

LECTURE #11

M/G/1

204528

Queueing Theory and
Applications in Networks

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Outline

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- More on M/G/1
- Busy period and its duration
- M/G/1 with Vacations

M/G/1

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$$W_q = \frac{\lambda \bar{X}^2}{2(1-\rho)}$$

$$N_q = \frac{\lambda^2 \bar{X}^2}{2(1-\rho)}$$

$$W_T = \bar{X} + \frac{\lambda \bar{X}^2}{2(1-\rho)}$$

$$N_T = \rho + \frac{\lambda^2 \bar{X}^2}{2(1-\rho)}$$

The coefficient of variation (C_v)

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- A normalized measure of dispersion of a probability distribution
- Defined as the ratio of the standard deviation to the mean

$$C_v = \frac{\sigma}{\mu}$$

- Only defined for *non-zero* mean
- C_v should only be computed for data measured on a **ratio scale**

dispersion = การกระจาย

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The coefficient of variation (C_V)

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- **Example** (from wikipedia)
 - For a group of temperatures
 - An object changes its temperature by 1 K also changes its temperature by 1 C
 - The standard deviation does not depend on whether the Kelvin or Celsius scale
 - However, the mean temp would differ in each scale by 273
 - So, the coefficient of variation would differ
- **Investment Dictionary** (<http://www.answers.com/topic/coefficient-of-variation>)
 - The lower the ratio, the better your risk-return tradeoff
 - The higher the ratio, the higher the risk

The coefficient of variation (C_V)

- In queueing theory
 - Exponential Dist. is often more important than the Normal Dist.
- For $C_V = 1$
 - E.g. Exponential distribution \rightarrow ($\sigma = \mu$)
- For $C_V < 1 \rightarrow$ low-variance
 - E.g. Erlang distribution [r-stage Erlangian server (E_r)]
- For $C_V > 1 \rightarrow$ high-variance
 - E.g. Hyper-Exponential distribution [r-Stage Parallel Servers (H_r)]
- Some formulas are expressed using **squared coefficient of variation (SCV)**

More on P-K Formula

Squared coefficient of variation C_v^2

$$C_v^2 = \frac{\text{Var}[X]}{(\bar{X})^2}$$

$$\begin{aligned}\overline{X^2} &= \text{Var}[X] + (\bar{X})^2 \\ &= (1 + C_v^2) \cdot (\bar{X})^2\end{aligned}$$

More on P-K Formula

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$$W_q = \frac{\lambda \bar{X}^2}{2(1-\rho)}$$

$$= \frac{1 + C_v^2}{2} \cdot \frac{\rho}{1-\rho} \cdot \bar{X}$$

$$W_T = \bar{X} + \frac{\lambda \bar{X}^2}{2(1-\rho)}$$

$$= \left(1 + \frac{1 + C_v^2}{2} \cdot \frac{\rho}{1-\rho} \right) \cdot \bar{X}$$

More on P-K Formula

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$$N_q = \frac{\lambda^2 \bar{x}^2}{2(1-\rho)}$$

$$= \frac{1 + C_v^2}{2} \cdot \frac{\rho^2}{1-\rho}$$

$$N_T = \rho + \frac{\lambda^2 \bar{x}^2}{2(1-\rho)}$$

$$= \rho + \frac{1 + C_v^2}{2} \cdot \frac{\rho^2}{1-\rho}$$

More on P-K Formula

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- Mean values depend only on the expectation $E[X]$ and variance $\text{Var}[X]$ of the service time distribution but not on higher moments.
- Mean values increase linearly with the variance.
- Randomness, ‘disarray’, leads to an increased waiting time and queue length.
- The formula are similar to those of the $M/M/1$ queue;
 - the only difference is the extra factor $\frac{1 + C_v^2}{2}$

M/G/1 Steady-State

$$W_q = \frac{\lambda (\bar{X}^2 + \sigma^2)}{2(1 - \rho)}$$

$$N_q = \frac{\lambda^2 (\bar{X}^2 + \sigma^2)}{2(1 - \rho)}$$

$$W_T = \bar{X} + \frac{\lambda (\bar{X}^2 + \sigma^2)}{2(1 - \rho)}$$

$$N_T = \rho + \frac{\lambda^2 (\bar{X}^2 + \sigma^2)}{2(1 - \rho)}$$

$$p_0 = 1 - \rho$$

M/G/1 Example

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- There are two workers competing for a job.
- **Paul** claims that an average service time is **faster** than **Jar**'s
- But **Jar** claims to be **more consistent**, if not as fast.
- The arrivals is a Poisson process at a rate of $\lambda = 2$ per hour. (1/30 per minute).
- **Paul**'s service statistics are an average service time of 24 minutes with a standard deviation of 20 minutes.
- **Jar**'s service statistics are an average service time of 25 minutes, but a standard deviation of only 2 minutes.

M/G/1 Example

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- If the average length of the queue is the criterion for hiring, which worker should be hired (Paul or Jar)?

M/G/1 Example

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- For Paul,

$$\lambda = 1/30 \text{ (per min)}$$

$$1/\mu = 24 \text{ min.}$$

$$\rho = \lambda/\mu = 24/30 = 4/5$$

$$\sigma^2 = 20^2 = 400 \text{ min}^2$$

$$N_q = \frac{\lambda^2(\overline{X^2} + \sigma^2)}{2(1 - \rho)} = \frac{(1/30)^2 (24^2 + 400)}{2(1 - 4/5)} = 2.711 \text{ customers}$$

M/G/1 Example

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- For Jar,

$$\lambda = 1/30 \text{ (per min)}$$

$$1/\mu = 25 \text{ min.}$$

$$\rho = \lambda/\mu = 25/30 = 5/6$$

$$\sigma^2 = 2^2 = 4 \text{ min}^2$$

$$N_q = \frac{\lambda^2(\overline{X^2} + \sigma^2)}{2(1 - \rho)} = \frac{(1/30)^2 (25^2 + 4)}{2(1 - 5/6)} = 2.097 \text{ customers}$$

M/G/1 Example

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- Although working faster on the average,
 - Paul's greater service variability results in an average queue length about **30% greater** than Jar's.
- On the other hand, the proportion of arrivals who would find Paul **idle** and thus experience no delay is $p_0 = 1 - \rho = 1/5 = \mathbf{20\%}$
- While the proportion who would find Jar **idle** and thus experience no delay is $p_0 = 1 - \rho = 1/6 = \mathbf{16.7\%}$.
- On the basis of average **queue length**, N_q ,
 - Jar wins.

Busy and Idle period

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- To derive the distribution for the M/G/1 queue
 - the length of the Idle period
 - the length of the busy period

Busy and Idle period

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- C_n = the n^{th} customer to enter the system
- τ_n = arrival time of C_n
- $t_n = \tau_n - \tau_{n-1}$ = interarrival time between C_{n-1} and C_n
- x_n = Service time of C_n

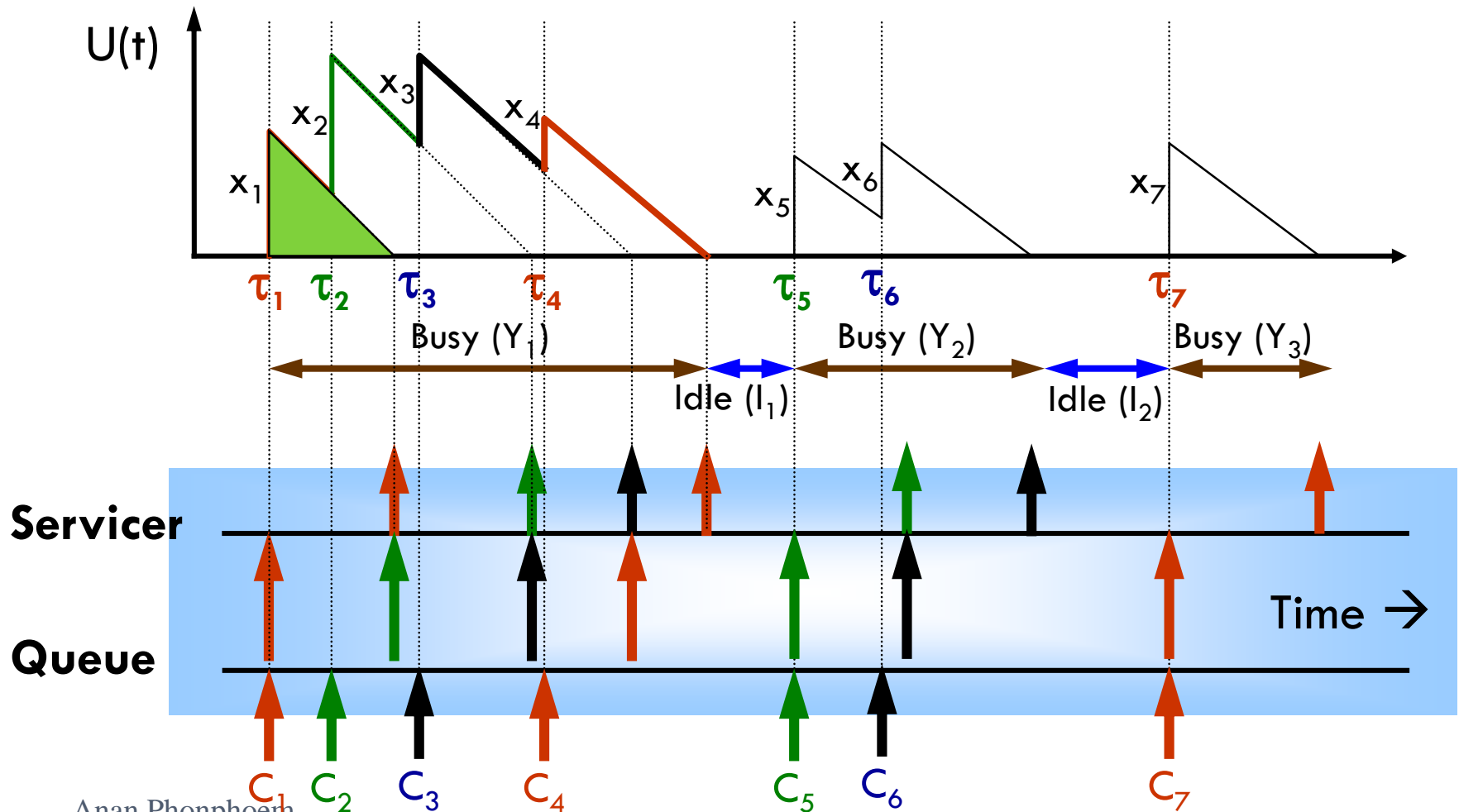
Busy and Idle period

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- $U(t)$ = Unfinished work in the system
= Virtual waiting time at time t
- Y_n = Busy period
- I_n = Idle period

The Unfinished Work (FCFS)

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M/G/1 (FCFS)

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- $A(t) = P[t_n \leq t] = 1 - e^{-\lambda t} \quad t \geq 0$
- $B(x) = P[x_n \leq x]$
- $A(t)$ and $B(x)$ are independent on n
- $F(y) =$ Idle period distribution
 $= P[I_n \leq y]$
- $G(y) =$ Busy period distribution
 $= P[Y_n \leq y]$

M/G/1 (FCFS)

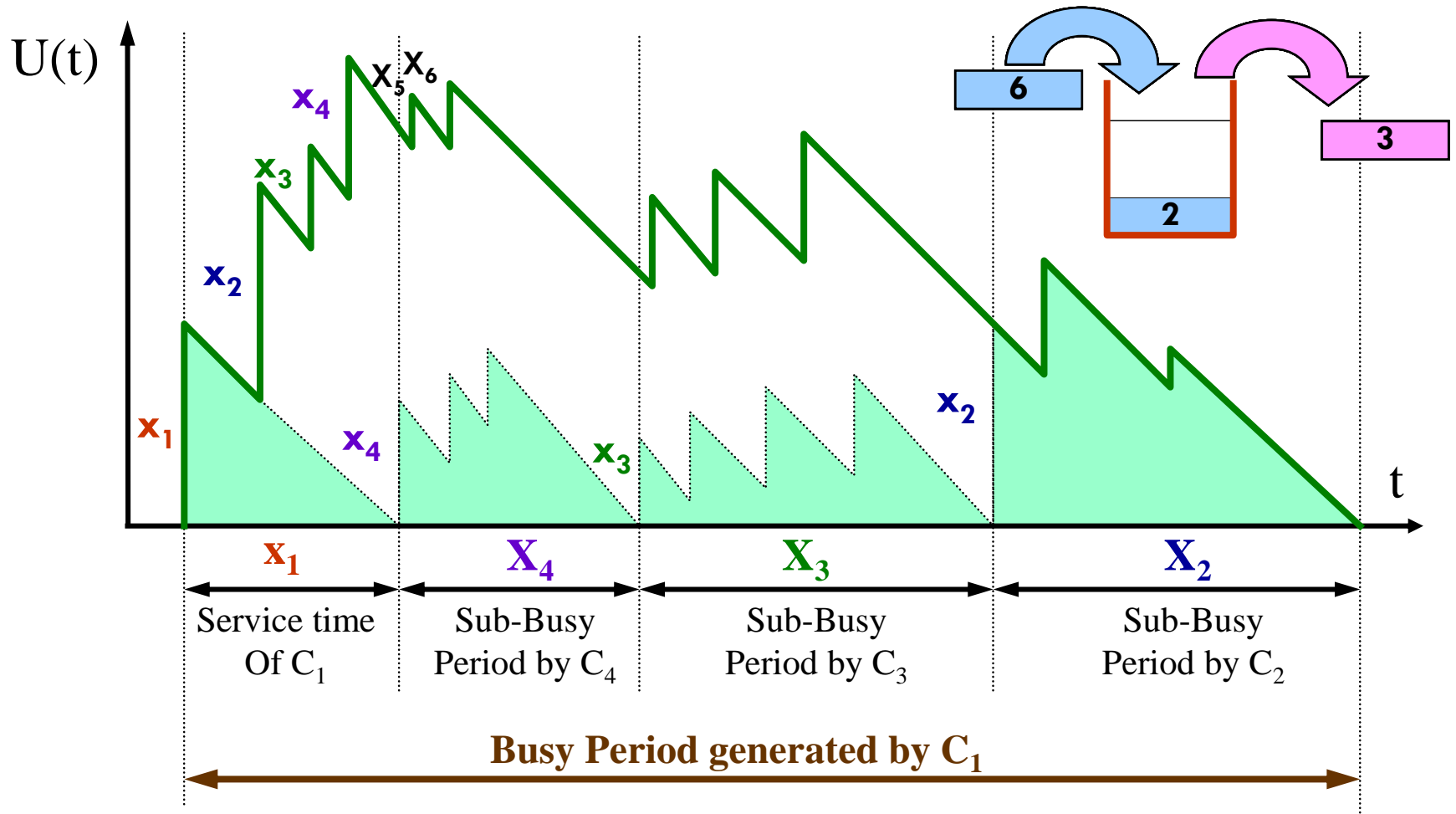
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- For Idle period $F(y)$
 - After busy period \rightarrow start the idle period
 - A new idle period will stop immediately when the new customer arrives
 - Therefore, from the memoryless distribution, Idle period distribution $F(y)$

$$F(y) = 1 - e^{-\lambda y} \quad y \geq 0$$

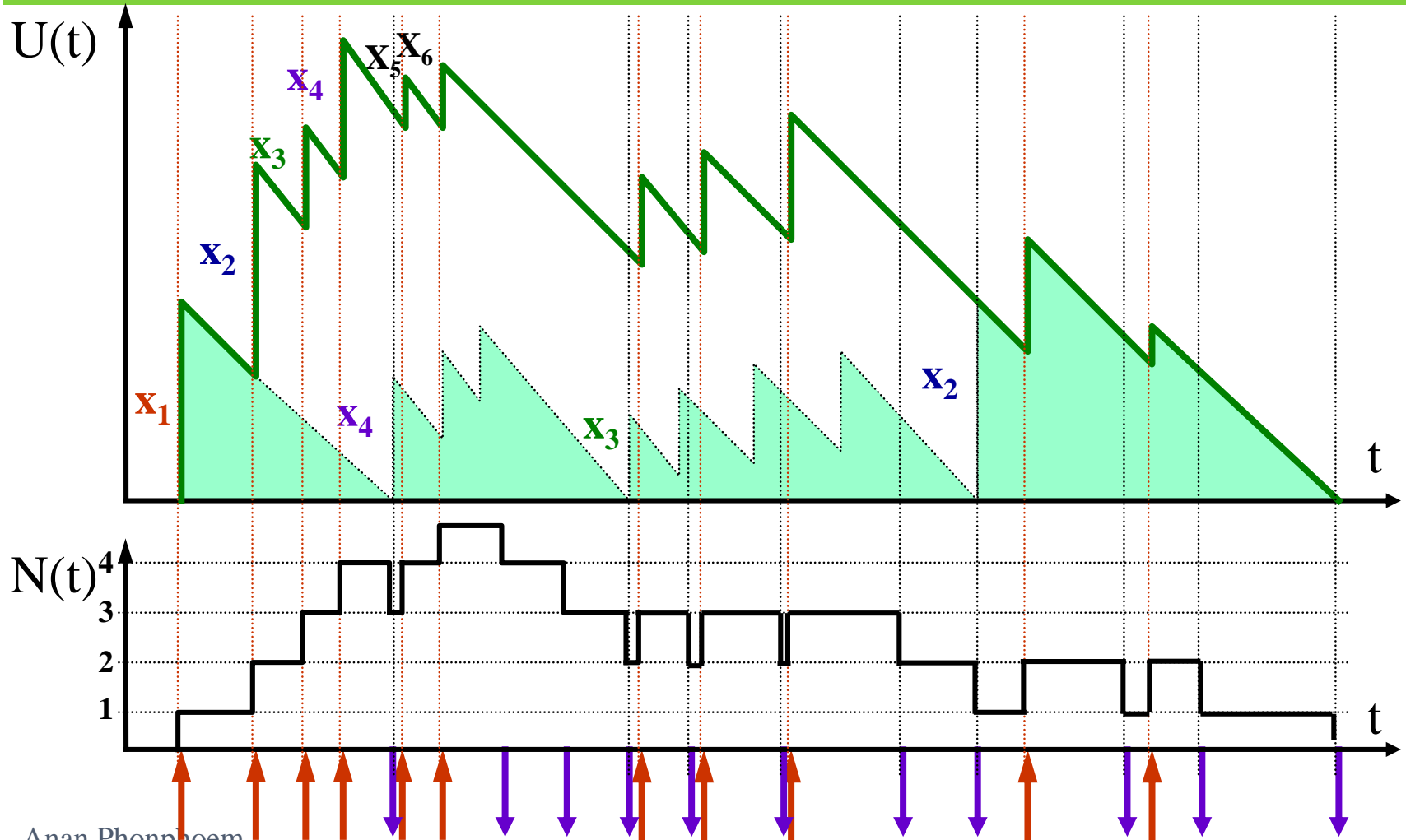
The Unfinished Work (LCFS)

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Number of Users (LCFS)

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M/G/1 (LCFS)

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- Each sub-busy period behaves statistically the same as the major busy period

- The duration of busy period Y

$$Y = X_1 + X_{v+1} + X_{v+2} + \dots + X_3 + X_2$$

- $X_v =$ sub-busy period
- $v =$ an RV = # of customer arrives during C_1 service interval

M/G/1 (LCFS)

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- For Busy period $G(y)$

$$G(y) = P[Y_n \leq y] \quad y \geq 0$$

- Transform of M/G/1 busy-period distribution

$$G^*(s) = B^*[s + \lambda - \lambda G^*(s)]$$

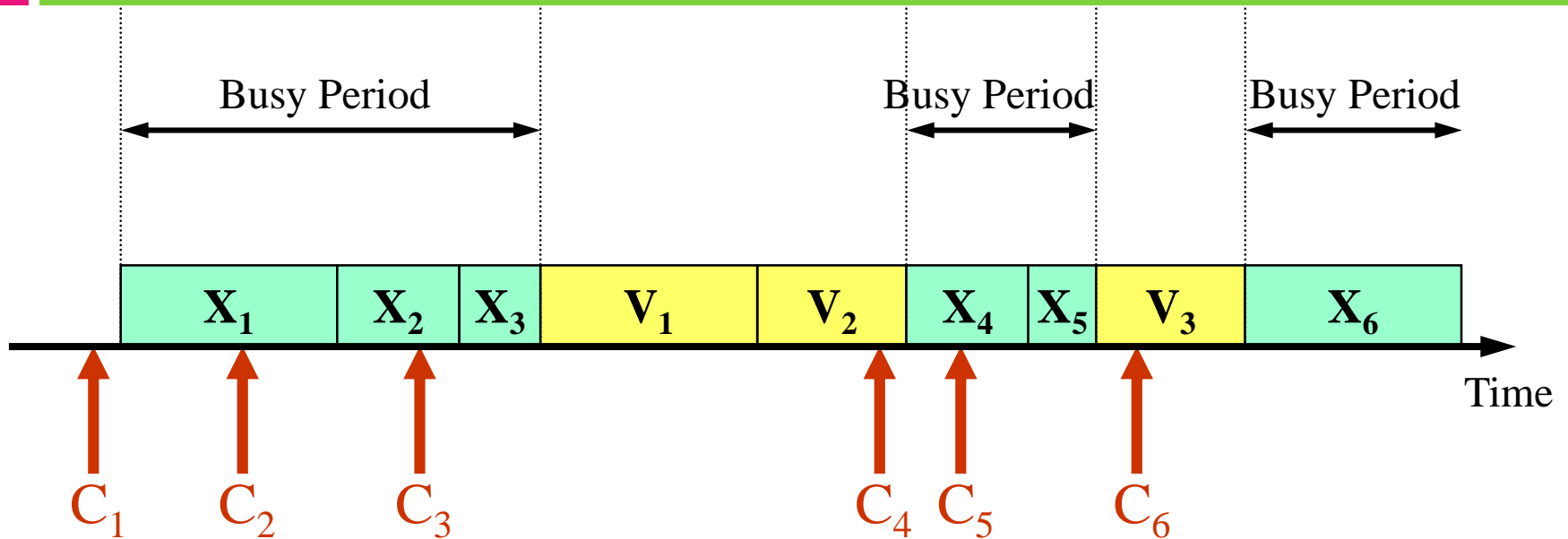
M/G/1 with Vacations

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- At the end of busy period
 - The server goes on “vacation”
 - The vacation period = random interval of time
 - A new arrive during vacation has to wait until the end of vacation period
 - If the system is idle after vacation, a new vacation starts right away

M/G/1 with Vacations

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- $V_n =$ Vacation period with \bar{V} and \bar{V}^2
= IID random variable and independent of customer interarrival and service time
- $X_n =$ Service period

M/G/1 with Vacations

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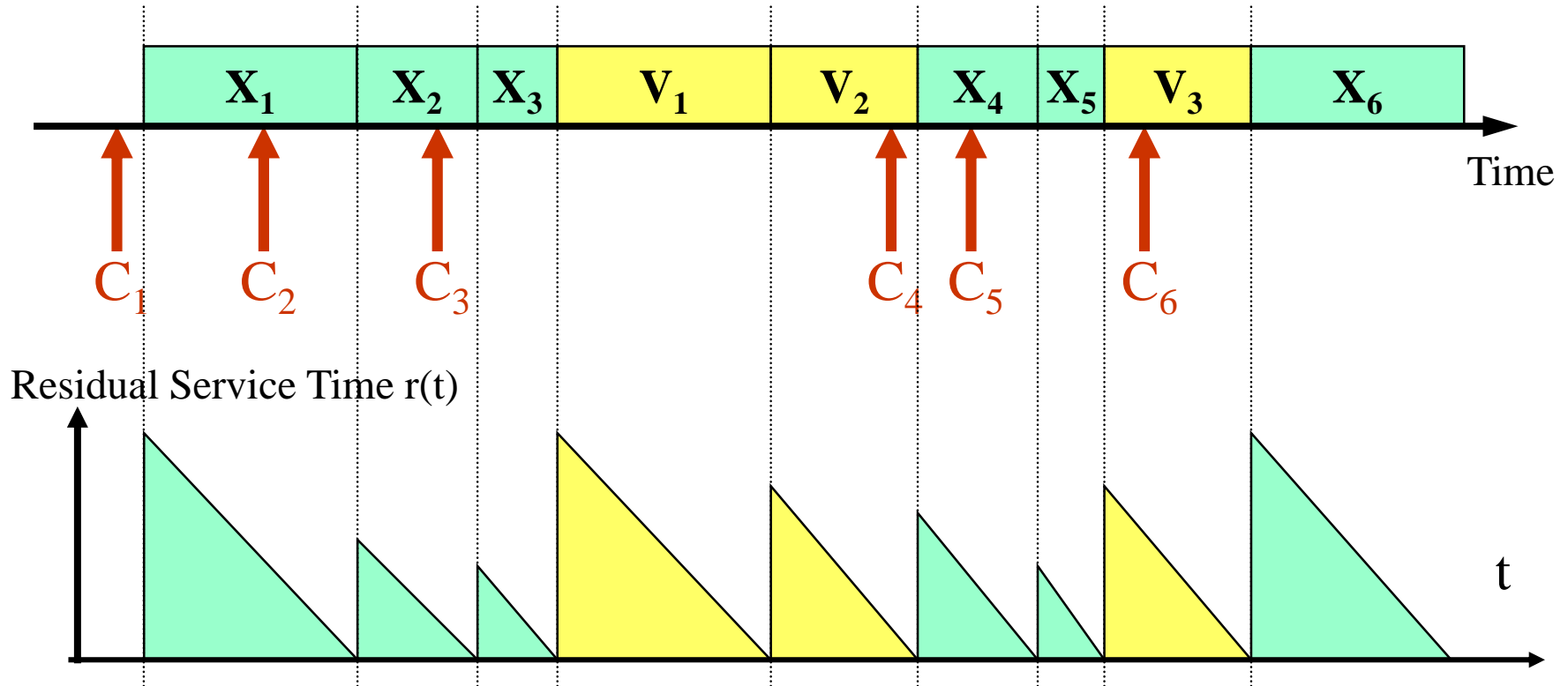
- A new customer is Poisson arrival and service time is general distribution
- The waiting time for customer is W

$$W = \frac{R}{1 - \rho}$$

- $R =$ Residual Time

M/G/1 with Vacations

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M/G/1 with Vacations

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$$\begin{aligned} R &= \frac{1}{t} \int_0^t r(\tau) d\tau = \frac{1}{t} \sum_{i=1}^{M(t)} \frac{1}{2} X_i^2 + \frac{1}{t} \sum_{i=1}^{L(t)} \frac{1}{2} V_i^2 \\ &= \frac{M(t)}{t} \frac{\sum_{i=1}^{M(t)} \frac{1}{2} X_i^2}{M(t)} + \frac{L(t)}{t} \frac{\sum_{i=1}^{L(t)} \frac{1}{2} V_i^2}{L(t)} \end{aligned}$$

M/G/1 with Vacations

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$$t \rightarrow \infty, \quad \frac{M(t)}{t} = \lambda \quad \text{and} \quad \frac{L(t)}{t} = \frac{(1-\rho)}{\bar{V}}$$

$$R = \frac{1}{2} \lambda \overline{X^2} + \frac{(1-\rho)\overline{V^2}}{2\bar{V}}$$

$$W = \frac{\lambda \overline{X^2}}{2(1-\rho)} + \frac{\overline{V^2}}{2\bar{V}}$$