

LECTURE #6

BIRTH-DEATH PROCESS

204528

Queueing Theory and
Applications in Networks

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Outline

2

- Birth-Death Process
- Markov Process Property
- Continuous Time Birth-Death Markov Chains
- State Transition Diagram
- A Pure Birth System
- A Pure Death System
- A Birth-Death Process
- Equilibrium Solution

Birth-Death Process

3

- A Markov Process
- Homogeneous, aperiodic, and irreducible
- Discrete time / Continuous time
- State changes can only happen between neighbors

Birth-Death Process

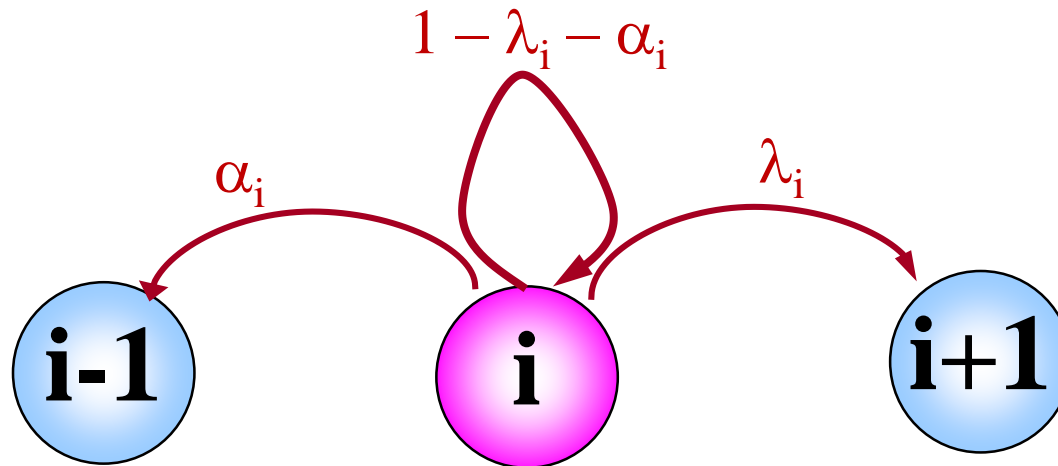
4

- Size of population
 - System is in state E_k when consists of k members
 - Changes in population size occur by at most one
 - Size has been increased by one \rightarrow “*Birth*”
 - Size has been decreased by one \rightarrow “*Death*”
- Transition probabilities p_{ij} do not change with time

Birth-Death Process

5

$$p_{ij} = \begin{cases} \alpha_i & j = i - 1 \\ 1 - \lambda_i - \alpha_i & j = i \\ \lambda_i & j = i + 1 \\ 0 & \text{Otherwise} \end{cases}$$



Birth-Death Process

6

- $\alpha_i =$ death (less one in population size)
- $\alpha_0 = 0$ (no population \rightarrow no death)
- $\lambda_i =$ birth (increase one in population)
- $\lambda_i > 0$ (birth is allowed)
- Pure Birth = no decrement, only increment
- Pure Death = no increment, only decrement

Queueing Theory Model

7

- **Population** = customers in the queueing system
- **Death** = a customer departures from the system
- **Birth** = a customer arrives to the system

Transition matrix

8

$$P = \begin{bmatrix} 1 - \lambda_0 & \lambda_0 & 0 & 0 & 0 & 0 & \dots \\ \alpha_1 & 1 - \lambda_1 - \alpha_1 & \lambda_1 & 0 & 0 & 0 & \\ 0 & \alpha_2 & 1 - \lambda_2 - \alpha_2 & \lambda_2 & & & \\ 0 & & \dots & & & & \\ 0 & & & & & & \\ \dots & & & \alpha_i & 1 - \lambda_i - \alpha_i & \lambda_i & \end{bmatrix}$$

Discrete Time Markov Chains

9

- One can stay in a *Discrete state (position)* and is permitted to change state at *Discrete time*.

Discrete Time Markov Chains

10

$$\begin{aligned} P\{X_n = j \mid X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\} \\ = P\{X_n = j \mid X_{n-1} = i_{n-1}\} \quad \text{Where } n = 1, 2, 3, \dots \end{aligned}$$

- X_n : The system is in state j at time n
- The system can begin at *state 0* with *initial probability* $P[X_0 = x]$
- $P\{X_n = j \mid X_{n-1} = i_{n-1}\}$ is the *one-step transition probability*

Discrete Time Markov Chains

11

- From *initial probability* and *one-step transition probability*,
- we can find *probability of being in various states at time n*

Continuous Time Markov Chains

12

$$\begin{aligned} P\{X(t_{n+1}) = j \mid X(t_1) = i_1, X(t_2) = i_2, \dots, X(t_n) = i_n\} \\ = P\{X(t_{n+1}) = j \mid X(t_n) = i_n\} \end{aligned}$$

Where $n = 1, 2, 3, \dots$ $t_1 < t_2 < \dots < t_n$

- One can stay in a *Discrete state (position)* and is permitted to change state at *Arbitrary time*

Markov Process Property

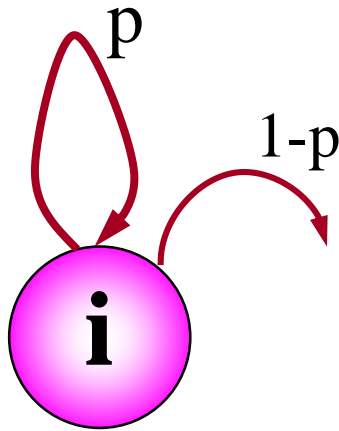
13

- Time that the process spends in any state must be “Memoryless”
- Discrete Time Markov Chains
 - Geometrically distributed state times
- Continuous Time Markov Chains
 - Exponentially distributed state times

Markov Process Property

14

For Discrete Time Markov Chain

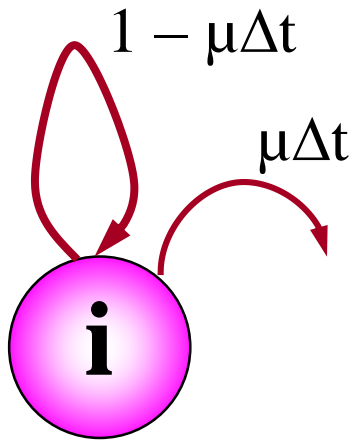


- $P[\text{system in state } i \text{ for } N \text{ time units} \mid \text{system in current state } i] = p^N$
- $P[\text{system in state } i \text{ for } N \text{ time units before exiting from state } i] = p^N (1-p)$
- Geometrically distributed state times

Markov Process Property

15

For Continuous Time Markov Chain



- $P[\text{system in state } i \text{ for time } T \mid \text{system in current state } i]$
 $= (1 - \mu\Delta t)^{T/\Delta t}$
 $= e^{-\mu T}$ where $\Delta t \rightarrow 0$
- Exponentially distributed state times

Continuous Time Birth-Death Markov Chains

16

- Let $\lambda_i =$ birth rate in state i
 $\mu_i =$ death rate in state i

- Then

$$P[\text{state } i \text{ to state } i - 1 \text{ in } \Delta t] = \mu_i \Delta t$$

$$P[\text{state } i \text{ to state } i + 1 \text{ in } \Delta t] = \lambda_i \Delta t$$

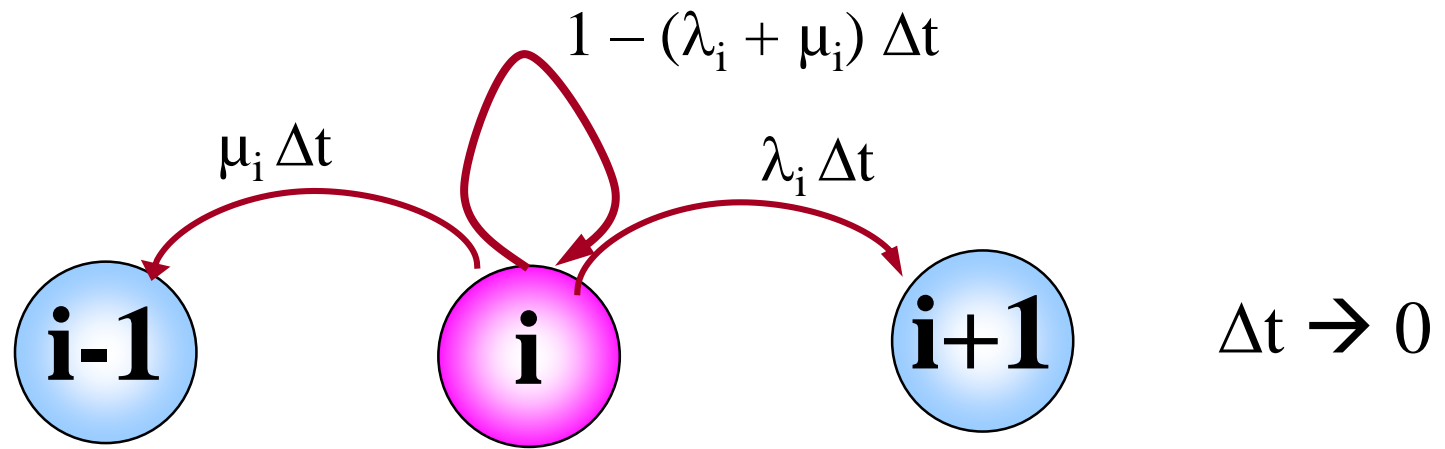
$$P[\text{state } i \text{ to state } i \text{ in } \Delta t] = 1 - (\lambda_i + \mu_i) \Delta t$$

$$P[\text{state } i \text{ to other state in } \Delta t] = 0$$

Continuous Time Birth-Death Markov Chains

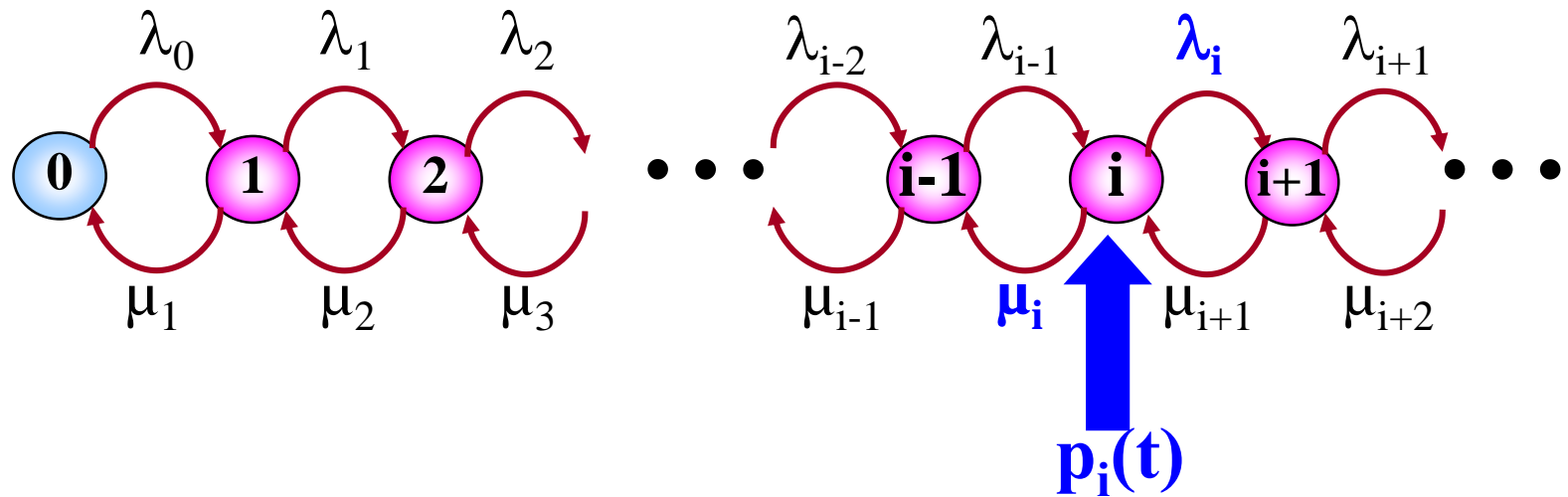
17

$$p_{ij} = \begin{cases} \mu_i \Delta t & j = i - 1 \\ 1 - (\lambda_i + \mu_i) \Delta t & j = i \\ \lambda_i \Delta t & j = i + 1 \\ 0 & \text{Otherwise} \end{cases}$$



State Transition Diagram

18



- $X(t)$ = #customers in the system at time t
= birth – death in $(0, t)$
- $p_i(t) = P[X(t) = i]$
= Prob. that system is in state i at time t

State Transition Diagram

19

- From t to $t + \Delta t$

$$p_0(t+\Delta t) = p_0(t)[1 - \lambda_0\Delta t] + p_1(t)\mu_1\Delta t$$

$$p_i(t+\Delta t) = p_i(t)[1 - (\lambda_i+\mu_i)\Delta t] + p_{i+1}(t)\mu_{i+1}\Delta t + p_{i-1}(t)\lambda_{i-1}\Delta t$$

- $\Delta t \rightarrow 0$

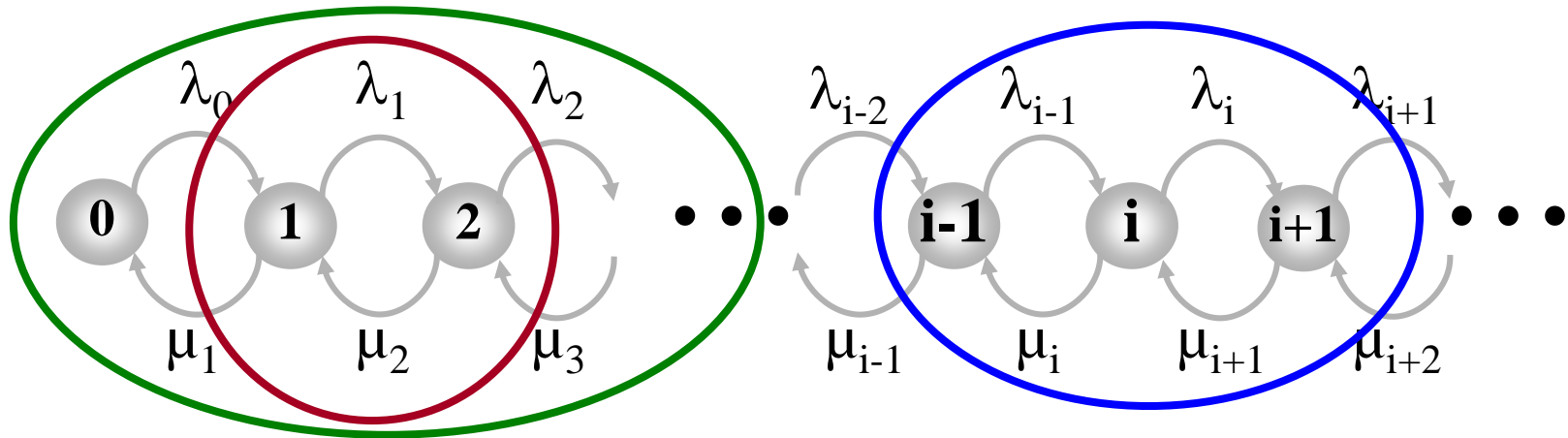
$$dp_0(t)/dt = -\lambda_0p_0(t) + \mu_1p_1(t)$$

$$dp_i(t)/dt = -(\lambda_i+\mu_i)p_i(t) + \mu_{i+1}p_{i+1}(t) + \lambda_{i-1}p_{i-1}(t)$$

- $\sum_{i=0}^{\infty} p_i(t) = 1$

Flow Balance Method

20

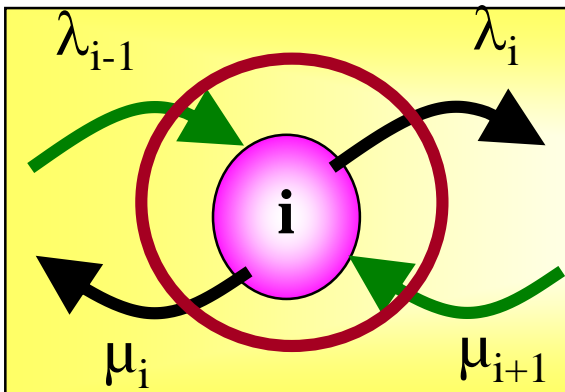


- Draw a closed boundary
- Observe all flows (*In* and *Out*) across the boundary

Flow Balance Method

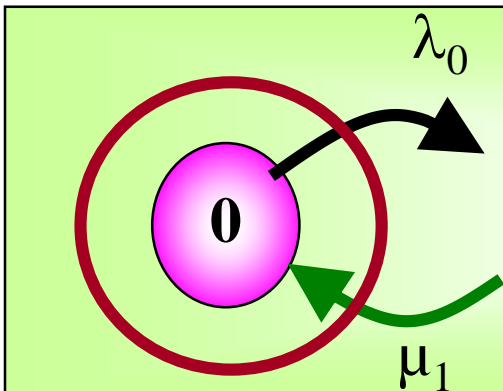
21

- Flow Out = Flow In



- Draw a closed boundary around state i

$$(\lambda_i + \mu_i) p_i = \mu_{i+1} p_{i+1} + \lambda_{i-1} p_{i-1}$$

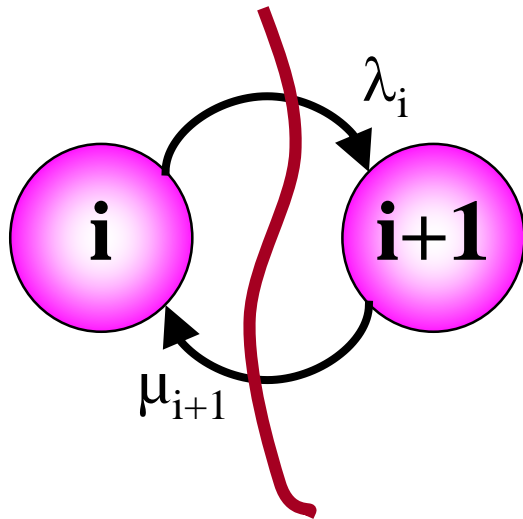


- Draw a closed boundary around state 0

$$\lambda_0 p_0 = \mu_1 p_1$$

Flow Balance Method

22

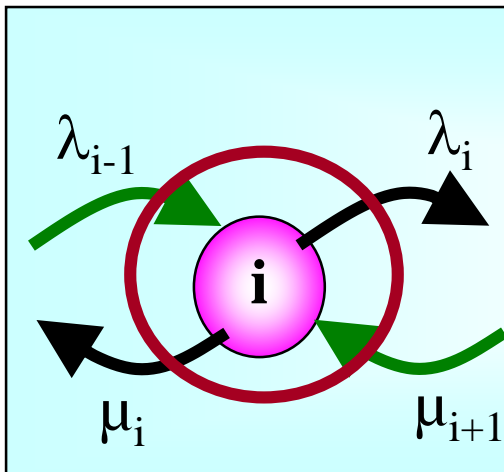


- Draw a closed boundary around state i at infinity

$$\lambda_i p_i = \mu_{i+1} p_{i+1}$$

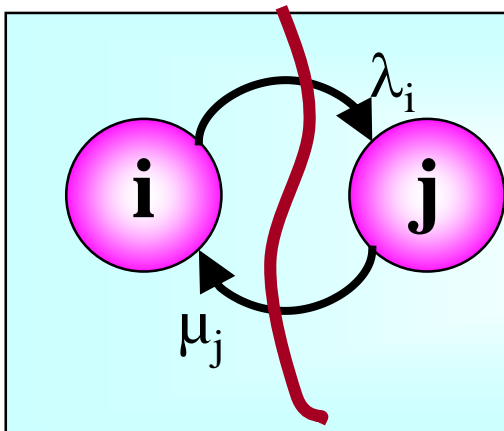
Flow Balance General Form

23



- Draw a closed boundary around state i
- ***Global Balance Equation***

$$\sum_{i \neq j} p_i p_{ij} = p_j \sum_{i \neq j} p_{ji}$$

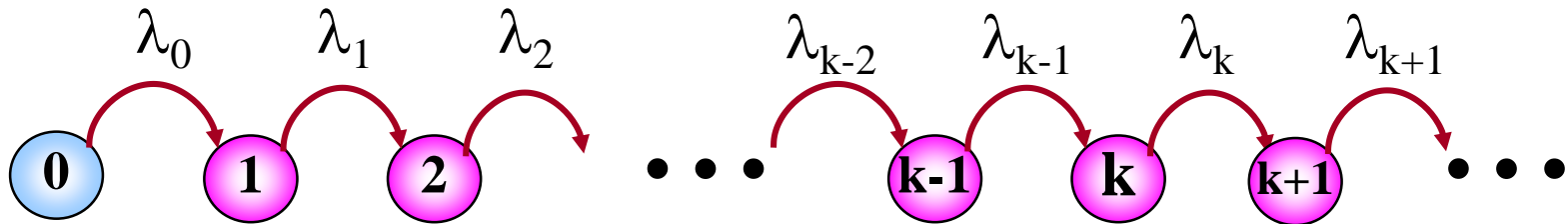


- Draw a closed boundary between state i and j
- ***Detailed Balance Equation***

$$p_i p_{ij} = p_j p_{ji}$$

A Pure Birth System

24



- Assumption

- $\mu_k = 0$ for all k
- $\lambda_k = \lambda$ for all k
- The system begins at time t_0 with 0 member

$$p_k(0) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

A Pure Birth System

25

- $dp_0(t)/dt = -\lambda_0 p_0(t) + \mu_1 p_1(t)$
→ $dp_0(t)/dt = -\lambda p_0(t)$
- $dp_k(t)/dt = -(\lambda_k + \mu_k) p_k(t) + \mu_{k+1} p_{k+1}(t) + \lambda_{k-1} p_{k-1}(t)$
→ $dp_k(t)/dt = -\lambda p_k(t) + \lambda p_{k-1}(t)$
- Solution for $p_0(t)$
→ $p_0(t) = e^{-\lambda t}$

$$\frac{d\text{☺}}{dt} = -\lambda \text{☺}$$

$$\text{☺} = e^{-\lambda t}$$

A Pure Birth System

26

- For $k = 1$

$$\begin{aligned}\rightarrow dp_1(t)/dt &= -\lambda p_1(t) + \lambda p_0(t) \\ &= -\lambda p_1(t) + \lambda e^{-\lambda t}\end{aligned}$$

$$\rightarrow p_1(t) = \lambda t e^{-\lambda t}$$

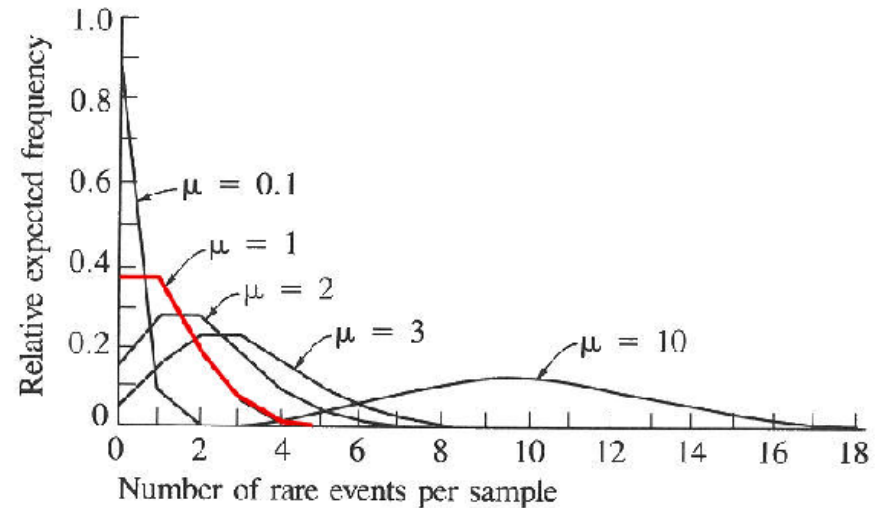
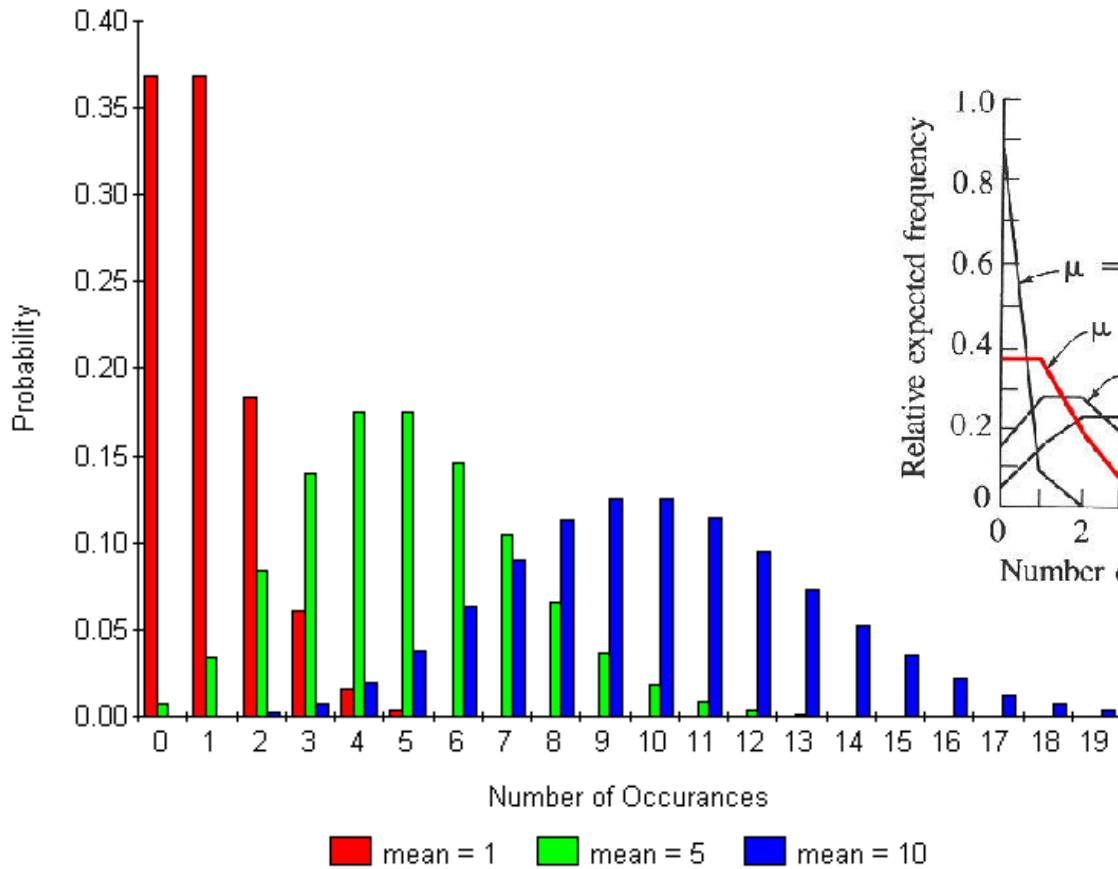
- For $k \geq 0, t \geq 0$

$$\rightarrow p_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Poisson Distribution

Poisson Distribution

27



http://www.mun.ca/biology/scarr/Poisson_distribution3.jpg

http://www.boost.org/doc/libs/1_35_0/libs/math/doc/sf_and_dist/graphs/poisson.png

A Poisson Process

28

- The arrival of customers
- λ = the average rate that customer arrives
- $p_k(t)$ = Prob. that k arrivals occur during $(0, t)$
- K = # of arrivals in the interval t
- The average # of arrivals in an interval t , $E[K] = ?$

A Poisson Process

29

$$\begin{aligned} E[K] &= \sum_{k=0}^{\infty} k p_k(t) &= e^{-\lambda t} \sum_{k=0}^{\infty} k \frac{(\lambda t)^k}{k!} \\ & &= e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!} \\ & &= e^{-\lambda t} \lambda t \left(\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \right) && \nearrow e^{\lambda t} \\ & &= \lambda t \end{aligned}$$

Pure Birth Process Example

30

- Linear Birth Process
- Yule-Furry Process
- Consider **cells** which **reproduce** according to the following rules:
 - 1) A cell presented at time t has probability $\lambda\Delta t + o(\Delta t)$ of splitting in two in the interval $(t, t + \Delta t)$
 - 2) This probability is independent of age
 - 3) Events between different cells are independent

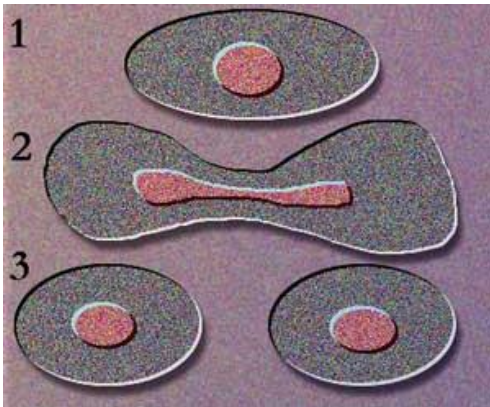
Modified from

1. <http://www.bibalex.org/supercourse/supercourseppt/19011-20001/19531.pdf>
2. The theory of stochastic processes By D. R. Cox, H. D. Miller

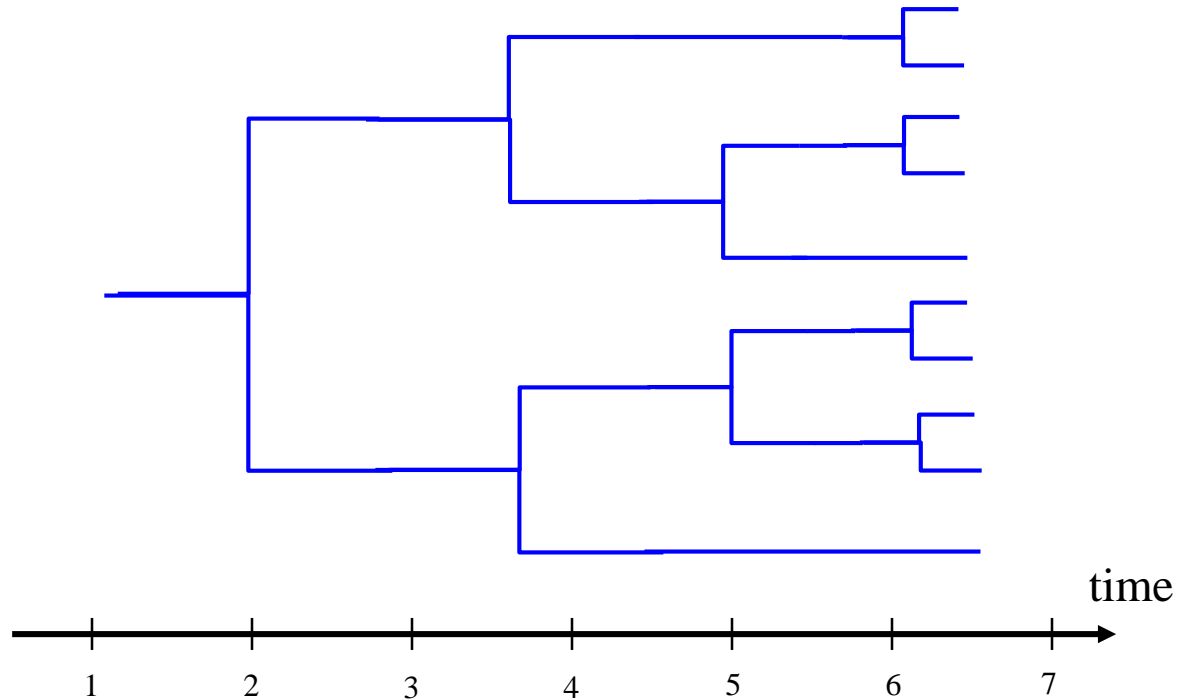
Pure Birth Process Example

31

Cell Division



<http://www.dmturner.org/Teacher/Library/5thText/SimplePart3.html>



Pure Birth Process Example

Non-Probabilistic Analysis

32

- $n(t)$ = no. of cells at time t
- λ = birth rate per single cell
- $n(t)\lambda\Delta t$ births occur in $(t, t + \Delta t)$

$$n(t + \Delta t) = n(t) + n(t)\lambda\Delta t$$

$$n'(t) = \frac{n(t + \Delta t) - n(t)}{\Delta t} = n(t)\lambda$$
$$\frac{n'(t)}{n(t)} = \frac{d}{dt} \log n(t) = \lambda$$

$$\log n(t) = \lambda t + c$$

$$n(t) = Ke^{\lambda t}, \quad n(0) = n_0$$

$$n(t) = n_0 e^{\lambda t}$$

Pure Birth Process Example

Probabilistic Analysis

33

- $N(t)$ = no. of cells at time t
- $P\{N(t) = n\} = P_n(t)$
- Prob. of birth in $(t, t + \Delta t)$ if $\{N(t) = n\} = n\Delta t + o(\Delta t)$

$$P_n(t + \Delta t) = P_n(t)(1 - n\lambda\Delta t + o(\Delta t)) + P_{n-1}(t)((n-1)\lambda\Delta t + o(\Delta t))$$

$$P_n(t + \Delta t) - P_n(t) = -n\lambda\Delta t P_n(t) + P_{n-1}(t)(n-1)\lambda\Delta t + o(\Delta t)$$

$$\frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = -n\lambda P_n(t) + P_{n-1}(t)(n-1)\lambda + o(\Delta t) \quad \text{as } \Delta t \rightarrow 0$$

$$P'_n(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)$$

Pure Birth Process Example

Probabilistic Analysis

34

$$P'_n(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)$$

- Initial condition: $P_{n_0}(0) = P\{n(0) = n_0\} = 1$

$$P_n(t) = \binom{n-1}{n-n_0} e^{-\lambda n_0 t} (1 - e^{-\lambda t})^{n-n_0} \quad n = n_0, n_0 + 1, \dots$$

- Solution is negative binomial distribution
 - = Probability of obtaining exactly n_0 successes in n trials

Pure Birth Process Example

Probabilistic Analysis

35

- Suppose $p = \text{prob. of success}$
and $q = 1 - p = \text{prob. of failure}$
- Then in the first $(n - 1)$ trials results in $(n_0 - 1)$ successes and $(n - n_0)$ failures followed by success on n^{th} trial

$$P_n(t) = \binom{n-1}{n_0-1} p^{n_0-1} q^{n-n_0} \cdot p = \binom{n-1}{n-n_0} p^{n_0} q^{n-n_0}$$

- If $p = e^{-\lambda t}$ and $q = (1 - e^{-\lambda t})$ $n = n_0, n_0 + 1, \dots$
 - $\rightarrow P_n(t)$ is as same as previous equation

Yule-Furry Process

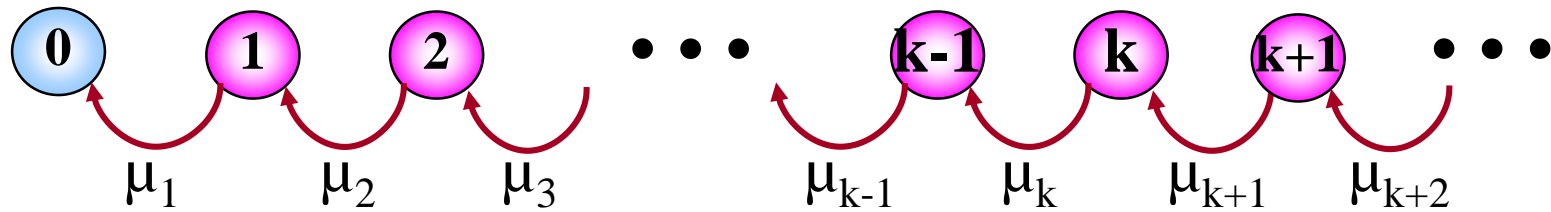
36

- Yule studied this process in connection with theory of evolution
 - i.e. population consists of the species within a genus and creation of new element is due to mutations
 - Neglects probability of species dying out and size of species
- Furry used same model for radioactive transmutations

a **genus** is a low-level taxonomic rank used in the classification of living

A Pure Death System

37



- Example
 - Microbial (a bacterium that causes disease) risk analysis
- Assumption
 - $\mu_k = \mu \geq 0$ for all k
 - $\lambda_k = 0$ for all k
 - The system begins with N members
 - $k = 1, 2, 3, \dots, N$

A Pure Death Process

38

$$p_k(t) = \frac{(\mu t)^{N-k}}{(N-k)!} e^{-\mu t} \quad 0 < k \leq N$$

$$\frac{dp_0(t)}{dt} = \frac{\mu(\mu t)^{N-1}}{(N-1)!} e^{-\mu t} \quad k = 0$$

Erlang Distribution