

LECTURE #7

EQUILIBRIUM BEHAVIOR OF BIRTH-DEATH QUEUEING SYSTEM

204528

Queueing Theory and
Applications in Networks

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Outline

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- M/M/1
- Discouraged Arrivals
- Responsive Servers (M/M/ ∞)
- Finite Storage (M/M/1/K)

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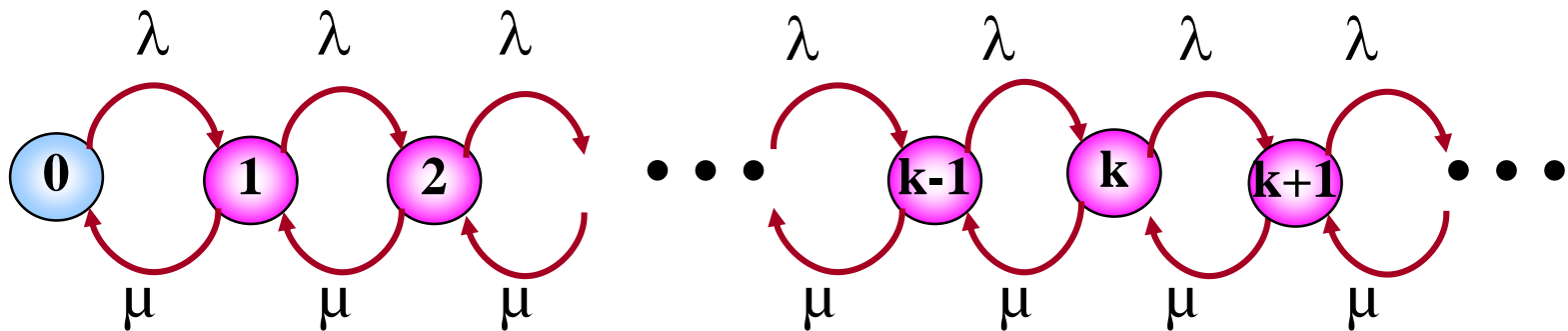
M/M/1

A Birth-Death Process : M/M/1

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- Assumption

- $\lambda_k = \lambda$ for $k \geq 0$
- $\mu_k = \mu$ for $k \geq 1$
- The system begins at time t_0 with 0 member



A Birth-Death Process : M/M/1

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- A Birth-Death Process
 - Constant coefficients λ and μ
- Interarrival Time / Service Time / #Servers
- Memoryless / Memoryless / 1 Server
- M/M/1 = A single-server queue with a Poisson arrival and an exponential distribution for service time

A Birth-Death Process : M/M/1

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- $A(t)$ = CDF of the arrival time
 - $\lambda_k = \lambda$ for $k \geq 0$
 - $A(t) = 1 - e^{-\lambda t}$
- $B(x)$ = CDF of the service time
 - $\mu_k = \mu$ for $k \geq 1$
 - $B(x) = 1 - e^{-\mu x}$

A Birth-Death Process : M/M/1

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- $dp_0(t)/dt = -\lambda p_0(t) + \mu p_1(t)$
- $dp_k(t)/dt = -(\lambda + \mu)p_k(t) + \lambda p_{k-1}(t) + \mu p_{k+1}(t)$
- Now we try to find the solution!
 - Hint: Solved by using z -transforms

A Birth-Death Process : M/M/1

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$$p_k(t) = e^{-(\lambda+\mu)t} \left(\begin{array}{l} \rho^{(k-i)/2} I_{k-i}(at) \\ + \rho^{(k-i-1)/2} I_{k+i+1}(at) \\ + (1-\rho)\rho^k \sum_{j=k+i+2}^{\infty} \rho^{-j/2} I_j(at) \end{array} \right)$$

- Where $\rho = \lambda/\mu$ and $a = 2\mu\rho^{1/2}$
- $I_k(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{k+2m}}{(k+m)!m!} \quad k \geq -1$
- Time dependent behavior of the state prob.

Equilibrium Solution

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- The time-dependent solution is unmanageable
- $p_k(t) \rightarrow$ not too useful \rightarrow transient
- Let $p_k \equiv$ limiting probability (system = k members)
$$= \lim_{t \rightarrow \infty} p_k(t) = \text{System in state } E_k$$
- p_k is not time-dependent

Equilibrium Solution

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- From

$$dp_0(t)/dt = -\lambda_0 p_0(t) + \mu_1 p_1(t)$$

$$dp_k(t)/dt = -(\lambda_k + \mu_k) p_k(t) + \lambda_{k-1} p_{k-1}(t) + \mu_{k+1} p_{k+1}(t)$$

- If set $\lim_{t \rightarrow \infty} dp_k(t)/dt = 0$

- Obtain the result

$$0 = -\lambda_0 p_0 + \mu_1 p_1 \quad k = 0$$

$$0 = -(\lambda_k + \mu_k) p_k + \lambda_{k-1} p_{k-1} + \mu_{k+1} p_{k+1} \quad k \geq 1$$

Equilibrium Solution

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- From conservation relation $\sum_{k=0}^{\infty} p_k(t) = 1$
- Find the answer for p_0 and p_i

Equilibrium Solution

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- From

$$0 = -\lambda_0 p_0 + \mu_1 p_1 \quad k = 0$$

$$0 = -(\lambda_k + \mu_k) p_k + \lambda_{k-1} p_{k-1} + \mu_{k+1} p_{k+1} \quad k \geq 1$$

- Yield

$$\lambda_0 p_0 = \mu_1 p_1$$

$$p_1 = (\lambda_0 / \mu_1) p_0$$

Equilibrium Solution

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- And for $k = 1$

$$(\lambda_k + \mu_k)p_k = \lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1}$$

$$(\lambda_1 + \mu_1)p_1 = \lambda_0 p_0 + \mu_2 p_2$$

$$(\lambda_1 + \mu_1)(\lambda_0 / \mu_1)p_0 = \lambda_0 p_0 + \mu_2 p_2$$

$$(\lambda_1 \lambda_0 / \mu_1)p_0 + \lambda_0 p_0 = \lambda_0 p_0 + \mu_2 p_2$$

$$(\lambda_1 \lambda_0 / \mu_1)p_0 = \mu_2 p_2$$

$$p_2 = \frac{\lambda_1 \lambda_0}{\mu_1 \mu_2} p_0$$

Equilibrium Solution

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$$p_i = p_0 \left(\prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}} \right) \quad \forall i \geq 1$$

$$p_0 = \left(1 + \sum_{i=0}^{\infty} \prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}} \right)^{-1}$$

M/M/1 Equilibrium Solution

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$$p_i = p_0 \left(\prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}} \right) \quad \forall i \geq 1$$

- From $\lambda_k = \lambda$ for $k \geq 0$ and $\mu_k = \mu$ for $k \geq 1$

$$p_i = p_0 \left(\prod_{k=0}^{i-1} \frac{\lambda}{\mu} \right) \quad \forall i \geq 1$$

$$p_i = p_0 \left(\frac{\lambda}{\mu} \right)^i \quad \forall i \geq 0$$

M/M/1 Equilibrium Solution

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$$p_0 = \left(1 + \sum_{i=1}^{\infty} \prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}} \right)^{-1}$$

- From $\lambda_k = \lambda$ for $k \geq 0$ and $\mu_k = \mu$ for $k \geq 1$

$$p_0 = \left(1 + \sum_{i=1}^{\infty} \left(\frac{\lambda}{\mu} \right)^i \right)^{-1}$$

Birth-Death Process Classification

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Define: $S_1 \triangleq \sum_{i=0}^{\infty} \prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}}$ From p_0

$$S_2 \triangleq \sum_{i=0}^{\infty} \left(\lambda_i \prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}} \right)^{-1}$$

Ergodic: $S_1 < \infty$ and $S_2 = \infty$

Recurrent Null: $S_1 = \infty$ and $S_2 = \infty$

Transient: $S_1 = \infty$ and $S_2 < \infty$

Ergodicity

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- $E_j = \text{Ergodic}$ if
 - $E_j = \textit{Aperiodic}$ and *Recurrent Nonnull*
- $f_j = 1$, $M_j < \infty$, and $\beta = 1$
- A Markov Chain is **ergodic**
 - If **all** states of a Markov Chain are **ergodic**
 - If number of states is **finite** and **all** states of a Markov Chain are **aperiodic**, and **irreducible**

Stability Condition

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- $p_0 > 0$
- The sufficient condition for ergodicity in M/M/1 is $\lambda < \mu$

$$p_0 = \left(1 + \sum_{i=1}^{\infty} \left(\frac{\lambda}{\mu} \right)^i \right)^{-1}$$
$$= \frac{1}{1 + \frac{\lambda / \mu}{1 - \lambda / \mu}} = 1 - \frac{\lambda}{\mu}$$

M/M/1

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- $\rho = \lambda / \mu$
- For stability conditions $0 \leq \rho < 1$

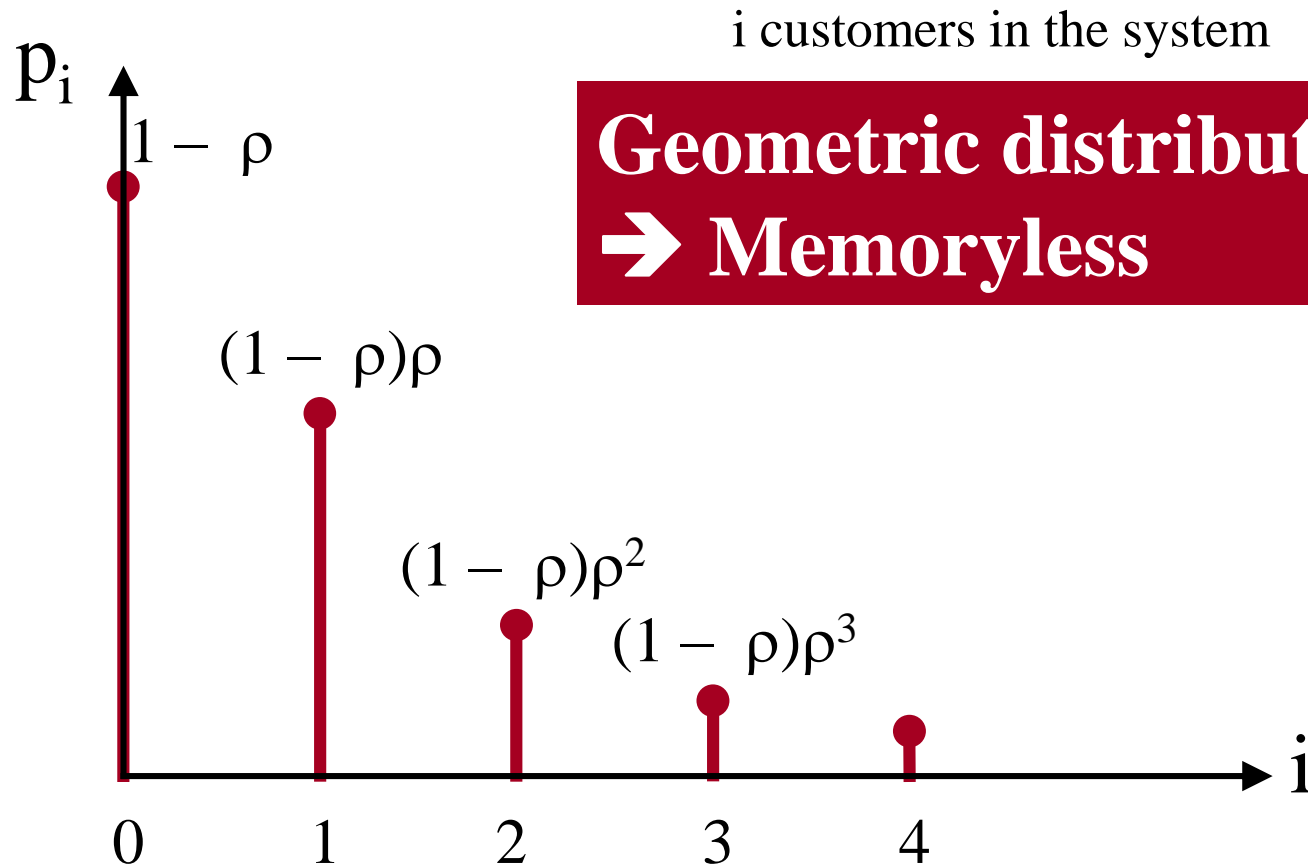
$$p_0 = 1 - \rho$$

$$p_i = p_0 \left[\frac{\lambda}{\mu} \right]^i$$

$$p_i = (1 - \rho)\rho^i$$

The Solution of p_i

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The average # of customers

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$$\begin{aligned}\bar{N} &= \sum_{i=0}^{\infty} i p_i &= (1 - \rho) \sum_{i=0}^{\infty} i \rho^i \\ & &= (1 - \rho) \rho \frac{\partial}{\partial \rho} \sum_{i=0}^{\infty} \rho^i \\ & &= (1 - \rho) \rho \frac{\partial}{\partial \rho} \frac{1}{(1 - \rho)} \\ & &= (1 - \rho) \rho \frac{1}{(1 - \rho)^2} \\ & &= \frac{\rho}{(1 - \rho)}\end{aligned}$$

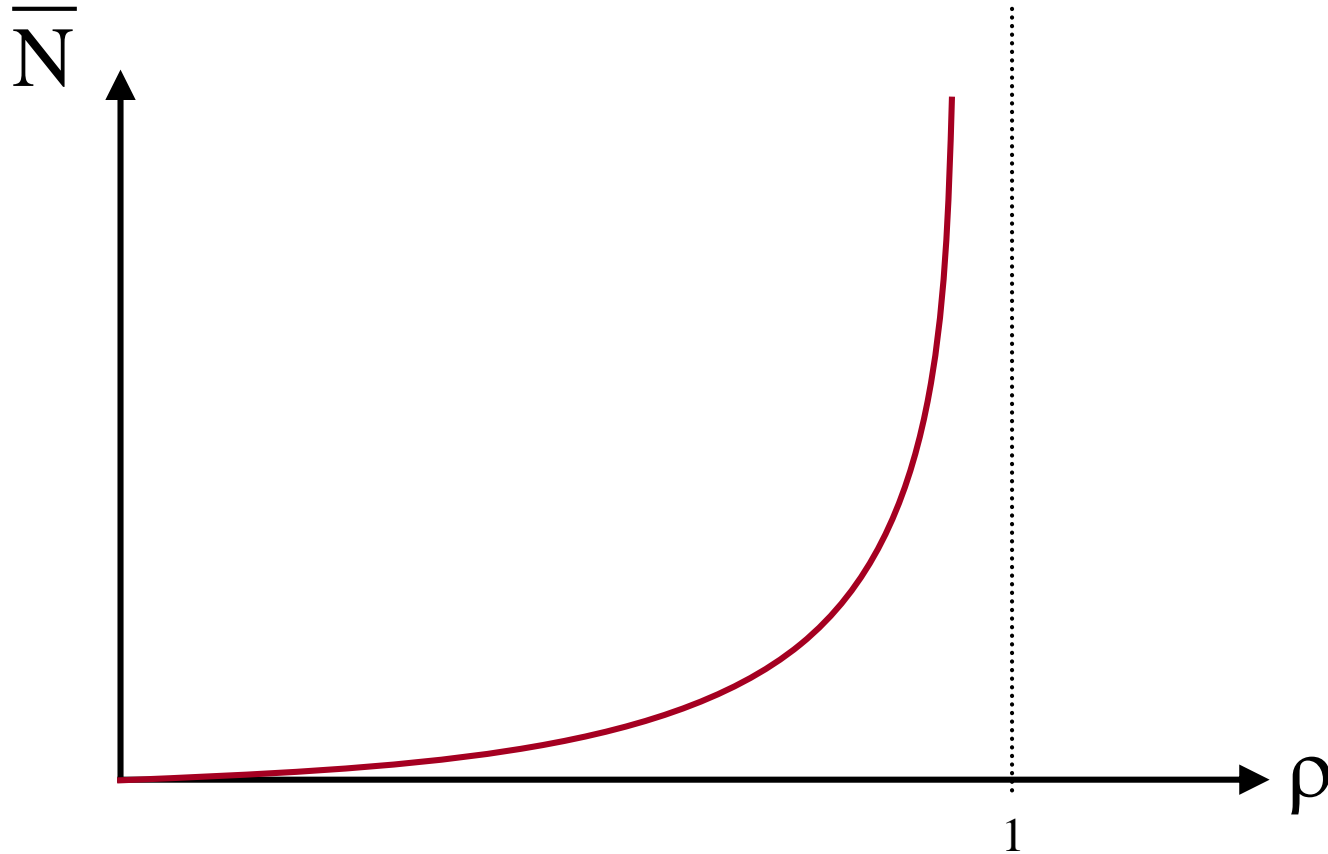
The average # of customers

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$$\begin{aligned}\bar{N} &= \frac{\rho}{(1 - \rho)} \\ &= \frac{\lambda/\mu}{(1 - \lambda/\mu)} \\ &= \frac{\lambda}{(\mu - \lambda)}\end{aligned}$$

The average # of customers

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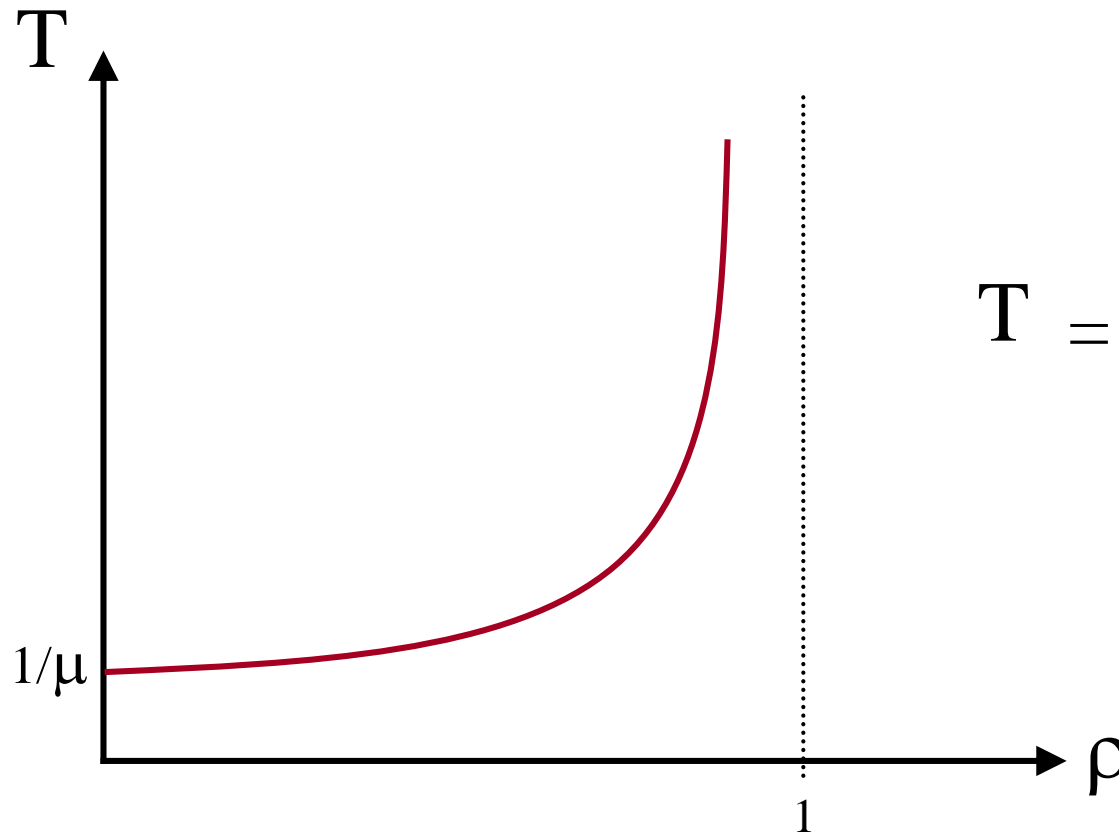
The average delay

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$$\begin{aligned} T &= \frac{\bar{N}}{\lambda} \\ &= \frac{1}{\lambda} \left(\frac{\rho}{(1-\rho)} \right) \\ &= \frac{1/\mu}{(1-\rho)} \\ &= \frac{1}{(\mu - \lambda)} \end{aligned}$$

The average delay

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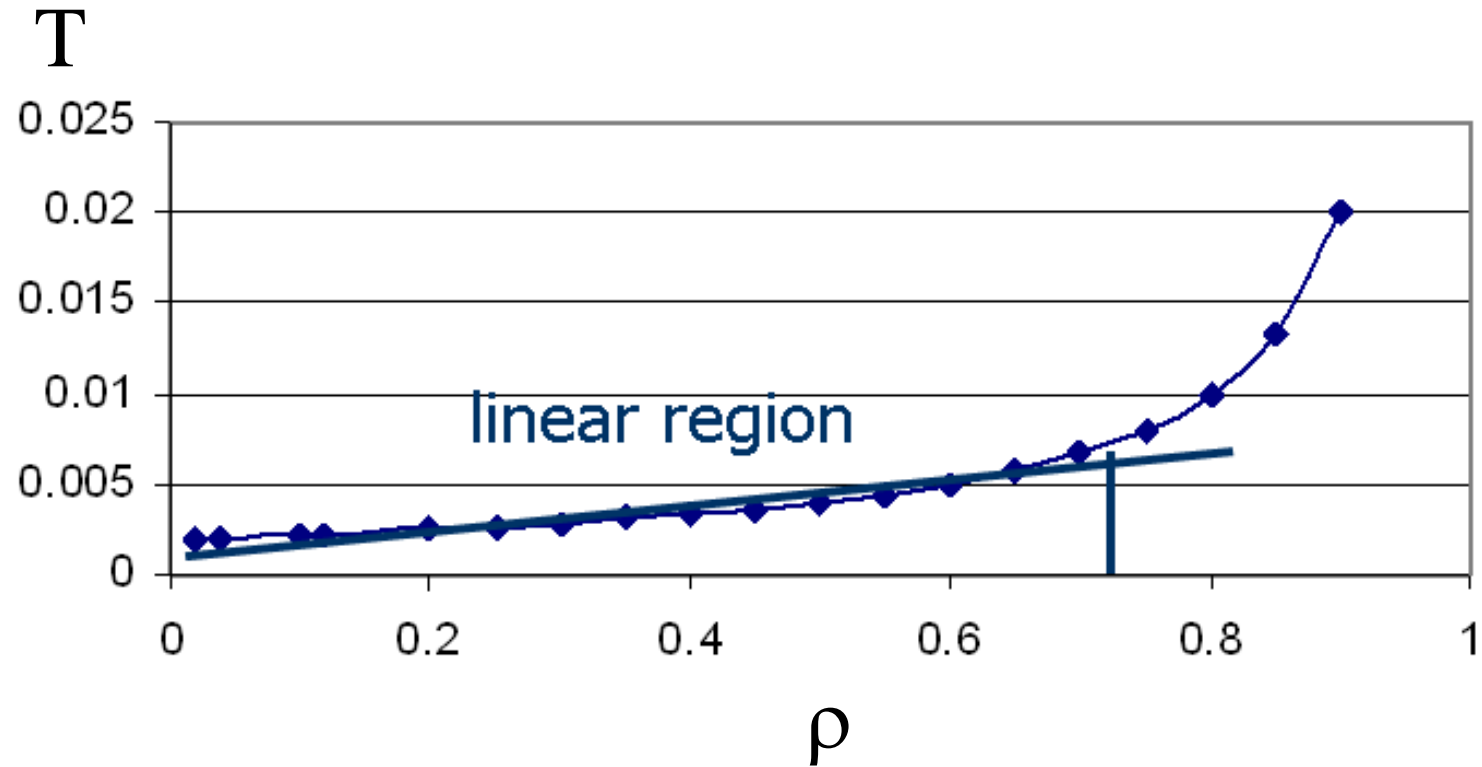


$$T = \frac{1 / \mu}{(1 - \rho)}$$

$\rho = 0 \rightarrow$ only service time

Stable Region

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The average # of customers in Q

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- $W = \text{Total Delay} - \text{Service time} = T - S$

$$\overline{W} = \frac{1 / \mu}{(1 - \rho)} - (1 / \mu)$$

$$= \frac{\rho / \mu}{(1 - \rho)}$$

$$\overline{N}_Q = \lambda \overline{W} = \frac{\rho^2}{(1 - \rho)}$$

Example 1

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- A wireless channel of IEEE 802.11b = 11Mbps
- For a video file needed to be sent at 2 Mbps
- $\lambda = 2$ Mbps
- $\mu = 11$ Mbps
- $\rho = \lambda/\mu = 2/11 = 0.182$

Example 1

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- The avg. Tx time $= \frac{1 / \mu}{(1 - \rho)}$
 $= (1/11)/(1 - 0.182)$
 $= 111.14 \text{ ms}$
- The avg.# of waiting bits $= 0.182^2/(1 - 0.182)$
 $= 0.04049$

Example 2

- On a network gateway, measurements show that the packets arrive at a mean rate of 125 packets per second (pps)
- Gateway takes about 2 milliseconds to forward them
- Assuming an M/M/1 model
 - What is the probability of buffer overflow if the gateway had only 13 buffers ?
 - How many buffers are needed to keep packet loss below one packet per million?

Example 2

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- Measurement of a network gateway:
 - mean arrival rate : 125 Packets/s
 - mean response time : 2 ms
- Assuming exponential arrivals:
 - What is the gateway's utilization?
 - What is the probability of n packets in the gateway?
 - mean number of packets in the gateway?
 - The number of buffers so $P(\text{overflow})$ is $<10^{-6}$?

Example 2

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- Arrival rate $\lambda =$
- Service rate $\mu =$
- Gateway utilization $\rho = \lambda/\mu =$
- Prob. of n packets in gateway =
- Mean number of packets in gateway =

Example 2

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- Arrival rate $\lambda = 125$ pps
- Service rate $\mu = 1/0.002 = 500$ pps
- Gateway utilization $\rho = \lambda/\mu = 0.25$
- Prob. of n packets in gateway

$$(1 - \rho)\rho^n = 0.75(0.25)^n$$

- Mean number of packets in gateway

$$\frac{\rho}{1 - \rho} = \frac{0.25}{0.75} = 0.33$$

Example 2

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- Probability of buffer overflow:
= P(more than 13 packets in gateway)
= $\rho^{13} = 0.25^{13} = 1.49 \times 10^{-8}$
= 15 packets per billion packets
- To limit the probability of loss to less than 10^{-6} :
$$\rho^n \leq 10^{-6}$$
$$n > \log(10^{-6}) / \log(0.25)$$
$$= 9.96$$

HW: M/M/1

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- A Bangkok gas station named “Smile Pump”
- Normally, they provide all kinds of Diesel and Gasohal (91, 95).
- Due to the Fuel crisis, the station installs a NGV gas pump which is the only one NGV in the area.

HW: M/M/1

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- Car (mostly Taxi) arrives at Poisson with rate 6 per hour. Each car take 5 min. exponentially distributed time for filling up.
 - What is the system utilization ?
 - How long does each car spend at the gas station ?
 - How big is the waiting area that Gas station needs to prepare ?
 - How long does each car have to wait in the Queue?
 - How many car in the queue ?

HW: M/M/1

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- During a Bangkok Music Festival, a lot of people coming to Bangkok. The arrival rate of cars increase to 10 per hour.
 - What is the system utilization ?
 - How long does each car spend at the gas station ?
 - How long does each car have to wait in the queue?
 - How many car in the queue ?

HW: M/M/1

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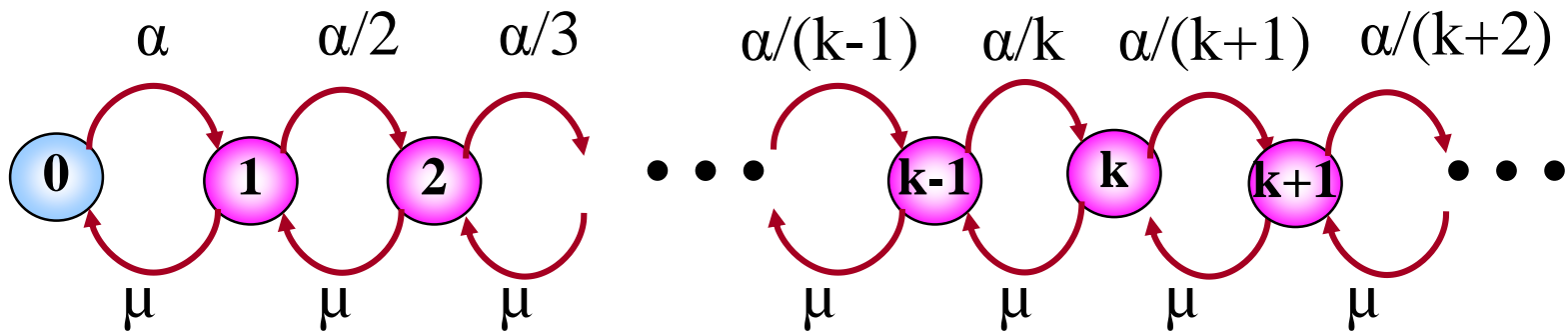
- During a Songkran Festival, people leaving Bangkok. The arrival rate of cars drop to 3 per hour.
 - What is the system utilization ?
 - How long does each car spend at the gas station ?
 - How long does each car have to wait in the queue?
 - How many car in the queue ?

DISCOURAGED ARRIVALS



Discouraged Arrivals

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- Assumption

- $\lambda_k = \alpha / (k+1)$ for $k \geq 0$
- $\mu_k = \mu$ for $k \geq 1$

Discouraged Arrivals

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$$\begin{aligned} p_k &= p_0 \left(\prod_{i=0}^{k-1} \frac{\alpha/(i+1)}{\mu} \right) \\ &= p_0 \left(\frac{\alpha}{\mu} \right)^k \left(\frac{1}{k!} \right) \\ p_0 &= \left(1 + \sum_{k=1}^{\infty} \left(\frac{\alpha}{\mu} \right)^k \left(\frac{1}{k!} \right) \right)^{-1} \\ &= e^{-\alpha/\mu} \end{aligned}$$

Discouraged Arrivals

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$$\rho = 1 - e^{-\alpha/\mu}$$

- The ergodic condition is $\alpha/\mu < \infty$

$$\begin{aligned} p_k &= p_0 \frac{(\alpha/\mu)^k}{k!} \\ &= \frac{(\alpha/\mu)^k}{k!} e^{-\alpha/\mu} \end{aligned}$$

Poisson Distribution

Discouraged Arrivals

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$$\bar{N} = \frac{\alpha}{\mu}$$

$$\lambda = \mu\rho = \mu(1 - e^{-\alpha/\mu})$$

- From Little's result

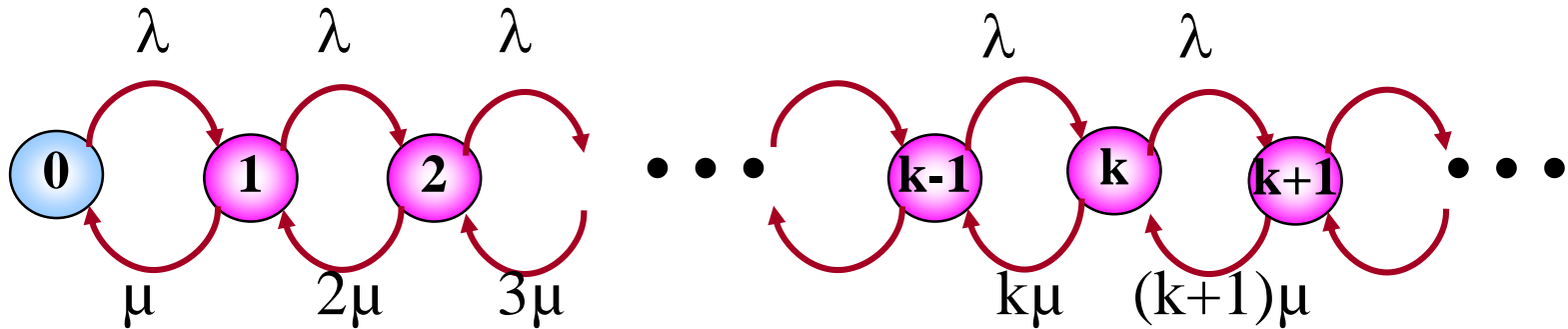
$$T = \frac{\alpha}{\mu^2 (1 - e^{-\alpha/\mu})}$$

RESPONSIVE SERVERS (M/M/∞)



Responsive Servers (M/M/∞)

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- Assumption
 - $\lambda_k = \lambda$ for $k \geq 0$
 - $\mu_k = k\mu$ for $k \geq 1$
- Infinite number of servers

Responsive Servers (M/M/∞)

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$$\begin{aligned} p_k &= p_0 \left(\prod_{i=0}^{k-1} \frac{\lambda}{(i+1)\mu} \right) \\ &= p_0 \left(\frac{\lambda}{\mu} \right)^k \left(\frac{1}{k!} \right) \\ p_0 &= \left(1 + \sum_{k=1}^{\infty} \left(\frac{\lambda}{\mu} \right)^k \left(\frac{1}{k!} \right) \right)^{-1} \\ &= e^{-\lambda/\mu} \end{aligned}$$

Responsive Servers (M/M/∞)

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$$\rho = 1 - e^{-\lambda/\mu}$$

- The ergodic condition is $\lambda/\mu < \infty$

$$\begin{aligned} p_k &= p_0 \frac{(\lambda/\mu)^k}{k!} \\ &= \frac{(\lambda/\mu)^k}{k!} e^{-\lambda/\mu} \end{aligned}$$

Poisson Distribution

Responsive Servers (M/M/∞)

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$$\bar{N} = \frac{\lambda}{\mu}$$

- From Little's result

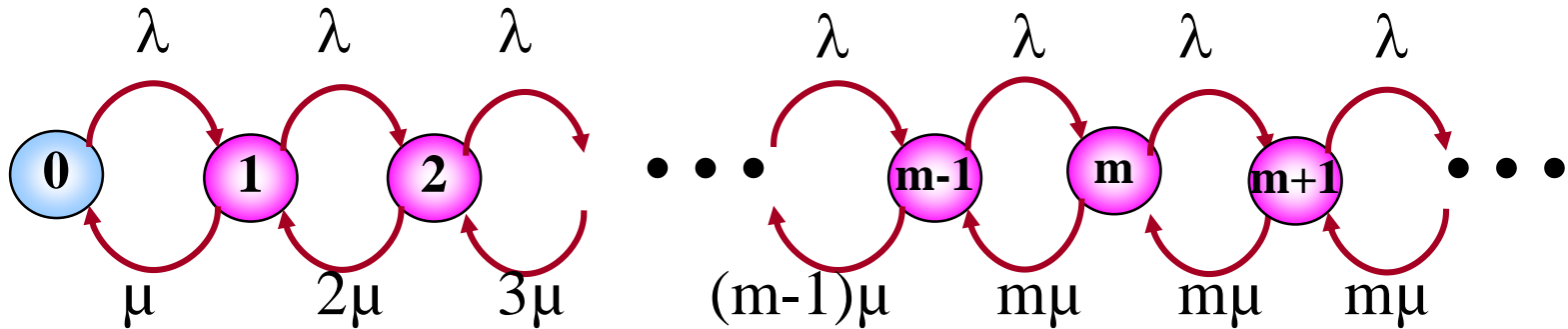
$$T = \frac{\bar{N}}{\lambda} = 1 / \mu \quad \leftarrow \text{Always get serve}$$

M-SERVER (M/M/M)



m-Server (M/M/m)

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- Assumption

- $\lambda_k = \lambda$ for $k \geq 0$

- $\mu_k = \min [k\mu, m\mu]$

$$= \begin{cases} k\mu & 0 \leq k \leq m \\ m\mu & m \leq k \end{cases}$$

m-Server (M/M/m)

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- For $k \leq m$

$$p_k = p_0 \left(\prod_{i=0}^{k-1} \frac{\lambda}{(i+1)\mu} \right)$$

$$= p_0 \left(\frac{\lambda}{\mu} \right)^k \left(\frac{1}{k!} \right)$$

m-Server (M/M/m)

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- For $k \geq m$

$$\begin{aligned} p_k &= p_0 \prod_{i=0}^{m-1} \frac{\lambda}{(i+1)\mu} \prod_{j=m}^{k-1} \frac{\lambda}{m\mu} \\ &= p_0 \left(\frac{\lambda}{\mu} \right)^k \frac{1}{m! m^{k-m}} \end{aligned}$$

m-Server (M/M/m)

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$$p_k = \begin{cases} p_0 \frac{(m\rho)^k}{k!} & k \leq m \\ p_0 \frac{(\rho)^k m^m}{m!} & k \geq m \end{cases}$$

$$\rho = \frac{\lambda}{m\mu} < 1$$

m-Server (M/M/m)

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$$p_0 = \left(1 + \sum_{k=1}^{m-1} \frac{(m\rho)^k}{k!} + \sum_{k=m}^{\infty} \frac{(m\rho)^k}{m!} \frac{1}{m^{k-m}} \right)^{-1}$$

$$p_0 = \left(\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \left(\frac{(m\rho)^m}{m!} \right) \left(\frac{1}{1-\rho} \right) \right)^{-1}$$

m-Server (M/M/m)

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$$\begin{aligned} P[\text{queueing}] &= \sum_{k=m}^{\infty} p_k \\ &= \sum_{k=m}^{\infty} p_0 \frac{(m\rho)^k}{m!} \frac{1}{m^{k-m}} \end{aligned}$$

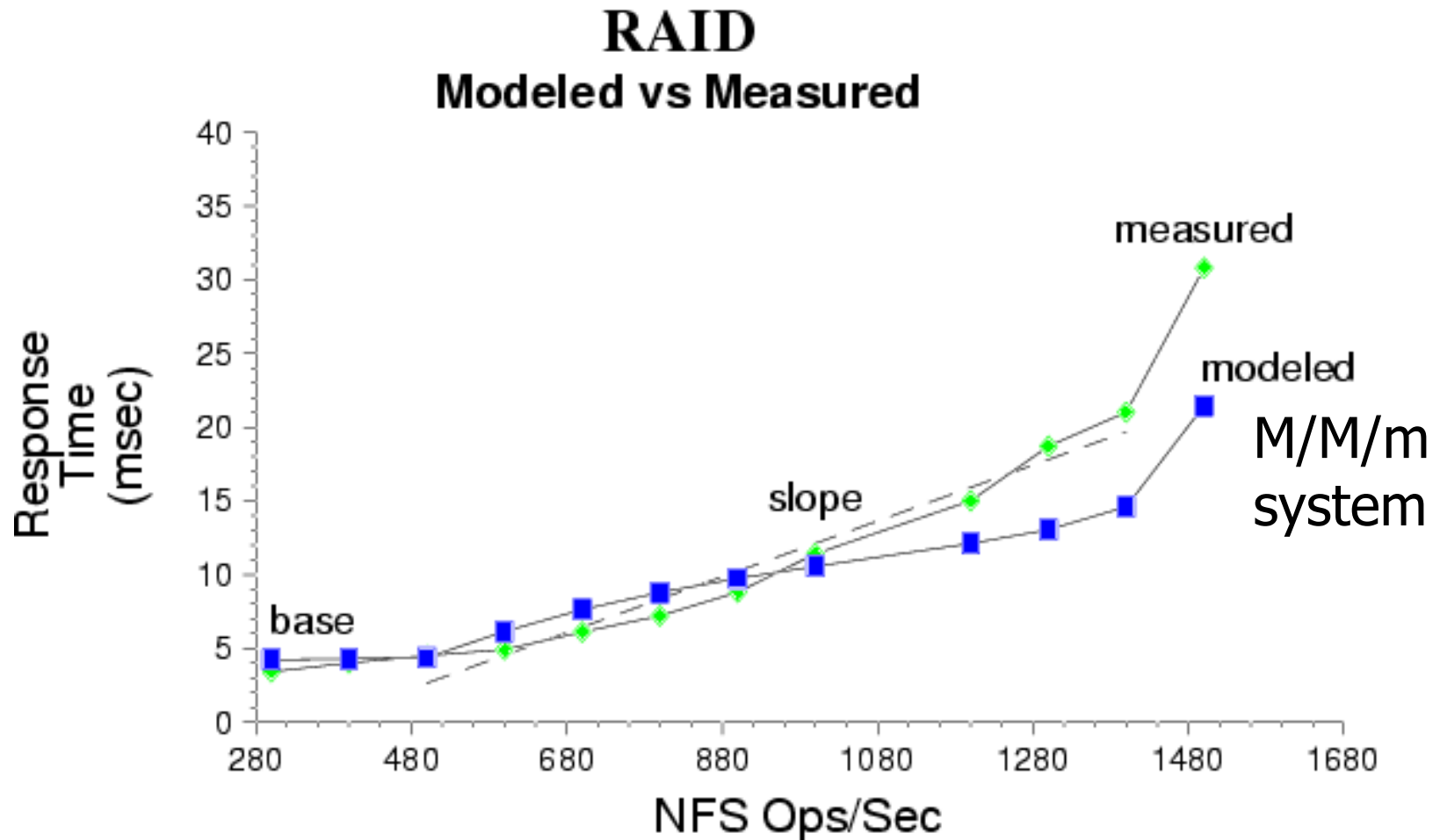
$$P[\text{queueing}] = \frac{\left(\frac{(m\rho)^m}{m!} \right) \left(\frac{1}{1-\rho} \right)}{\left(\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \left(\frac{(m\rho)^m}{m!} \right) \left(\frac{1}{1-\rho} \right) \right)}$$

Erlang's C formula
 $C(m, \lambda/\mu)$

Empirical Example

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Example from CS352, Rutgers University

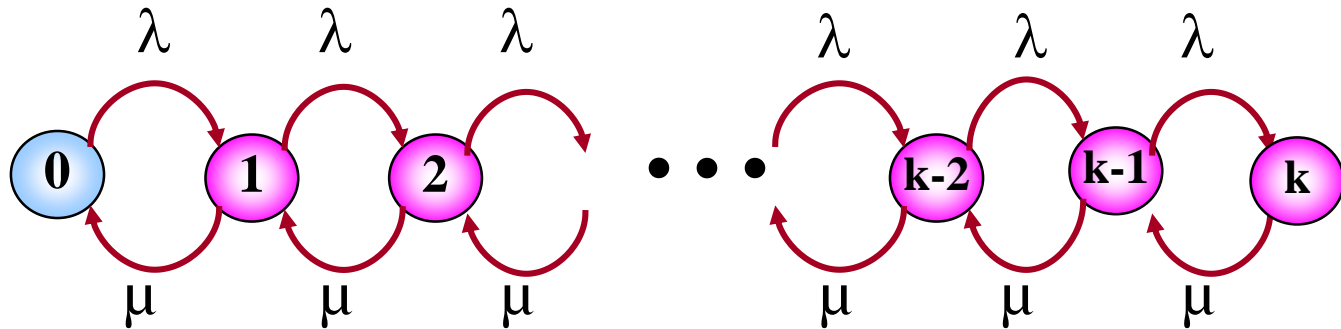


FINITE STORAGE (M/M/1/K)



Finite Storage (M/M/1/K)

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- Assumption

- $\mu_k = \mu$

$$1 \leq k \leq K$$

- $\lambda_k = \begin{cases} \lambda \\ 0 \end{cases}$

$$k < K$$

$$k \geq K$$

Finite Storage (M/M/1/K)

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- For $k \leq K$

$$p_k = p_0 \prod_{i=0}^{k-1} \frac{\lambda}{\mu}$$

$$= p_0 \left(\frac{\lambda}{\mu} \right)^k$$

- For $k > K$

$$p_k = 0$$

Finite Storage (M/M/1/K)

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$$\begin{aligned} p_0 &= \left(1 + \sum_{k=1}^K \left(\frac{\lambda}{\mu} \right)^k \right)^{-1} \\ &= \left(1 + \frac{(\lambda/\mu)(1 - (\lambda/\mu)^K)}{1 - (\lambda/\mu)} \right)^{-1} \\ &= \frac{1 - (\lambda/\mu)}{1 - (\lambda/\mu)^{K+1}} \end{aligned}$$

Finite Storage (M/M/1/K)

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$$p_k = \begin{cases} \frac{1 - (\lambda/\mu)}{1 - (\lambda/\mu)^{K+1}} \left(\frac{\lambda}{\mu}\right)^k & 0 \leq k \leq K \\ 0 & \text{Otherwise} \end{cases}$$

Finite Storage (M/M/1/K)

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- For $K = 1$ (Blocked call cleared)

$$p_k = \begin{cases} \frac{1}{1 + (\lambda/\mu)} & k = 0 \\ \frac{\lambda/\mu}{1 + (\lambda/\mu)} & k = 1 = K \\ 0 & \text{Otherwise} \end{cases}$$

Note on M/M/1/K

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- $\rho > 1$ is possible