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Probability Theory and Random Processes

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Lecture #8

Continuous Random Variable

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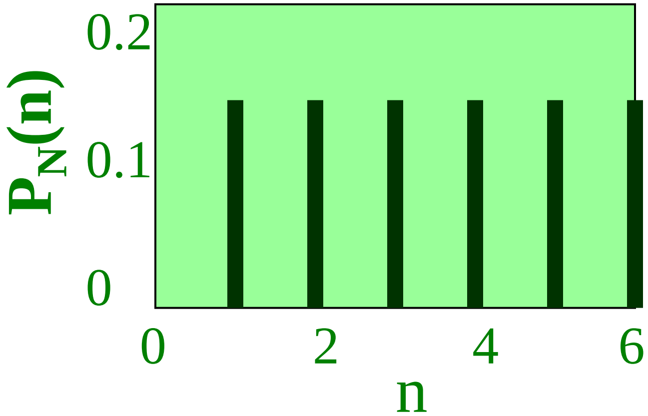
Some Useful Continuous RVs

- Uniform
- Exponential
- Gaussian

Uniform Discrete RV

- Example: PMF of **N**

$$P_N(n) = \begin{cases} 1/6 & n = 1,2,3,\ldots,6 \\ 0 & \text{Otherwise} \end{cases}$$

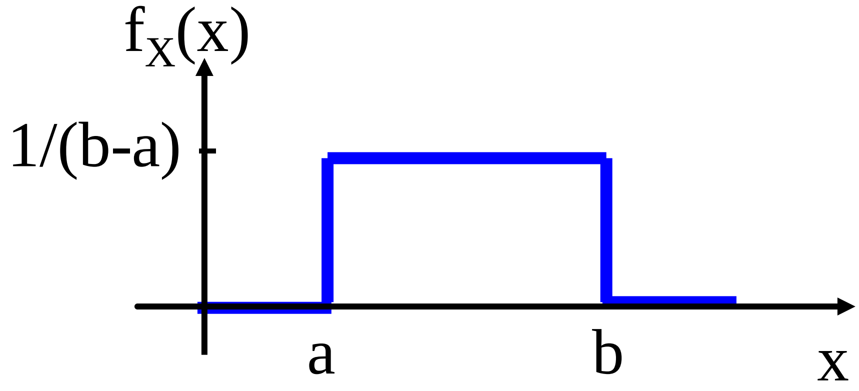


Uniform Continuous RV

Definition:

$$f_X(x) = \begin{cases} 1/(b-a) & a \le x < b \\ 0 & \text{Otherwise} \end{cases}$$

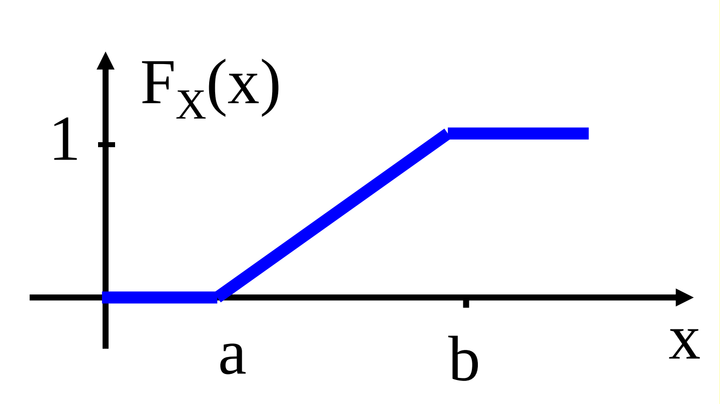
where b > a



Uniform Continuous RV

Theorem:

- $$F_X(x) = \begin{cases} 0 & x \le a \\ (x-a)/(b-a) & a < x \le b \\ 1 & x > b \end{cases}$$
- $E[X] = (b + a)/2$
- $Var[X] = (b - a)^2/12$

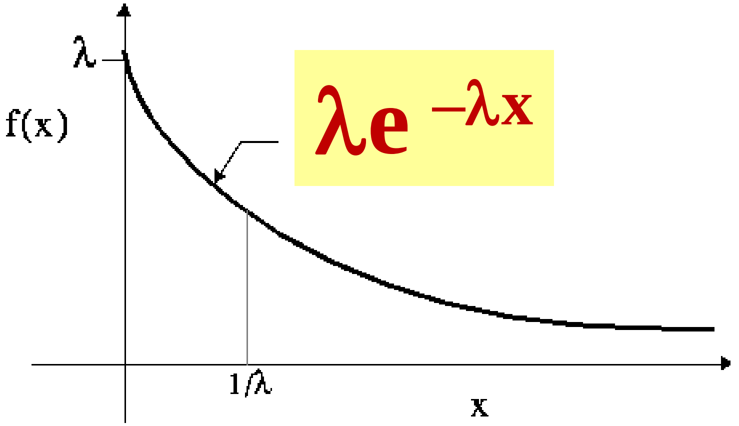


Exponential Continuous RV

Definition:

$$f_X(x) = \begin{cases} a\, e^{-ax} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

where a > 0



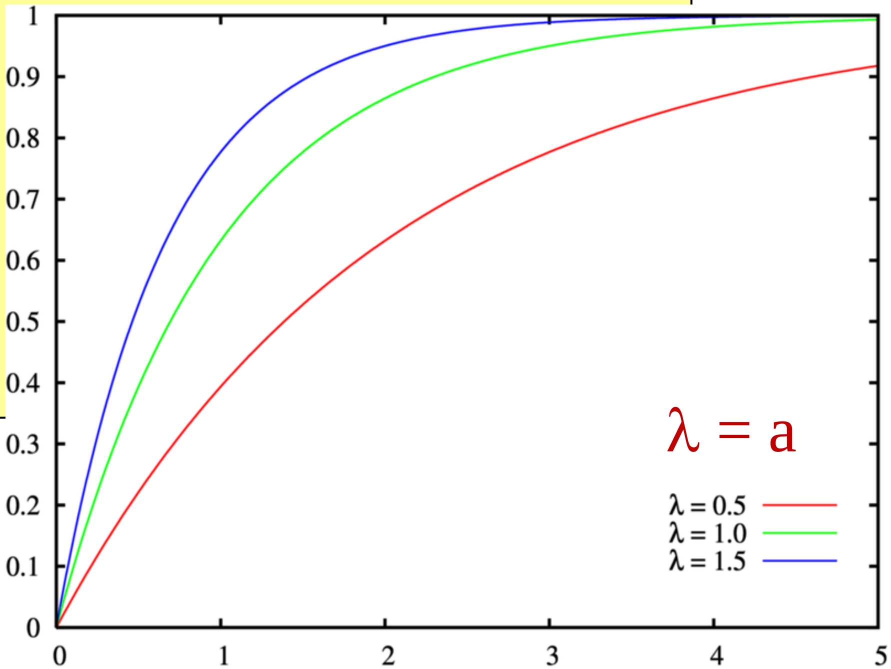
www.rzg.mpg.de/.../mc/node18.html

Exponential Continuous RV

Theorem:

$$F_X(x) = \begin{cases} 1 - e^{-ax} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

- $E[X] = 1/a$
- $Var[X] = 1/a^2$

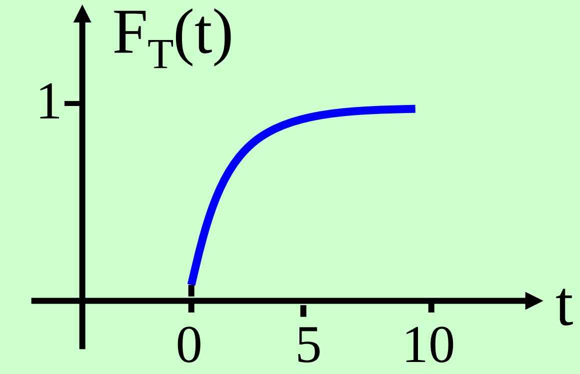


commons.wikimedia.org/wiki/File:Exponential_d...

Exponential Example

Telephone call lasts no more than t minutes

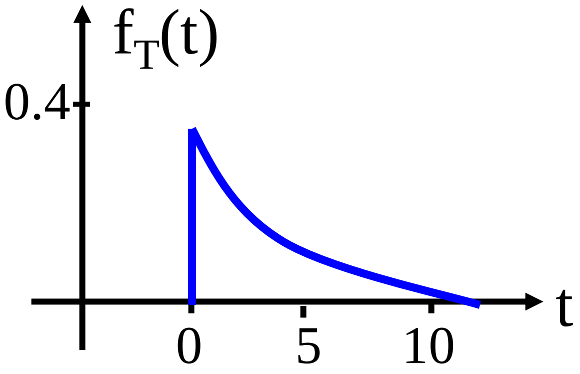
$$F_T(t) = \begin{cases} 1 - e^{-t/3} & t \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$



Find PDF

$$f_T(t) = \frac{dF_T(t)}{dt}$$

$$= \begin{cases} (1/3)\, e^{-t/3} & t \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$



Exponential Example

Find E[T]

$$E[T] = \int_{-\infty}^{\infty} t\, f_T(t)\, dt$$

$$= \int_{0}^{\infty} t\,(1/3)e^{-t/3}\, dt$$

By Parts: $uv - \int v\,du$

Let $u = t$ $\quad du = dt$

$dv = e^{-t/3}\, dt$ $\quad v = -3e^{-t/3}$

$$= (1/3)\left[(t)(-3e^{-t/3})\Big|_0^{\infty} - \int_0^{\infty} (-3e^{-t/3})dt \right]$$

$$= -t\, e^{-t/3}\Big|_0^{\infty} - (1/3)(-3)\int_0^{\infty} e^{-t/3}\, dt$$

$$= 0 + \int_0^{\infty} e^{-t/3}\, dt$$

$$= 3$$

Exponential Example

Find Var[T] $\mathrm{Var}[T] = E[T^2] - (E[T])^2$

$$E[T^2] = \int_{-\infty}^{\infty} t^2 f_T(t)\, dt$$

$$= (1/3) \int_{0}^{\infty} t^2 e^{-t/3}\, dt$$

$$= -t^2 e^{-t/3} \Big|_{0}^{\infty} + \int_{0}^{\infty} (2t) e^{-t/3}\, dt$$

$$= 2 \int_{0}^{\infty} t e^{-t/3}\, dt = 2(3E[T]) = 18$$

Exponential Example

$$\mathrm{Var}[T] = E[T^2] - (E[T])^2$$

$$= 18 - 3^2 \quad = \quad 9 \text{ min}$$

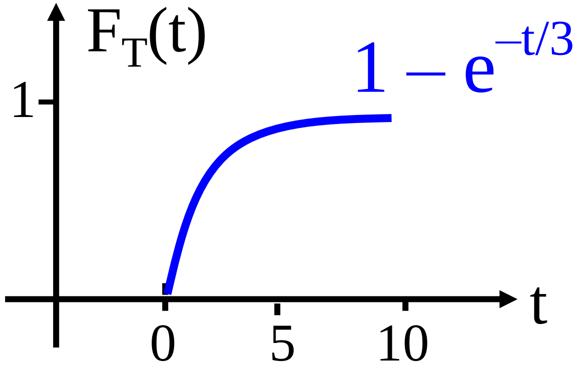
$$\sigma_T = \sqrt{\mathrm{Var}[X]} \; = 3 \text{ min}$$

Find Prob. that call duration is within <u>1 standard variation</u> ($\pm 1\ \sigma_T$)

$E[T] = 3$

$P[0 \le T \le 6] = F_T(6) - F_T(0)$

$= 1 - e^{-2} \; = 0.865$



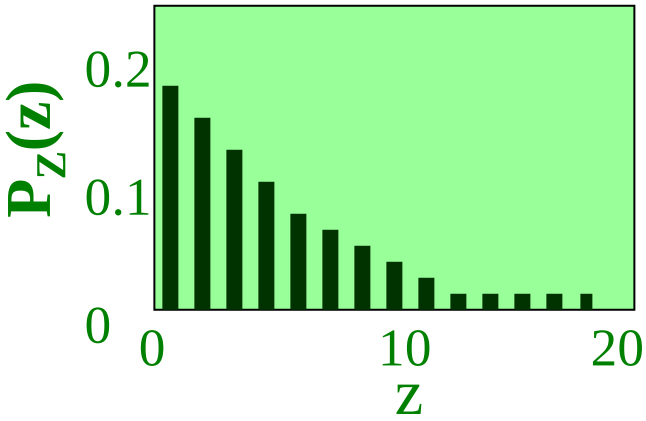
Geometric & Exponential RV

- Any relationship between Geometric & Exponential RV ?

Geometric RV

$$P_Z(z) = \begin{cases} p(1-p)^{z-1} & z = 1,2,3,\ldots \\ 0 & \text{Otherwise} \end{cases}$$

$$P_Z(z) = \begin{cases} 0.2(0.8)^{z-1} & z = 1,2,3,\ldots \\ 0 & \text{Otherwise} \end{cases}$$



Geometric & Exponential RV

Theorem: If X = Exponential RV with parameter a
Then $K = \lceil X \rceil$ is a Geometric RV
with parameter $p = 1 - e^{-a}$

Exponential RV

$$f_X(x) = \begin{cases} a e^{-ax} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

Geometric RV

$$P_K(k) = \begin{cases} p(1-p)^{k-1} & k = 1,2,3,\ldots \\ 0 & \text{Otherwise} \end{cases}$$

Geometric & Exponential RV

Theorem: If **X** = Exponential RV with parameter **a**
Then **K** = $\lceil X \rceil$ is a Geometric RV
with parameter **p = 1 – e^–a^** $p = 1 - e^{-a}$

$$P_K(k) = P[K=k] = P[k-1 < X \le k]$$

$$= F_X(k) - F_X(k-1)$$

$$= 1 - e^{-ak} - (1 - e^{-a(k-1)})$$

$$= -e^{-ak} + e^{-a(k-1)}$$

$$= e^{-a(k-1)}\left(1 - \frac{e^{-ak}}{e^{-a(k-1)}}\right)$$

$$= e^{-a(k-1)}(1 - e^{-a})$$

$$= (1-p)^{k-1}\, p$$

$$F_X(x) = \begin{cases} 1 - e^{-ax} & x \ge 0 \\ 0 & \text{Otherwise} \end{cases}$$

$$p = (1 - e^{-a}) \Rightarrow e^{-a} = (1 - p)$$

$$P_K(k) = \begin{cases} p(1-p)^{k-1} & k = 1,2,3,\ldots \\ 0 & \text{Otherwise} \end{cases}$$

Example

- Phone Company A:
  - 3 Baht / min.
  - With full min. charge
- Phone Company B:
  - 3 Baht / min.
  - With exact charge
- Let T = duration of call
- T: exponential with a = 1/3

http://4.bp.blogspot.com/_zTL2b_tTBzc/SRw6zH813PI/AAAAAAAAAcI/vtJoIFOTe2c/s400/telephone_call.bmp

Example

- E[T] = 1/a = 3 min.
- R: money received per call
- For Company B:

  E[R] = 3 E[**T**] = **9** Baht/Call
- For Company A:

  E[R] = 3 E[**K**]

  where K = $\lceil T \rceil$ → geometric with $p = 1 - e^{-1/3}$

  E[R] = 3 (1/p) = 3 (3.53)

  = **10.59** Baht/Call

http://cdn.cutestpaw.com/wp-content/uploads/2011/11/Surprise-l.jpg