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Probability Theory and Random Processes

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Lecture #13

Law of large number & Central limit theorem

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Outline

- Law of Large Numbers
  - Weak law
  - Strong law
- Central Limit Theorem

Law of Large Numbers (LLN)

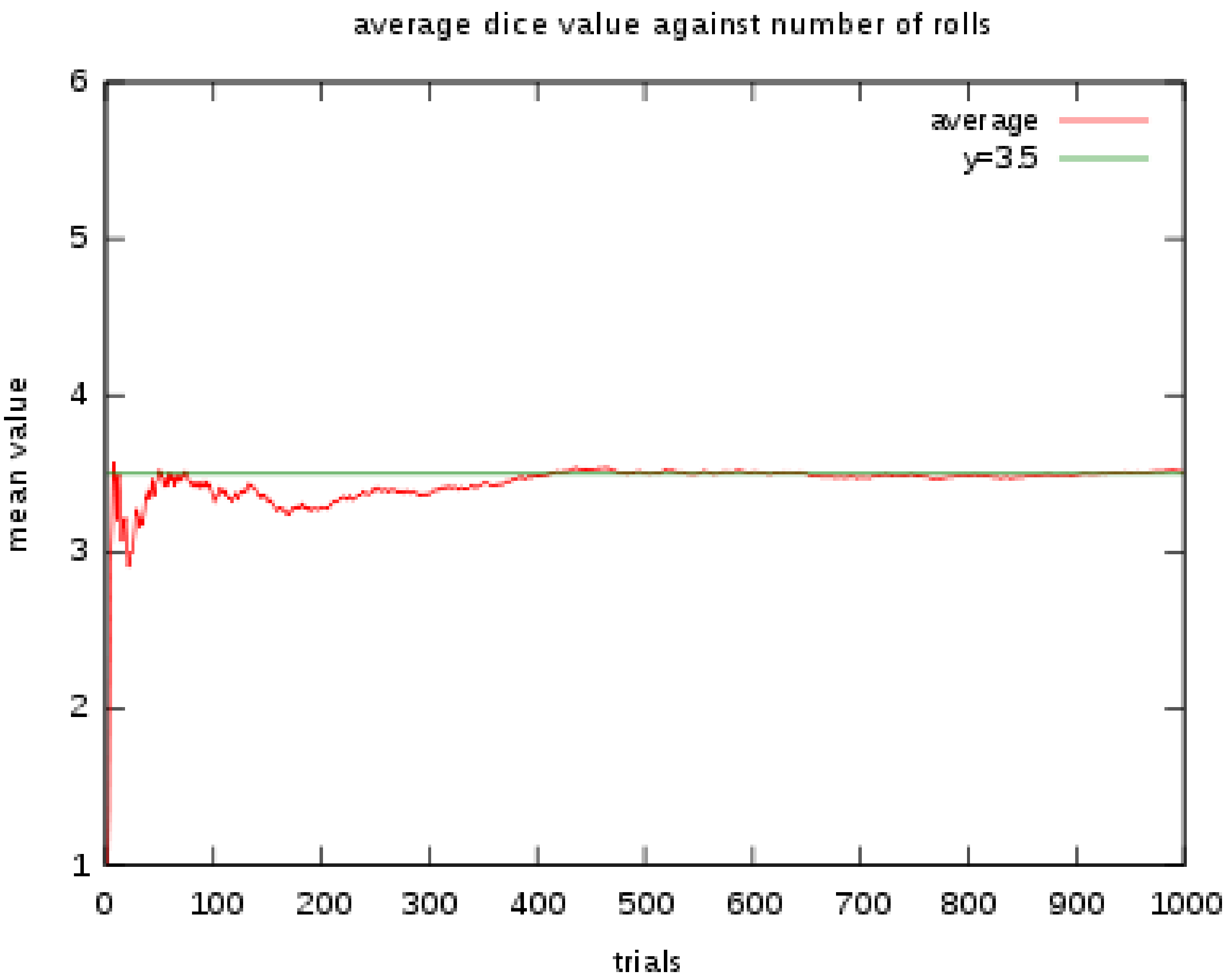
- Performing the same experiment a large number of times
- The average of the results obtained from a large number of trials should be closed to the expected value
- Tend to become closer as more trials are performed

Example 1

- Row a fair die
- 5 times, 10 times, 50 times
- The expect value is
  - (1+2+3+4+5+6)/6 = 3.5
- From the law of large numbers
  - If a large number of dice are rolled
  - The average is likely to be close to 3.5
  - More trials → More accuracy

Example 1

http://en.wikipedia.org/wiki/Law_of_large_numbers



Law of Large Numbers

- Let X be a random variable
  - Unknown mean, $E(X) = \mu$
- For n independent trials
  - $X_1, X_2, X_3, \ldots, X_n$
  - X_j are IID

Sample mean, $$M_n = \frac{1}{n}\sum_{j=1}^{n} X_j$$

Sample Mean

Two properties of a good estimator

1) On average, should provide the correctness

$$E[M_n] = \mu$$

2) Should not vary too much

$$E[(M_n - \mu)^2] \rightarrow \varepsilon$$

Variance of Sample Mean

$$M_n = \frac{S_n}{n} \Rightarrow Var[M_n] = \frac{1}{n^2} Var[S_n]$$

$$Let \qquad S_n = X_1 + X_2 + \ldots + X_n$$

$$Var[S_n] = Var[X_1] + Var[X_2] + \ldots + Var[X_n]$$

$$= nVar[X_j] = n\sigma^2$$

$$Var[M_n] = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

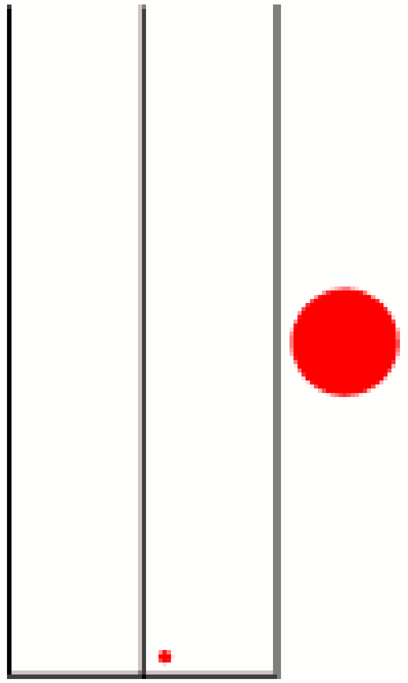
For n →∞, Var[M_n] → 0

Example 2

http://en.wikipedia.org/wiki/Law_of_large_numbers

Flip a coin

- Red for head
- Blue for tail
- A pie chart shows the proportion of red and blue so far
- The proportion varies a lot at first, but gradually approaches 50%

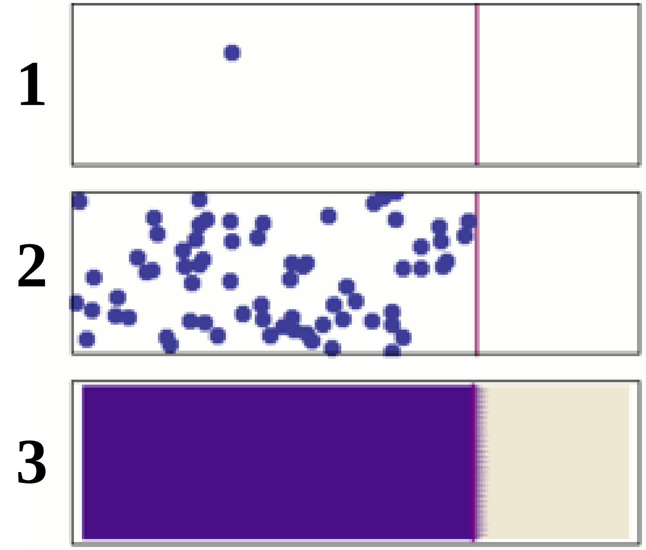


Example 3

http://en.wikipedia.org/wiki/Law_of_large_numbers

Diffusion in applied chemistry

1. With a **single** molecule
   - The motion appears to be quite random
2. With **more** molecules
   - Solute trends to fill the container more uniformly (random fluctuations)



3. With an **enormous** number of solute molecules
   - Randomness is essentially gone
   - Solute appears to move smoothly and systematically from high-concentration areas to low-concentration areas

Law of Large Numbers

- **Weak law**

$$E[M_n] \xrightarrow{p} \mu \quad for \quad n \to \infty$$ **Convergence in probability**

$$\lim_{n \to \infty} P[|M_n - \mu| < \varepsilon] = 1$$

- **Strong law**

$$E[M_n] \xrightarrow{a.s.} \mu \quad for \quad n \to \infty$$ **Convergence almost surely**

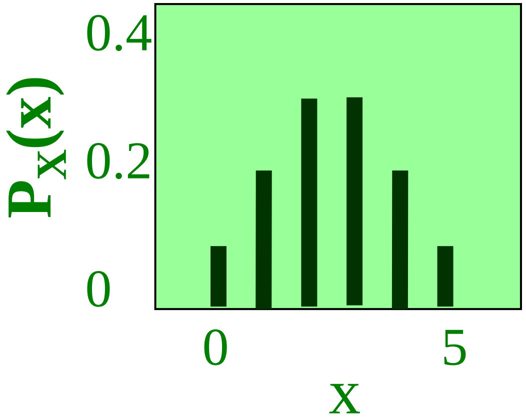
$$P[\lim_{n \to \infty} M_n = \mu] = 1$$

Central Limit Theorem (CLT)

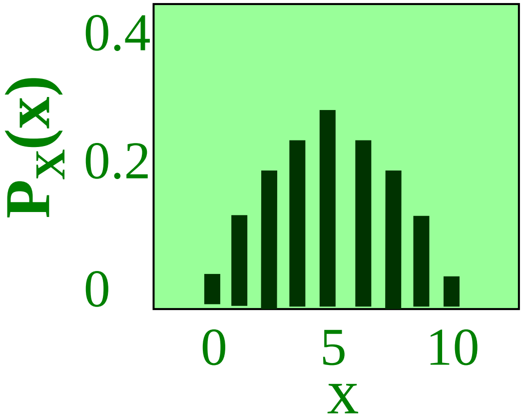
- The central limit theorem
  - **Mean of a sufficiently large number** of independent random variables
  - With finite mean and variance
  - Will be **approximately Normally Distributed (Bell-shaped curve)**
- Requires
  - Random variables to be **identically** distributed (unless certain conditions are met)

Example

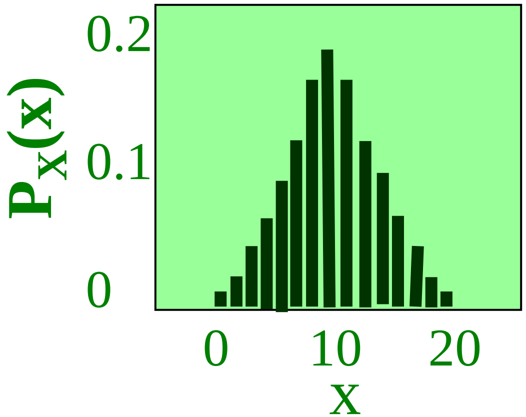
- Flip a coin
- X = # of heads in n coin flips
- Binomial (n,1/2)
- n → large, PMF → bell-shaped curve



n = 5



n = 10



n = 20

Central Limit Theorem

$$Let \qquad W_n = X_1 + X_2 + \ldots + X_n$$

$$For \quad n \to \infty$$

$$E[W_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$$

$$= n\mu_x$$

$$Var[W_n] = Var[X_1] + Var[X_2] + \ldots + Var[X_n]$$

$$= nVar[X]$$

Finding the convergence of CDF, $F_{W_n}(w)$, is not easy

Central Limit Theorem

Definition: in term of standardized RV. for all n

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu_X}{\sqrt{n\sigma_X^2}}$$

$$E[Z_n] = 0 \qquad Var[Z_n] = 1$$

Central Limit Theorem

Theorem: Given X_1, X_2,… is a sequence of IID random variable with expected value μ_X and variance σ^2_X, the **CDF of Z_n** has the property

$$\lim_{n \to \infty} F_{Z_n}(z) = \Phi(z)$$

To use the CLT

- For IID sum $W_n = X_1 + X_2 + \ldots + X_n$

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu_X}{\sqrt{n\sigma_X^2}} \Rightarrow Z_n = \frac{W_n - n\mu_X}{\sqrt{n\sigma_X^2}}$$

$$W_n = Z_n\sqrt{n\sigma_X^2} + n\mu_X$$

To use the CLT

The CDF of W_n can be expressed in terms of the CDF of Z_n as

$$F_{W_n}(w) = P[Z_n \sqrt{n\sigma_X^2} + n\mu_X \le w]$$

$$F_{W_n}(w) = P[Z_n \le \frac{w - n\mu_X}{\sqrt{n\sigma_X^2}}]$$

$$= F_{Z_n}\left(\frac{w - n\mu_X}{\sqrt{n\sigma_X^2}}\right) \Rightarrow \lim_{n \to \infty} F_{Z_n}(z) \approx \Phi(z)$$

Central Limit Theorem Approximation (Guassian approximation)

Definition: Let $W_n = X_1 + X_2 + \ldots + X_n$ be the sum of n IID random variables, each with expected value μ_X and variance σ^2_X, the **CLT approximation to the CDF of W_n** is

$$F_{W_n}(w) \approx \Phi\left(\frac{w - n\mu_X}{\sqrt{n\sigma_X^2}}\right)$$

Example

- A modem transmits one millions bits
- Each bit is “1” or “0” independently with equal probability
- Estimate the probability of at least 502,000 ones

Example

<u>Solution</u>

- Let X_i =be value of “1” or “0”

$$W = \sum_{i=1}^{10^6} X_i$$

- X_i is Bernoulli (0.5) → $E[X_i] = 0.5$ and $Var[X_i] = 0.25$
- $E[W] = 10^6 E[X_i] = 500,000$
- $Var[W] = 10^6 Var[X_i] = 250,000$
- $\sigma_W = 500$

Example

- By CLT approximation,

$$P[W \geq 502{,}000] = 1 - P[W \leq 502{,}000]$$

$$\approx 1 - \Phi\left(\frac{502{,}000 - 500{,}000}{500}\right) = 1 - \Phi(4)$$

- Using the table ➔ answer = Q(4) = 3.17×10^{-5}