y$

Example (1)

$P_{S,T}(s,t)$	$t = 40$	$t = 60$	$P_S(s)$
$s = 1$ sheet	0.15	0.1	0.25
$s = 2$ sheets	0.3	0.2	0.5
$s = 3$ sheets	0.15	0.1	0.25
$P_T(t)$	0.6	0.4	1

$$\begin{aligned}\mu_S &= \sum s P_S(s) \\ &= (0.25*1)+(0.5*2)+(0.25*3) \\ &= 2\end{aligned}$$

$$\begin{aligned}\mu_T &= \sum t P_T(t) \\ &= (0.6*40)+(0.4*60) \\ &= 48\end{aligned}$$

Example (1)

$P_{S,T}(s,t)$	$t = 40$	$t = 60$	$P_S(s)$
$s = 1$ sheet	0.15	0.1	0.25
$s = 2$ sheets	0.3	0.2	0.5
$s = 3$ sheets	0.15	0.1	0.25
$P_T(t)$	0.6	0.4	1

$$\mu_S = 2, \mu_T = 48$$

Definition: $\text{Cov}[X, Y] = E[(X - \mu_X)(Y - \mu_Y)]$

$$\text{Cov}(S, T) = \sum_{(s,t)} (s - 2)(t - 48) P_{S,T}(s, t)$$

$$\begin{aligned}
 &= (1 - 2)(40 - 48) * 0.15 & &= +1.2 & &= \mathbf{0} \\
 &+ (1 - 2)(60 - 48) * 0.1 & &-1.2 & & \\
 &+ (2 - 2)(40 - 48) * 0.3 & &+ 0 & & \\
 &+ (2 - 2)(60 - 48) * 0.2 & &+ 0 & & \\
 &+ (3 - 2)(40 - 48) * 0.15 & &-1.2 & & \\
 &+ (3 - 2)(60 - 48) * 0.1 & &+1.2 & &
 \end{aligned}$$

Example (1)

$P_{S,T}(s,t)$	$t = 40$	$t = 60$	$P_S(s)$
$s = 1$ sheet	0.15	0.1	0.25
$s = 2$ sheets	0.3	0.2	0.5
$s = 3$ sheets	0.15	0.1	0.25
$P_T(t)$	0.6	0.4	1

$$\mu_S = 2, \mu_T = 48$$

Theorem: $\text{Cov}[X, Y] = E[XY] - \mu_X \mu_Y$

$$\text{Cov}(S, T) = E[ST] - \mu_S \mu_T$$

$$\begin{aligned}
 &= ((1*40)*0.15 + (1*60)*0.1 + (2*40)*0.3 + (2*60)*0.2 + (3*40)*0.15 + (3*60)*0.1) - (2*48) \\
 &= (+6 + 6 + 24 + 24 + 18 + 18) - 96 \\
 &= (96) - 96 \\
 &= 0
 \end{aligned}$$

← Same as previous method

**S and T are
Not Strongly Related**

Example (2)

- Insurance Company
 - Automobile policy (\$100, \$250)
 - Homeowner's policy (\$0, \$100, \$200)
- Customer purchase both gets deductible (in dollars)
 - X = deductible amount on Auto Policy
 - Y = deductible amount on Homeowner's Policy
- $P_{X,Y}(x,y)$
- $\text{Cov}(X, Y)$

Example (2)

$P_{X,Y}(x,y)$	$y=0$	$y=100$	$y=200$
$x=100$	0.20	0.10	0.20
$x=250$	0.05	0.15	0.30

X = deduct. Auto Policy
Y = deduct. Home Policy

$$\begin{aligned}\mu_X &= \sum x P_X(x) \\ &= (0.5*100)+(0.5*250) \\ &= 175\end{aligned}$$

$$\begin{aligned}\mu_Y &= \sum y P_Y(y) \\ &= (0.25*0)+(0.25*100)+(0.5*200) \\ &= 125\end{aligned}$$

Example (2)

$P_{X,Y}(x,y)$	$y=0$	$y=100$	$y=200$	$P_X(x)$
$x=100$	0.20	0.10	0.20	0.50
$x=250$	0.05	0.15	0.30	0.50
$P_Y(y)$	0.25	0.25	0.5	1

Definition: $\text{Cov}[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$

$$\begin{aligned}\text{Cov}(X,Y) &= \sum_{(x,y)} (x - 175)(y - 125) P_{X,Y}(x,y) \\ &= (100 - 175)(0 - 125) * 0.2 \\ &\quad + (100 - 175)(100 - 125) * 0.1 \\ &\quad + (100 - 175)(200 - 125) * 0.2 \\ &\quad + (250 - 175)(0 - 125) * 0.05 \\ &\quad + (250 - 175)(100 - 125) * 0.15 \\ &\quad + (250 - 175)(200 - 125) * 0.3 \\ &= 1875\end{aligned}$$

**X and Y are
Strong Positive
Relationship**

Covariance  **Linear Dependence** between two RVs

Example (2)

- If the deductible **in cents** (not **in dollars**)
 - $X' \Rightarrow 100X$, $Y' \Rightarrow 100Y$
 - $\text{Cov}(X', Y') \Rightarrow \text{Cov}(100X, 100Y) \Rightarrow 100 * 100 * \text{Cov}(X, Y)$
- From 1,875 (in \$) \Rightarrow 18,750,000 (in cents)
- From 1,875 (in \$) \Rightarrow 0.1875 (in **hundreds of \$**)

Defect of Covariance:

Covariance critically **depends on Unit** measurement !!!

\Rightarrow We need **Dimensionless** measurement

\Rightarrow **Correlation Coefficient**

Covariance of X and Y

Theorem: $\text{Cov}[X, Y] = E[XY] - \mu_x \mu_y$

Correlation

$$\begin{aligned} \text{If } X = Y \Rightarrow \text{Cov}[X, X] &= E[XX] - \mu_x \mu_x \\ &= E[X^2] - \mu_x^2 \\ &= E[X^2 - 2\mu_x^2 + \mu_x^2] \\ &= E[X^2 - 2\mu_x X + \mu_x^2] \\ &= E[(X - \mu_x)^2] \\ &= \text{Var}[X] \end{aligned}$$

$$\text{If } \mu_x \text{ or } \mu_y = 0 \Rightarrow \text{Cov}[X, Y] = E[XY]$$

Correlation

Definition: The correlation of X and Y is $r_{X,Y}$

$$r_{X,Y} = E[XY]$$

Theorem: $\text{Cov}[X, Y] = r_{X,Y} - \mu_x \mu_y$

More Definition

Definition 1:

X and Y are **Orthogonal** if $r_{X,Y} = 0$; $E[XY]=0$

Definition 2:

X and Y are **Uncorrelated** if $Cov[X,Y] = 0$

Definition 3:

Correlation Coefficient of X and Y is

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}} = \frac{Cov[X,Y]}{\sigma_X\sigma_Y} = [-1, 1]$$

Correlation Coefficient

- $\rho_{X,Y}$
 - Describes the info about X by observing Y
- $\rho_{X,Y} > 0$
 - If X \uparrow (relative to mean) \Rightarrow Y \uparrow
 - If X \downarrow (relative to mean) \Rightarrow Y \downarrow
- $\rho_{X,Y} < 0$
 - If X \uparrow (relative to mean) \Rightarrow Y \downarrow
 - If X \downarrow (relative to mean) \Rightarrow Y \uparrow
- Example:
 - X = student's height, Y = student's weight $\rho_{X,Y} > 0$
 - X = cell phone distance, Y = received signal Strength $\rho_{X,Y} < 0$

Example (2) Revisit

$P_{X,Y}(x,y)$	$y=0$	$y=100$	$y=200$	$P_X(x)$
$x=100$	0.20	0.10	0.20	0.50
$x=250$	0.05	0.15	0.30	0.50
$P_Y(y)$	0.25	0.25	0.5	1

$$\mu_X = 175, \mu_Y = 125$$

$$\begin{aligned} E[X^2] &= \sum x^2 P_X(x) \\ &= (100^2 * 0.5) + (250^2 * 0.5) \\ &= 5,000 + 31,250 \\ &= 36,250 \end{aligned}$$

$$\begin{aligned} \sigma^2_X &= E[X^2] - \mu^2_X \\ &= 36,250 - (175)^2 \\ \sigma_X &= 75 \end{aligned}$$

$$\begin{aligned} E[Y^2] &= \sum y^2 P_Y(y) \\ &= (0^2 * 0.25) + (100^2 * 0.25) + (200^2 * 0.5) \\ &= 22,500 \end{aligned}$$

$$\begin{aligned} \sigma^2_Y &= E[Y^2] - \mu^2_Y \\ &= 22,500 - (125)^2 \\ \sigma_Y &= 82.92 \end{aligned}$$

Example (2) Revisit

$$\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sigma_X \sigma_Y} = \frac{1,875}{75 * 82.92}$$
$$= 0.301$$

From $\rho_{X,Y} > 0$

If $X \uparrow$ (relative to mean) $\Rightarrow Y \uparrow$

If $X \downarrow$ (relative to mean) $\Rightarrow Y \downarrow$

From $\text{Cov}(X,Y)$

X and Y are

Strong Positive Relationship

Uncorrelated

If X and Y are **Independent**, then

$$\Rightarrow \text{Cov}[X, Y] = 0 \Rightarrow \rho_{X, Y} = 0$$

$\Rightarrow X$ and Y are **Uncorrelated**

Note:

If X and Y are **Uncorrelated**,

$\Rightarrow X$ and Y **may or may not Independent**

Conditional Joint PMF by an Event

Conditional Joint PMF by an Event

$$P_{X,Y|B}(x,y) = \frac{P[(X=x, Y=y) \cap B]}{P[B]}$$

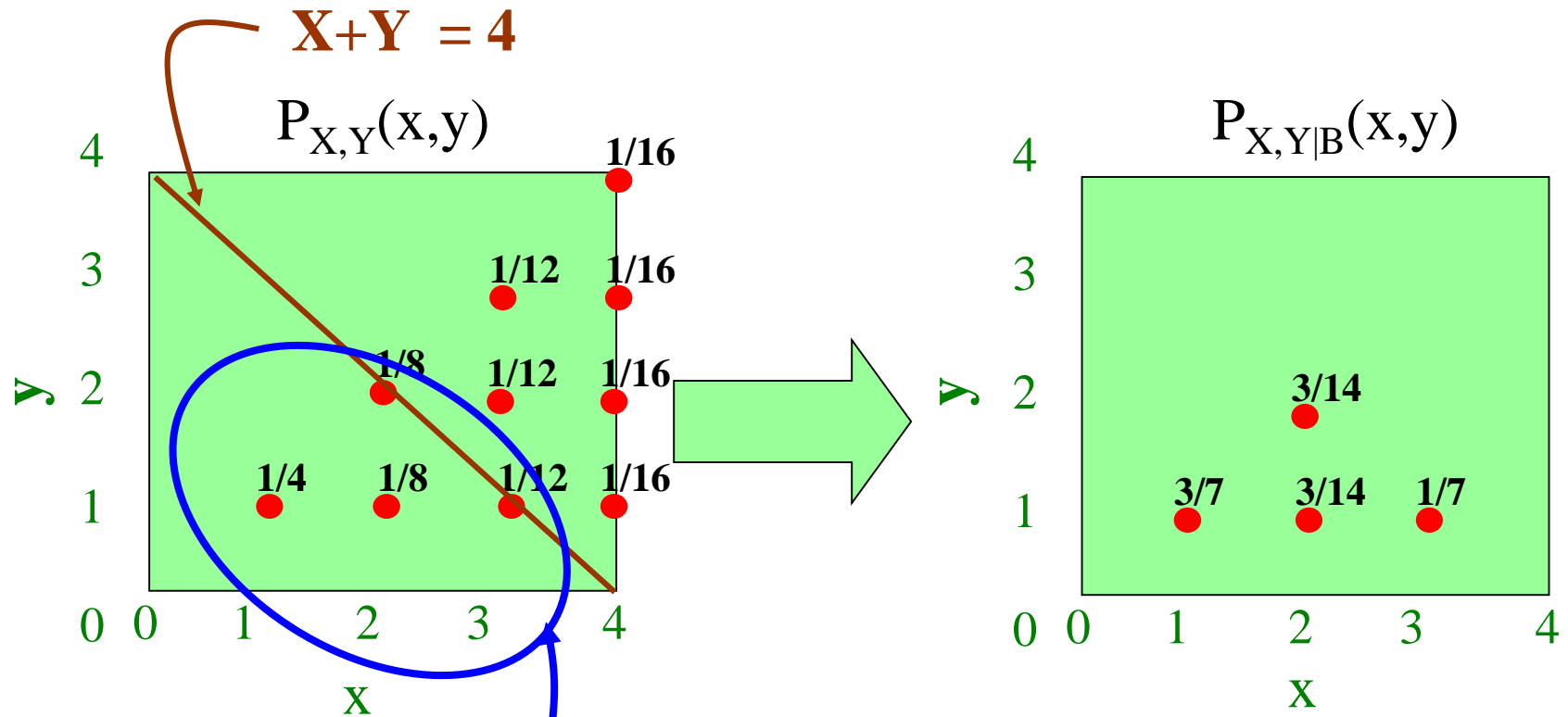
If $(X=x, Y=y) \in B \Rightarrow (X=x, Y=y) \cap B = (X=x, Y=y)$

$$P_{X,Y|B}(x,y) = \begin{cases} \frac{P[(X=x, Y=y)]}{P[B]} & (x,y) \in B \\ 0 & \text{Otherwise} \end{cases}$$

Example $P_{X,Y|B}(x,y)$

Let $B = \{X+Y \leq 4\}$

Find $P_{X,Y|B}(x,y)$



$$B = \{X+Y \leq 4\} \Rightarrow P[B] = 7/12$$

Conditional PMF

- Special case of Conditional Joint PMF by an Event
⇒ the Event is $X=x$ or $Y=y$
- $P_{X,Y|B}(x,y)$ when $B = \{Y=y\}$
⇒ $P_{X,Y|Y=y}(x,y) = P_{X|Y}(x|y)$

Definition: $P_{X|Y}(x|y) = P[X=x | Y=y]$

Conditional PMF

$$\begin{aligned} P_{X|Y}(x|y) &= P[X=x \mid Y=y] \\ &= \frac{P[X=x, Y=y]}{P[Y=y]} \\ &= \frac{P_{X,Y}(x,y)}{P_Y(y)} \end{aligned}$$

Theorem:

$$P_{X,Y}(x,y) = P_{X|Y}(x|y)P_Y(y) = P_{Y|X}(y|x)P_X(x)$$

Independent RVs

- From the independent definition
 - A and B are independent iff $P[AB] = P[A]P[B]$
- X and Y are independent RVs iff
 - $\{X=x\}$ and $\{Y=y\}$ are independent for all x, y in $S_{X,Y}$

Definition: $P_{X,Y}(x,y) = P_X(x)P_Y(y)$

Independent RVs

Theorem:

$$(a) r_{X,Y} = E[XY] = E[X]E[Y]$$

$$(b) E[X|Y = y] = E[X] \quad \text{for all } y \in S_Y$$

$$(c) E[Y|X = x] = E[Y] \quad \text{for all } x \in S_X$$

$$(d) \text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$$

$$(e) \text{Cov}[X,Y] = \rho_{X,Y} = 0$$

More than 2 RVs

Definition: Joint PMF of discrete RV X_1, \dots, X_N is

$$P_{X_1, \dots, X_N}(x_1, \dots, x_N) = P[X_1=x_1, \dots, X_N=x_N]$$

Summary

- Joint PMF
- Marginal PMF
- Covariance
- Correlation Coefficient
- Conditional Joint PMF by an Event