01204312
Probability Theory and Random Processes

Department of Computer Engineering, Faculty of Engineering,
Kasetsart University, THAILAND

1st semester 2012
(June – September)

Lecture #9 – 1
Mixed Random Variable

Assoc. Prof. Anan Phonphoem, Ph.D.
http://www.cpe.ku.ac.th/~anan

Department of Computer Engineering, Faculty of Engineering,
Kasetsart University, THAILAND

Mixed Random Variable

- Discrete RV → PMF & Summation
- Continuous RV → PDF & Integral
- Combination of Discrete and Continuous RV
  - Unit impulse function
  - Can use same formulas to describe both RVs

Unit Impulse Function

- Delta Function : $\delta(x)$

Definition:

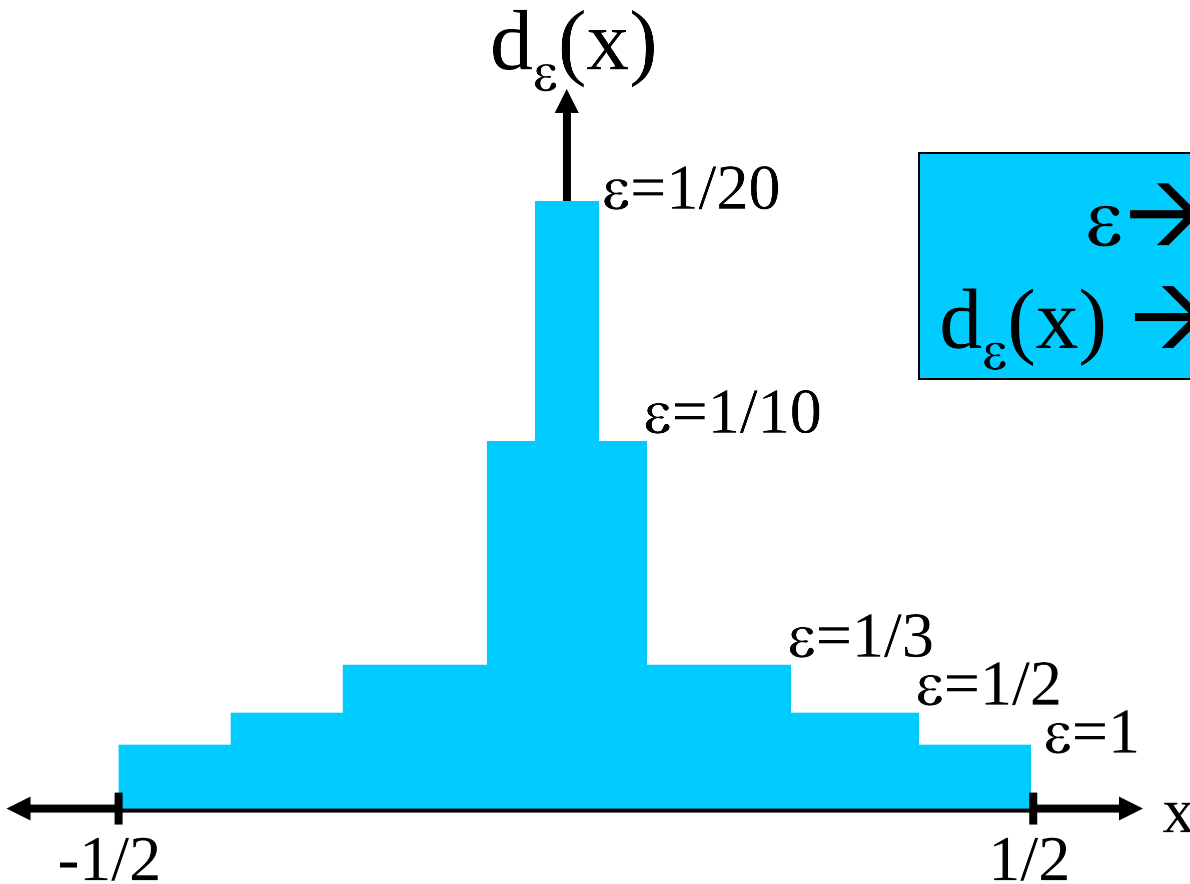
Let

$$d_\varepsilon(x) = \begin{cases} 1/\varepsilon & -\varepsilon/2 \le x \le \varepsilon/2 \\ 0 & \text{Otherwise} \end{cases}$$

Then

$$\delta(x) = \lim_{\varepsilon \to 0} d_\varepsilon(x)$$

Delta Function



$\varepsilon \to 0$

$d_\varepsilon(x) \to \delta(x)$

No mathematically meaning **but very useful**

Delta Function

$$\int_{-\infty}^{\infty} d_\varepsilon(x)\, dx = \int_{-\varepsilon/2}^{\varepsilon/2} \frac{1}{\varepsilon}\, dx = 1$$

As $\varepsilon \rightarrow 0$, $d_\varepsilon(x) \rightarrow \delta(x)$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

Special case of

1 0

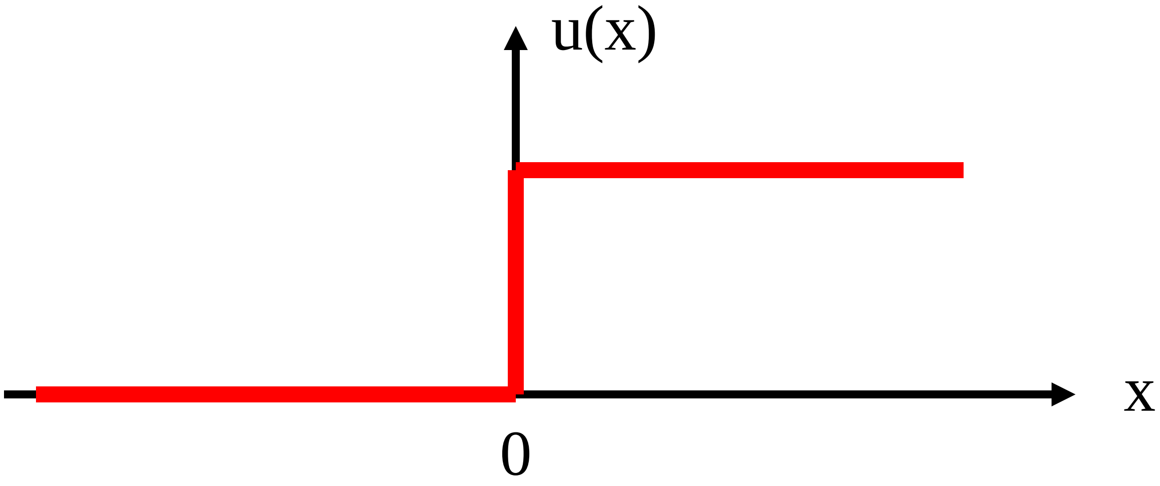
Theorem: (Sifting Property)

$$\int_{-\infty}^{\infty} g(x)\, \delta(x-x_0)\, dx = g(x_0)$$

Unit Step Function

Definition:

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



Unit Step Function

$$\int_{-\infty}^{-x} d_{\varepsilon}(v)\, dv = 0 \qquad \int_{-\infty}^{x} d_{\varepsilon}(v)\, dv = 1$$

For $x \neq 0$, $\varepsilon \rightarrow 0$

$$\int_{-\infty}^{x} d_{\varepsilon}(v)\, dv = u(x)$$

Theorem:

$$\int_{-\infty}^{x} \delta(v)\, dv = u(x)$$

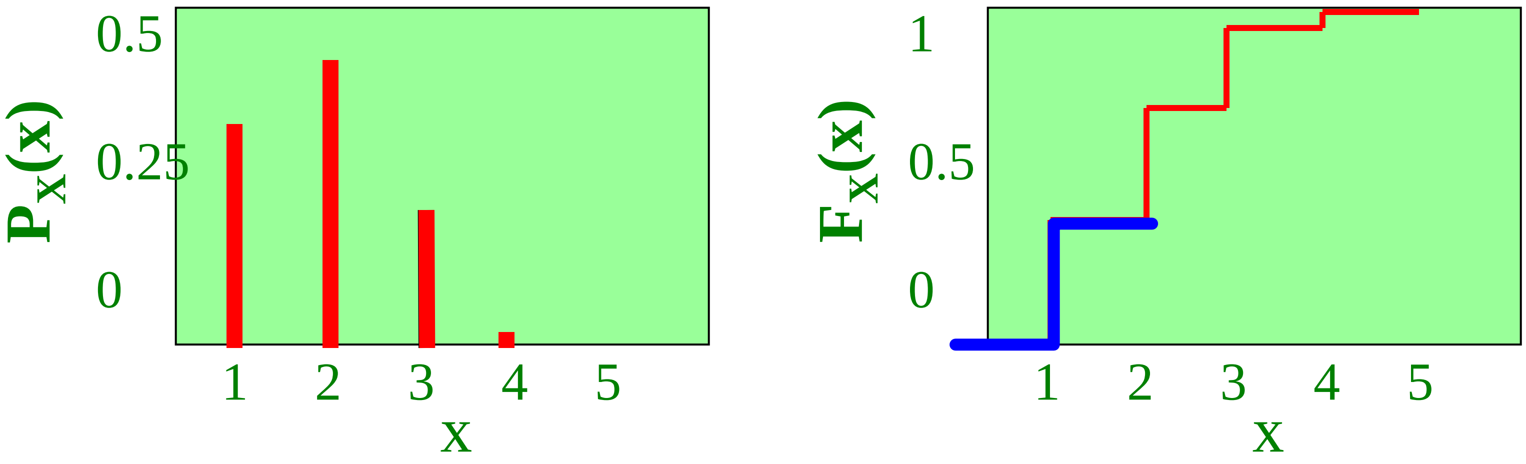
For x = 0 ??

Not exist

$$\delta(x) = \frac{d\, u(x)}{dx}$$

PMF → PDF

$$F_X(x) = \sum_{x_i \in S_X} P_X(x_i)\, u(x-x_i)$$

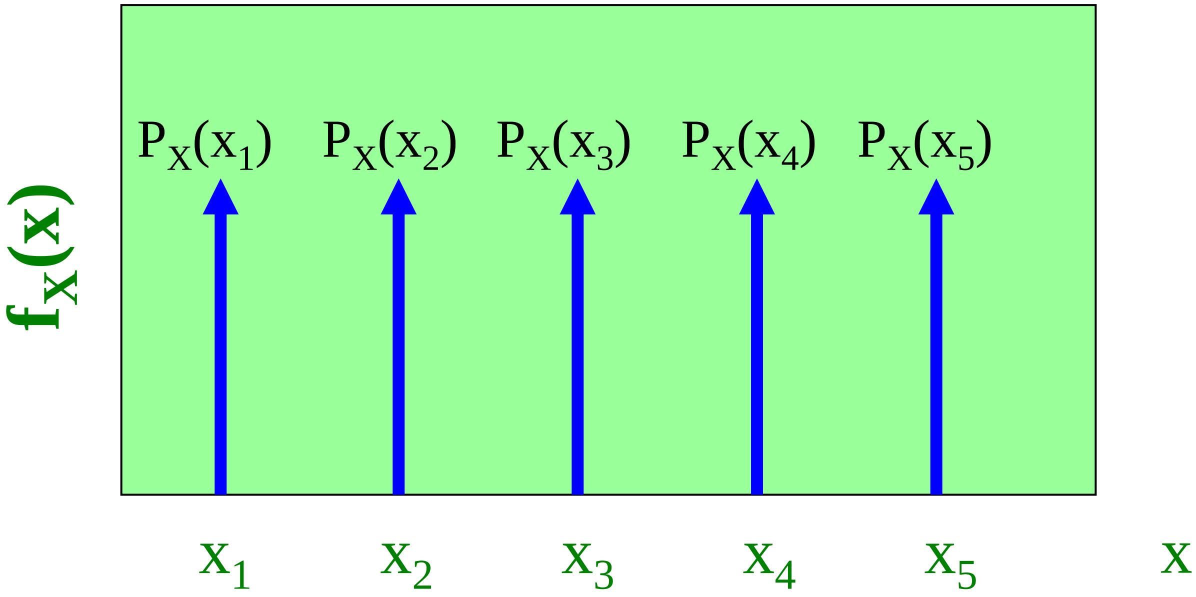


$u(x-x_i)$ → $u(x)$ shift to x_i

PMF → PDF

$$F_X(x) = \sum_{x_i \in S_X} P_X(x_i)\, u(x-x_i)$$

$$f_X(x) = \sum_{x_i \in S_X} P_X(x_i)\, \delta(x-x_i)$$



PMF → PDF

$$f_X(x) = \sum_{x_i \in S_X} P_X(x_i)\, \delta(x-x_i)$$

$$E[X] = \int_{-\infty}^{\infty} x \sum_{x_i \in S_X} P_X(x_i)\, \delta(x-x_i)\, dx$$

$$= \sum_{x_i \in S_X} \int_{-\infty}^{\infty} x\, P_X(x_i)\, \delta(x-x_i)\, dx$$

$$= \sum_{x_i \in S_X} x_i\, P_X(x_i)$$

Sifting property

$$\int_{-\infty}^{\infty} g(x)\, \delta(x-x_0)\, dx = g(x_0)$$

PMF ←→ PDF

Theorem :

- $P[X = x_0] = q$
- $P_X(x_0) = q$
- $F_X(x_0^+) - F_X(x_0^-) = q$ Discontinuity at x_0
- $f_X(x_0) = q\,\delta(0)$ ➔ $q\,\delta(x_0)$??

Mixed Random Variable

Definition: **X** is a mixed RV Iff

$\mathbf{f_x(x)}$ = both impulses and nonzero, finite values

Example

- Observe the period of telephone call
  - 1/3 of calls : never begin (no answer/busy)
  - For the success call, with probability of 2/3, call is uniformly [0,3]
- Find PDF, CDF and Mean of call holding time

Example

- Y: call holding time
- A: phone was answered → A^c: not answered
- $0 \le y \le 3$
- $F_Y(y) = P[Y \le y]$

1 **1/3** **y/3** **2/3**

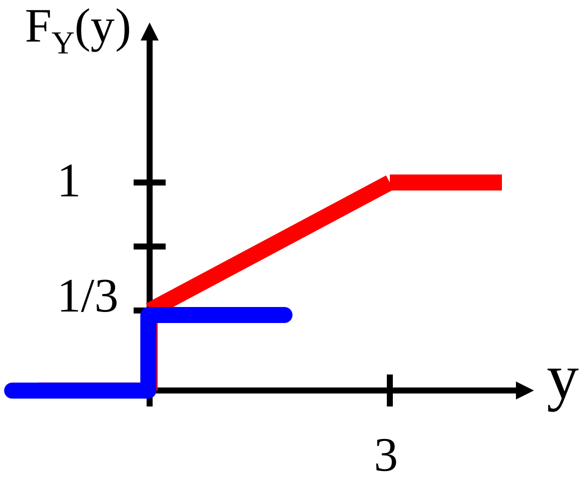
$$= P[Y \le y | A^c]P[A^c] + P[Y \le y | A]P[A]$$

$$= (1)(1/3) + (y/3)(2/3)$$

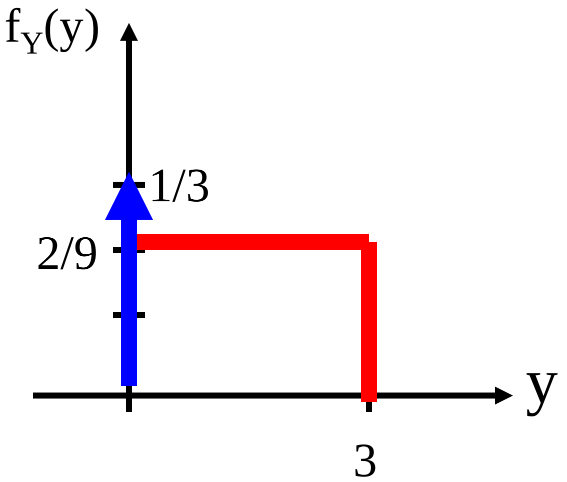
$$= 1/3 + 2y/9$$

Example

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 1/3 + 2y/9 & 0 \le y \le 3 \\ 1 & \text{Otherwise} \end{cases}$$



$$f_Y(y) = \begin{cases} \delta(y)/3 + 2/9 & 0 \le y \le 3 \\ 0 & \text{Otherwise} \end{cases}$$



Example

$$E[Y] = \int_{-\infty}^{\infty} y\,(1/3)\delta(y)\,dy + \int_{0}^{3} y\,(2/9)\,dy$$

$$= 0 + 1 \quad = 1$$

Note: for Uniform [0,3] → E[Y] = 1.5
But now has an effect of δ(y) → E[Y] = 1

Derived Random Variable

Y = aX

$$F_Y(y) = P[aX \le y] = P[X \le y/a] = F_X(y/a)$$

$$f_Y(y) = \frac{d\,F_Y(y)}{dy} = (1/a)\, f_X(y/a)$$

Theorem :

- $F_Y(y) = F_X(y/a)$
- $f_Y(y) = (1/a)\, f_X(y/a)$

Derived Random Variable

$Y = X + b$

$$F_Y(y) = P[X + b \le y] = P[X \le y - b] = F_X(y - b)$$

$$f_Y(y) = \frac{d\,F_Y(y)}{dy} = f_X(y - b)$$

Theorem :

- $F_Y(y) = F_X(y - b)$
- $f_Y(y) = f_X(y - b)$

Conditioning a continuous RV

$$P[A|B] = P[AB] / P[B]$$

$$P[x_1 < X \le x_2] = \int_{x_1}^{x_2} f_X(x)\, dx$$

Approx: $P[x < X \le x+dx] = f_X(x)\, dx$

$$f_{X|B}(x)\, dx = P[x < X \le x+dx\, |B] = \frac{P[x < X \le x+dx\, ,B]}{P[B]}$$

$$= \frac{P[x < X \le x+dx]}{P[B]}$$

← $x \in B,\ x+dx \in B$

$$= \frac{f_X(x)\, dx}{P[B]}$$

Conditioning a continuous RV

$$f_{X|B}(x)\,\cancel{dx} = \frac{f_X(x)\,\cancel{dx}}{P[B]}$$

Definition:

$$f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{P[B]} & x \in B \\ 0 & \text{Otherwise} \end{cases}$$

Definition:

$$E[g(X)|B] = \int_{-\infty}^{\infty} g(x)\, f_{X|B}(x)\, dx$$

Example

- Observe the period of telephone call (T) is an **exponential RV** with expected value 3 min.
- Find E[T|T>2]

- **Solution**:

$$f_T(t) = \begin{cases} (1/3)\, e^{-t/3} & t \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

$$P[T > 2] = \int_{2}^{\infty} f_T(t)\, dt = e^{-2/3}$$

Example

$$f_{T|T>2}(t) = \begin{cases} f_T(t) \,/\, P[T > 2] & t > 2 \\ 0 & \text{Otherwise} \end{cases}$$

$$= \begin{cases} (1/3)\, e^{-(t-2)/3} & t > 2 \\ 0 & \text{Otherwise} \end{cases}$$

$$E[T \mid T > 2] = \int_2^{\infty} t\,(1/3)\, e^{-(t-2)/3}\, dx$$

$$= 5 \text{ min.}$$

Lecture #9 – 2
Multiple Random Variables

Assoc. Prof. Anan Phonphoem, Ph.D.
http://www.cpe.ku.ac.th/~anan

Department of Computer Engineering, Faculty of Engineering,
Kasetsart University, THAILAND

Joint CDF

- Pairs of Random Variables
- Discrete:

  Joint PMF $P_{X,Y}(x,y) = P[X=x, Y=y]$
- Continuous:

  $$P_{X,Y}(x,y) = 0 \qquad (P_X(x) = 0,\ P_Y(y) = 0\ )$$

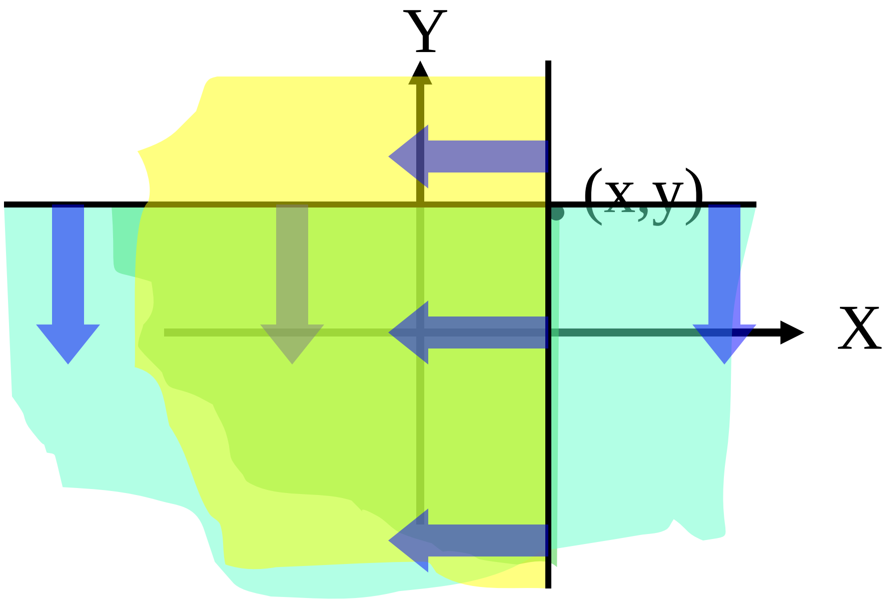
  For 1 RV → interval on real axis

  For 2 RVs → area in a plane

Joint CDF

Definition: Joint CDF of X and Y

$$F_{X,Y}(x,y) = P[X \leq x, Y \leq y]$$



Interesting Properties

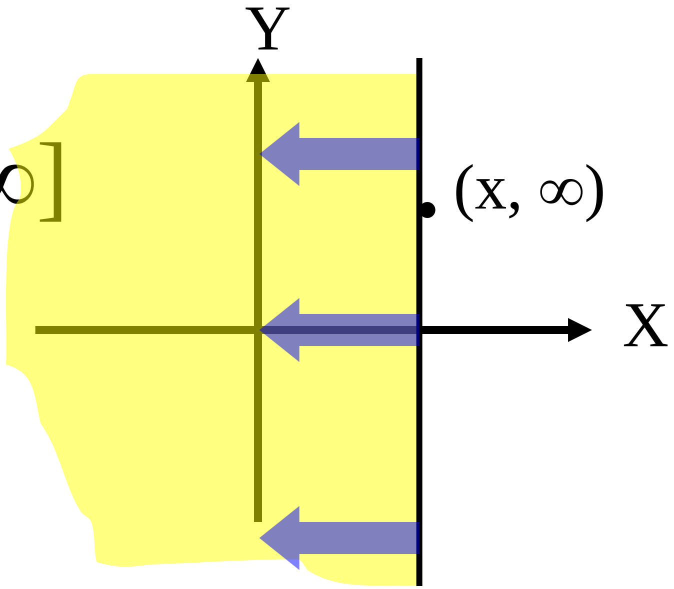
- For Event $\{X \le x\}$

$$F_X(x) = P[X \le x]$$

$$= P[X \le x, Y \le \infty]$$

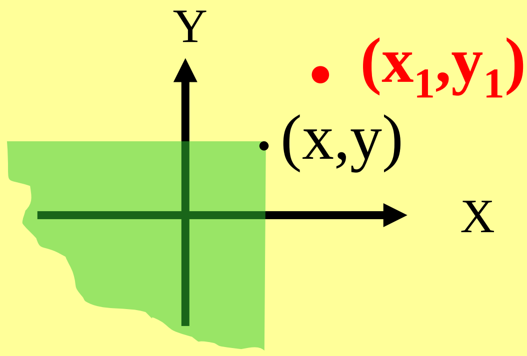
$$= \lim_{y \to \infty} F_{X,Y}(x,y)$$

$$= F_{X,Y}(x, \infty)$$



Joint CDF

Theorem :



(a) $0 \leq F_{X,Y}(x,y) \leq 1$

(b) $F_X(x) = F_{X,Y}(x, \infty)$

(c) $F_Y(y) = F_{X,Y}(\infty, y)$

(d) $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$

(e) If $x_1 \geq x$ and $y_1 \geq y$

then $F_{X,Y}(x_1,y_1) > F_{X,Y}(x,y)$

(f) $F_{X,Y}(\infty, \infty) = 1$

Joint PDF

Definition: Joint PDF of X and Y is satisfied

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)\, dv\, du$$

Theorem:

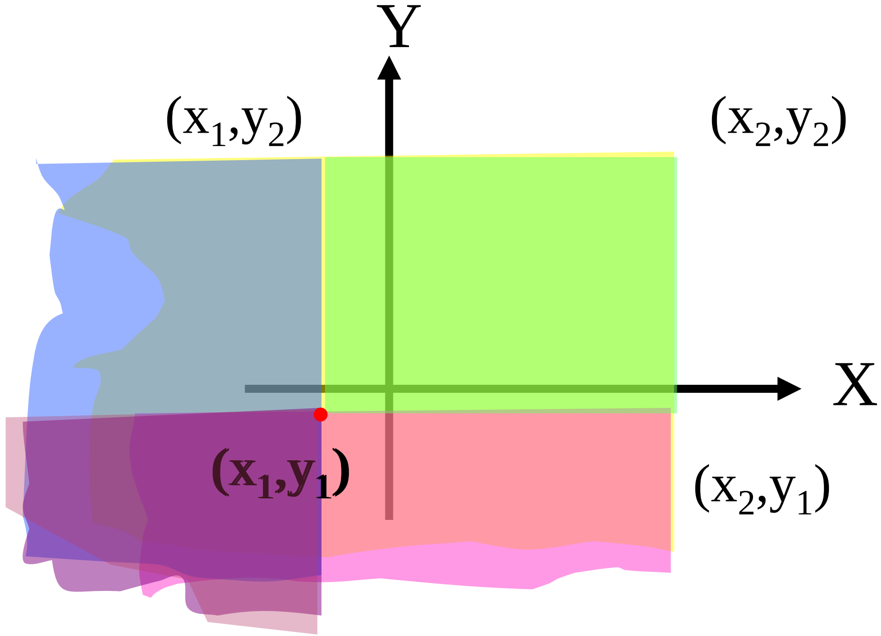
$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}$$

Joint CDF

Theorem:

$P[x_1 < X \leq x_2, y_1 < Y \leq y_2]$

$= F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1)$



Joint PDF

Theorem:

(a) $f_{X,Y}(x,y) \geq 0$ for all (x,y)

(b) $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$$

Theorem:

$$P[A] = \iint_{A} f_{X,Y}(x,y)\, dx\, dy$$

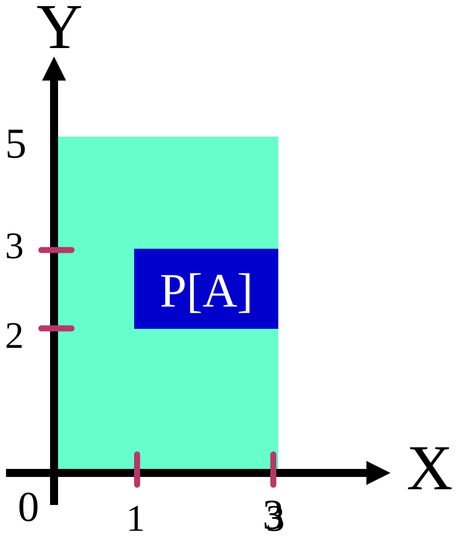
Example

$$f_{X,Y}(x,y) = \begin{cases} c & 0 \le x \le 3,\ 0 \le y \le 5 \\ 0 & \text{Otherwise} \end{cases}$$

Find constant c

$$\int_0^3 \int_0^5 c \, dy \, dx = 15c = 1$$

$$\Rightarrow c = 1/15$$



Find P[A] = P[1 ≤ x ≤ 3, 2 ≤ y ≤ 3]

$$P[A] = \int_1^3 \int_2^3 1/15 \, dv \, du = 2/15$$

Marginal PDF

Theorem:

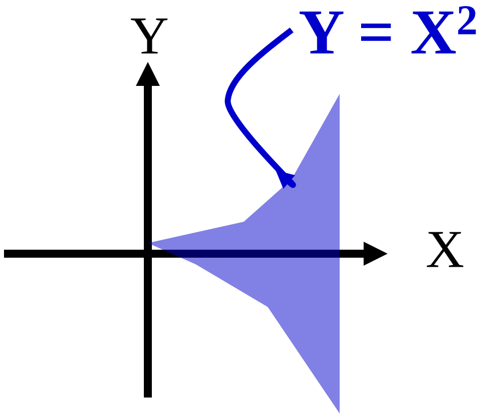
$$f_{X,}(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$f_{Y,}(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

Example

$$f_{X,Y}(x,y) = \begin{cases} cx & 0 \le x \le 1,\ |y| < x^2 \\ 0 & \text{Otherwise} \end{cases}$$

Find constant c



$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} cx\, dx\, dy = \int_0^1 \left(\int_{-x^2}^{x^2} cx\, dy \right) dx$$

$$= \int_0^1 cx\,(2x^2)\, dx = \frac{cx^4}{2} \Big|_0^1$$

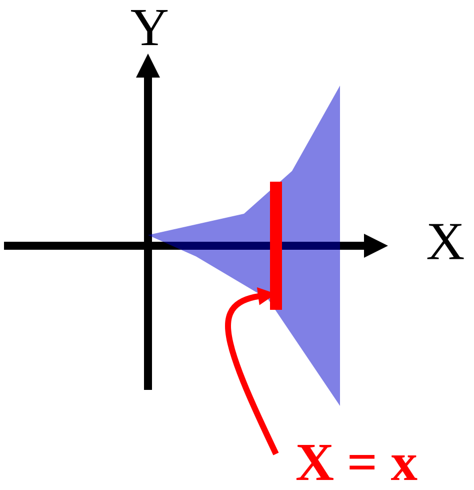
$$= \frac{c}{2} = 1 \quad \Rightarrow c = 2$$

Example

Find the marginal PDF $f_X(x)$ and $f_Y(y)$

Fixed x **(X = x)** then integrate all y

$$f_X(x) = \int_{-x^2}^{x^2} 2x \, dy = 4x^3$$

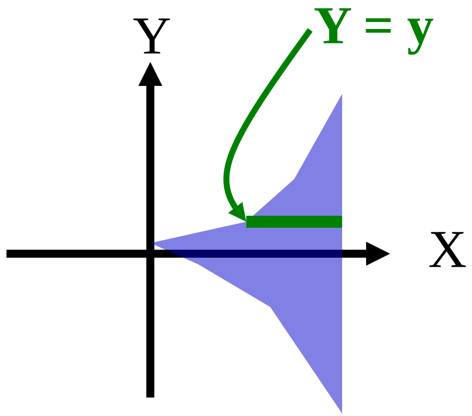


$$f_X(x) = \begin{cases} 4x^3 & 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

Example

Fixed y **(Y = y)** then integrate all x

$$f_{Y,}(y) = \int_{\sqrt{|y|}}^{1} 2x\,dx = 1 - |y|$$



$$f_Y(y) = \begin{cases} 1 - |y| & -1 \leq y \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

Functions of 2 RVs

Example:

Wireless based station with 2 antennas. X and Y are RVs of the signal

- Find the strongest signal

  $W = X$ if $|X| > |Y|$ or $W = Y$ otherwise

- Find the addition of 2 signals

  $W = X + Y$

- Find the addition of 2 signals with weight

  $W = aX + bY$

Functions of 2 RVs

$$F_W(w) = P[W \leq w] = \iint_{g(x,y) \leq w} f_{X,Y}(x,y)\, dx\, dy$$

Example

$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \le x \le 3,\ 0 \le y \le 5 \\ 0 & \text{Otherwise} \end{cases}$$

Find PDF of $W = \max(X,Y)$

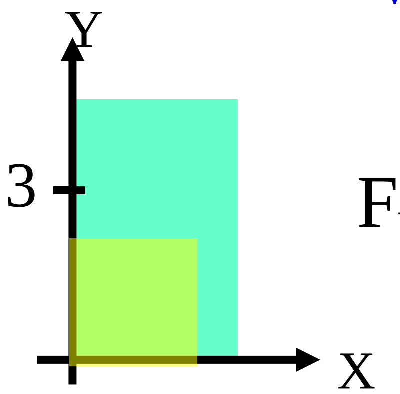
For $W = \max(X,Y)$ → $\{W \le w\} = \{X \le w,\ Y \le w\}$

$$F_W(w) = P[X \le w,\ Y \le w]$$

$$= \int_{-\infty}^{w}\int_{-\infty}^{w} f_{X,Y}(x,y)\, dx\, dy$$

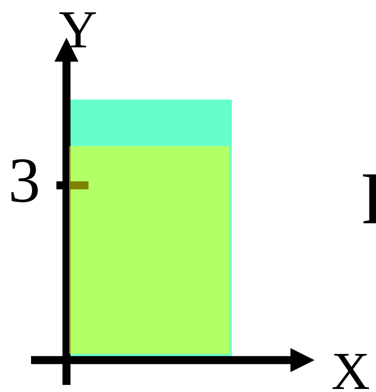
Example

We can divide into 2 cases



$0 \leq w \leq 3$

$$F_W(w) = \int_0^w \int_0^w 1/15 \, dx \, dy = w^2/15$$

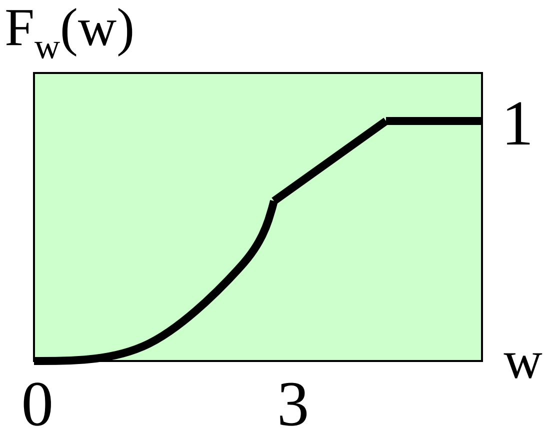


$3 \leq w \leq 5$

$$F_W(w) = \int_3^w \left(\int_0^3 1/15 \, dx \right) dy = w/5$$

Example

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^2/15 & 0 \le w \le 3 \\ w/5 & 3 < w \le 5 \\ 1 & w > 5 \end{cases}$$



$$f_W(w) = \begin{cases} 2w/15 & 0 \le w \le 3 \\ 1/5 & 3 < w \le 5 \\ 0 & \text{Otherwise} \end{cases}$$