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Probability Theory and Random Processes

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Lecture #13

Stochastic Process – II

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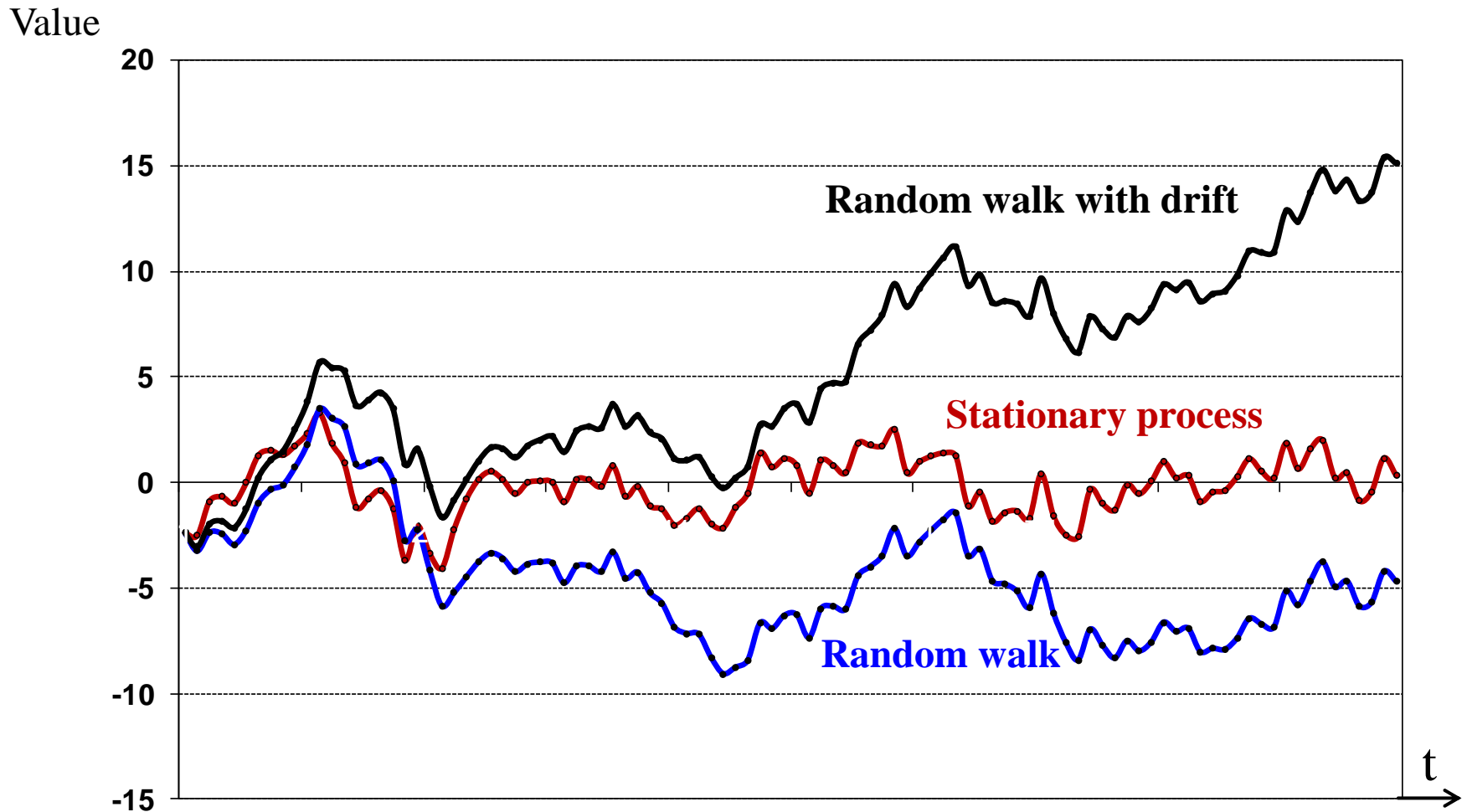
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Outline

- Stationary Process
- Wide-sense Stationary Process

Stationary and Non-stationary Process



Stationary Process

- For a random process $X(t)$,
 - At t_1 : $X(t_1)$ has pdf = $f_{X(t_1)}(\mathbf{x})$
 - Normally, pdf **depends** on t_1
 - However, pdf **may not depend** on t_1
- **Stationary Process**
 - Same random variable at all time
 - No statistical properties change with time

$$f_{X(t_1)}(\mathbf{x}) = f_{X(t_1 + \tau)}(\mathbf{x}) = f_X(\mathbf{x})$$

Stationary Process

Definition: A stochastic process $X(t)$ is stationary iff for all sets of time t_1, \dots, t_m and any time different τ ,

$$f_{X(t_1), \dots, X(t_m)}(x_1, \dots, x_m) = f_{X(t_1 + \tau), \dots, X(t_m + \tau)}(x_1, \dots, x_m)$$

Stationary Random Sequence

Definition: A random sequence X_n is stationary iff for any finite sets of time instants n_1, \dots, n_m and any time different k ,

$$f_{X(n_1), \dots, X(n_m)}(x_1, \dots, x_m) = f_{X(n_1+k), \dots, X(n_m+k)}(x_1, \dots, x_m)$$

Stationary Process

Theorem: A stationary process $X(t)$,

$$\begin{aligned}\mu_X(t) &= \mu_X \\ R_X(t, \tau) &= R_X(0, \tau) = R_X(\tau) \\ C_X(t, \tau) &= R_X(\tau) - \mu_X^2 = C_X(\tau)\end{aligned}$$

autocorrelation
of a stationary process
(independent of t)

autocovariance
of a stationary process
(independent of t)

Stationary Random Sequence

Theorem: A stationary random sequence X_n , for all m

$$E[X_m] = \mu_X$$

$$R_X[m,k] = R_X[0,k] = R_X[k]$$

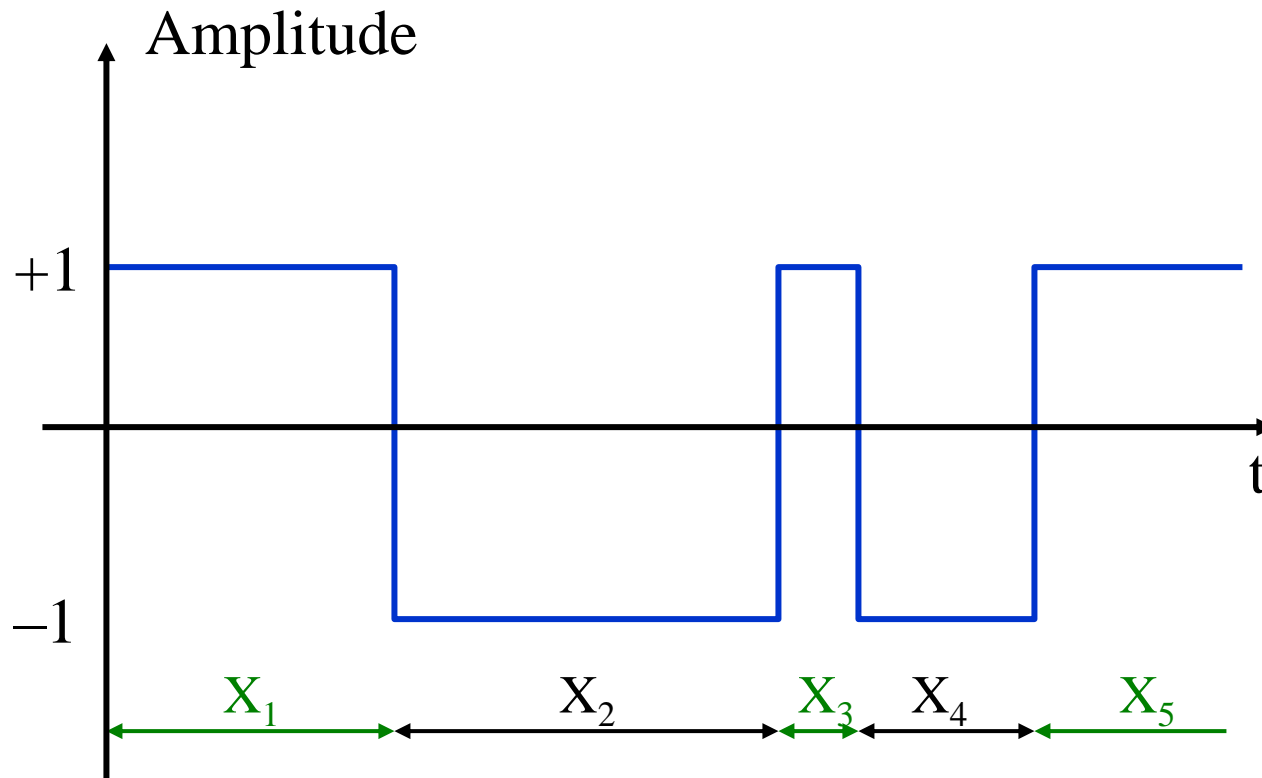
$$C_X[m,k] = R_X[k] - \mu_X^2 = C_X[k]$$

Example

From “Probability and Random Process for Electrical Engineering”,
2nd Edition, by Alberto Leon-Garcia

- Telegraph Signal, $X(t)$ take value ± 1
- $X(0) = \pm 1$ with probability = 0.5
- Let $X(t)$ toggles the polarity with each occurrence of an event in a Poisson process rate α

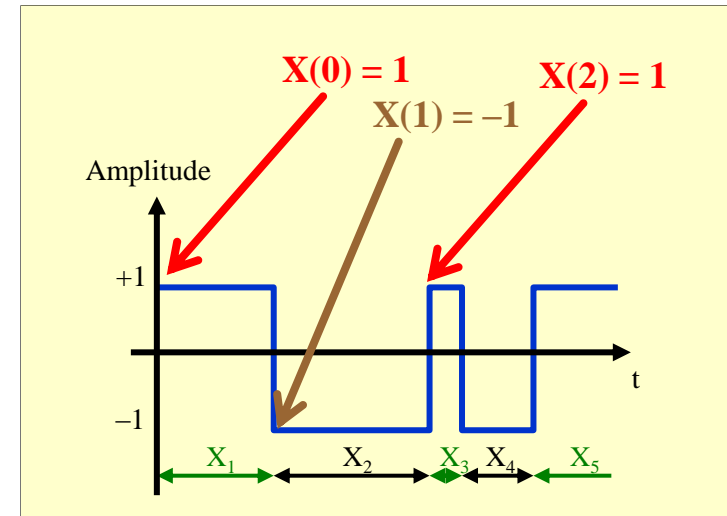
Example



Example

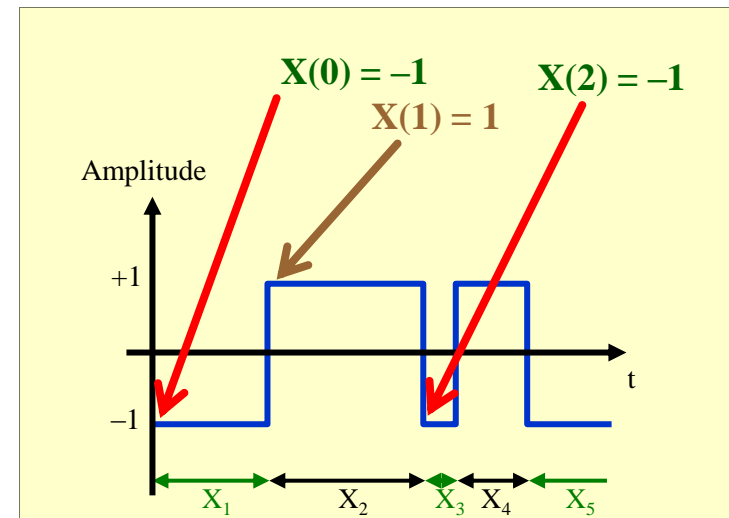
- Find PMF of $X(t)$, $f_{X(t)}(x)$
- $P[X(t) = 1] = P[X(t) = 1 | X(0) = 1] P[X(0) = 1] + P[X(t) = 1 | X(0) = -1] P[X(0) = -1]$

- $P[X(t) = 1 | X(0) = 1]$
 $= P[N(t) = \text{even}]$
 $= \sum_{j=0}^{\infty} \frac{(\alpha t)^{2j}}{(2j)!} e^{-\alpha t}$
 $= e^{-\alpha t} (1/2) (e^{\alpha t} + e^{-\alpha t})$
 $= (1/2) (1 + e^{-2\alpha t})$



Example

- $P[X(t) = 1 \mid X(0) = -1]$
 $= P[N(t) = \text{odd}]$
 $= \sum_{j=0}^{\infty} \frac{(\alpha t)^{2j+1}}{(2j+1)!} e^{-\alpha t}$
 $= e^{-\alpha t} (1/2) (e^{\alpha t} - e^{-\alpha t})$
 $= (1/2) (1 - e^{-2\alpha t})$



Example

- $P[X(t) = 1]$
= $P[X(t) = 1 \mid X(0) = 1] P[X(0) = 1]$
+ $P[X(t) = 1 \mid X(0) = -1] P[X(0) = -1]$
= $(1/2) (1 + e^{-2\alpha t})(1/2) + (1/2) (1 - e^{-2\alpha t})(1/2)$
= $1/2$
- $P[X(t) = -1]$
= $1 - P[X(t) = 1] = 1/2$

$$f_{X(t)}(x) = \begin{cases} 1/2 & x = -1, 1 \\ 0 & \text{Otherwise} \end{cases}$$

$X(t)$ is a stationary process

Example

- $\mu_X(t) = 1(1/2) + (-1)(1/2) = 0$
- $\text{Var} [X(t)] = E[X^2(t)] - \mu_X(t)$
 $= 1^2(1/2) + (-1)^2(1/2) = 1$

$$f_{X(t_1), \dots, X(t_m)}(x_1, \dots, x_m) =$$
$$f_{X(t_1 + \tau), \dots, X(t_m + \tau)}(x_1, \dots, x_m)$$

Wide Sense Stationary

Definition: $X(t)$ is a wide sense stationary random process iff for all t ,

$$E[X(t)] = \mu_X$$

$$R_X(t, \tau) = R_X(0, \tau) = R_X(\tau)$$

~~$$C_X(t, \tau) = R_X(\tau) - \mu_X^2 - C_X(\tau)$$~~

Definition: X_n is a wide sense stationary random sequence iff for all n ,

$$E[X_n] = \mu_X$$

$$R_X[n, k] = R_X[0, k] = R_X[k]$$

Wide Sense Stationary

- For every **stationary** process or sequence, it is also **wide sense stationary**.
- However, if it is a **wide sense stationary** it may or may not be **stationary**.

Example

- For $n = \text{even}$
 - $X_n = \pm 1$ with probability value = 0.5
- For $n = \text{odd}$
 - $X_n = -1/3$ with probability value = 0.9
 - $X_n = 3$ with probability value = 0.1

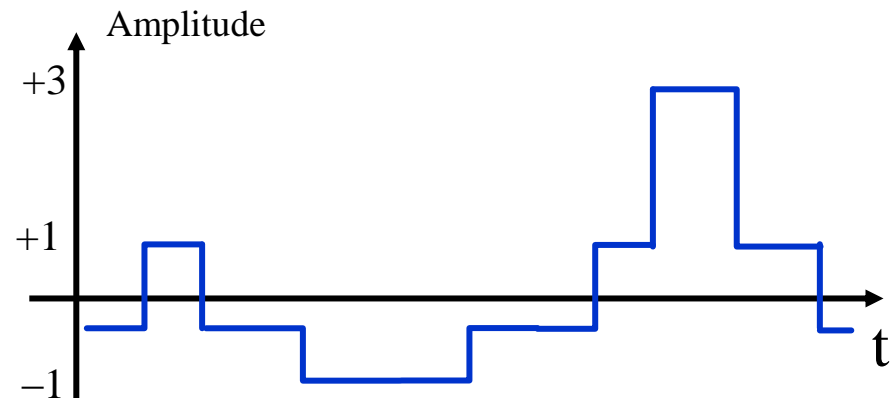
- Stationary ?

- **No**

- Wide sense stationary ?

- Mean = 0 for all n
 - $C_X(t, \tau) = 0$ for $\tau > 0$
 - $C_X(t, \tau) = 1$ for $\tau = 0$

- **Yes , it's wide sense stationary**



Wide Sense Stationary

Theorem: For a wide sense stationary process $X(t)$,

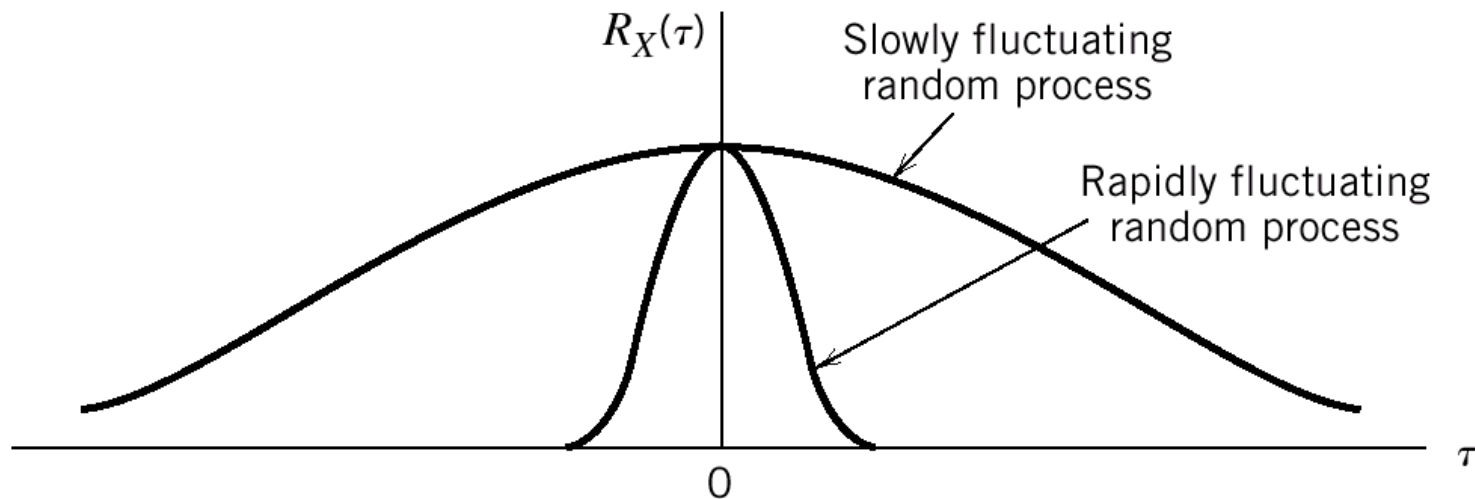
$$R_X(0) \geq 0$$

$$R_X(\tau) = R_X(-\tau)$$

$$|R_X(\tau)| \leq R_X(0)$$

$R_X(\tau)$

- The $R_X(\tau)$ provides the interdependence information of two random variables obtained from $X(t)$ at times τ seconds apart



<http://cc.ee.ntu.edu.tw/~wujsh/PC%20Chapter1.ppt>

Wide Sense Stationary

Theorem: For a wide sense stationary sequence X_n ,

$$R_X[0] \geq 0$$

$$R_X[k] = R_X[-k]$$

$$|R_X[k]| \leq R_X[0]$$

Average Power

- From Ohm's Law : $V = IR$
- For $v(t)$, $i(t)$, and R
 - The instantaneous power dissipated $P(t)$
$$P(t) = v^2(t)/R = i^2(t)R$$
- For $R = 1 \Omega$, $P(t) = v^2(t) = i^2(t)$
- For a voltage or current is a sample function of random process, $x(t,s)$
 - P across 1Ω resistor = $x^2(t,s)$

Average Power

- **Define** $x^2(t,s)$
 - as the instantaneous power of $x(t,s)$
- For a $X(t)$,
 - $X^2(t)$ is the instantaneous of power $X(t)$

Definition:

The **average power** of a wide sense stationary process $X(t)$ is

$$R_X(0) = E[X^2(t)]$$