LECTURE #8
GENERAL MARKOV PROCESS IN EQUILIBRIUM

204528

Queueing Theory and Applications in Networks

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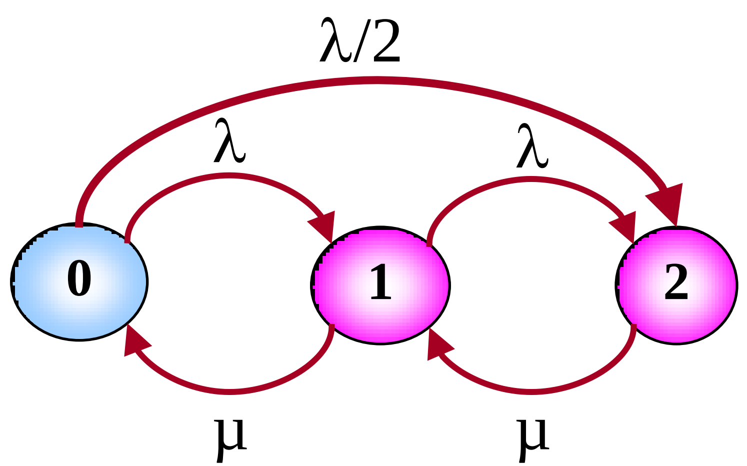
Outline

- General Markov Process
- Erlangian Distribution
  - 2-stage Erlangian E_2
  - r-stage Erlangian E_r
- $M/E_r/1$

General Markov Process

- More general than birth-death process
  - Transition beyond nearest neighbors are allowed
  - But not too complicated
- We are interested in the Equilibrium Solutions

Simple three-state Markov Chain



- From Flow conservation
  - Prob. Flow in = Prob. Flow out
- For state 0: $\boxed{(3/2)\lambda p_0 = \mu\, p_1}$
- For state 1: $\boxed{(\lambda + \mu)p_1 = \lambda p_0 + \mu\, p_2}$
- For state 2: $\mu p_2 = (\lambda/2)p_0 + \lambda p_1$

Simple three-state Markov Chain

- We need one more equation

$$p_0 + p_1 + p_2 = 1$$

- Solution

$$p_0 = [\ 1 + 2(\lambda/\mu) + (3/2)(\lambda/\mu)^2\ ]^{-1}$$

$$p_1 = 3/2(\lambda/\mu)\ p_0$$

$$p_2 = [\ (1/2)(\lambda/\mu) + (3/2)(\lambda/\mu)^2\ ]\ p_0$$

Limiting Probability

- As $t \rightarrow \infty$
  - We can find the p_k
- How about prob. that an arriving customer finds the system in state E_k ?
  - It is equal to p_k, isn’t it?

D/D/1

- Arrival is deterministic and occurs at every interval = $\bar{t}$ sec
- Service time is also deterministic and identical for all customers = $\bar{x}$ sec
- Let

r_k = [prob. that an arriving customer finds the system in state E_k]

D/D/1

- For stability of the system
  - It requires: $\overline{x} < \overline{t}$
  - $\rho = \overline{x} / \overline{t}$
- No customer has to wait !
  - $r_0 = 1$ and $r_k = 0$ for $k \geq 1$
- Therefore
  - $p_0 = 1 - \rho$ and $p_1 = \rho$
  - $p_k = 0$ for $k \geq 2$

D/D/1

- In summary
  - For D/D/1: $r_k \neq p_k$
- How about the other systems?
  - Many contains the property of $r_k = p_k$
  - Poisson arrival system is an example

Poisson Arrivals

- For

  $P_k(t)$ = Prob. that system is in state E_k

  $R_k(t)$ = Prob. that an arriving customer finds the system is in state E_k

- Let $A(t , t+\Delta t)$ = Event that arrival occurs in the interval $(t , t+\Delta t)$

Poisson Arrivals

- $$R_k(t) \triangleq \lim_{\Delta t \to 0} P[\ N(t) = k \mid A(t\ ,\ t+\Delta t)\ ]$$

$$= \lim_{\Delta t \to 0} \frac{P[\ N(t) = k\ ,\ A(t\ ,\ t+\Delta t)\ ]}{P[A(t\ ,\ t+\Delta t)\ ]}$$

$$= \lim_{\Delta t \to 0} \frac{P[A(t\ ,\ t+\Delta t) \mid N(t) = k]\ P[N(t) = k]}{P[A(t\ ,\ t+\Delta t)\ ]}$$

$$= \lim_{\Delta t \to 0} P[\ N(t) = k]$$

$$\boxed{P_k(t) = R_k(t)}$$

Erlangian Distribution E_r

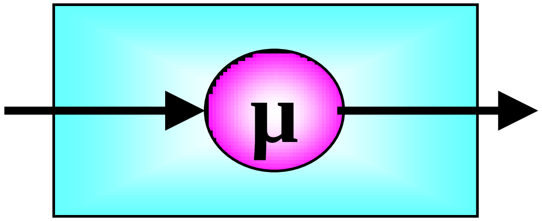
- A.K. Erlang proposed
  - The extreme simplicity of exponential distribution
  - Solving Markovian queueing systems
- The exponential distribution is not **always** appropriate for representing the true situation with regards to service times
- More general service distribution ➔ Markov chain becomes complicated.

Agner Krarup Erlang
Denmark
From: Wikipedia

Erlangian Distribution Er

- Erlang purposed
  - The decomposing the service time distribution into a collection of exponential distributions

An exponential server



Service Facility

- For an exponentially distributed service time

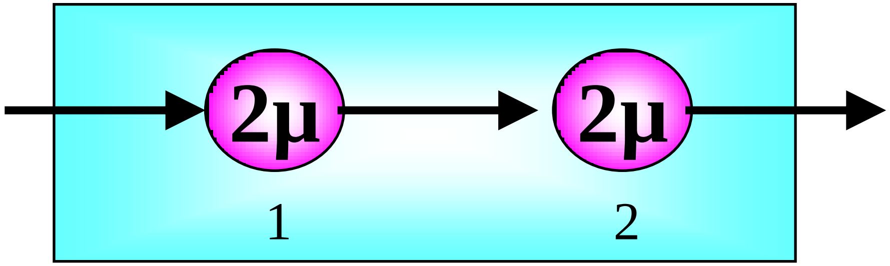
$$b(x) \triangleq \frac{dB(x)}{dx}$$

$$= \mu\, e^{-\mu x} \qquad x \geq 0$$

- For Mean and Variance of the service time

$$E[x] = \frac{1}{\mu} \qquad \sigma_b^2 = \frac{1}{\mu^2}$$

A series of two exponential servers



Service Facility

- A series of two small exponential servers
- Each internal server has parameter = 2μ
- So the pdf of each server

$$h(y) = 2\mu e^{-2\mu y} \qquad y \geq 0$$

A series of two exponential servers

- For Mean and Variance of the service time

$$E[x] = \frac{1}{2\mu} \qquad \sigma_h^2 = \frac{1}{(2\mu)^2}$$

- The functions of the system
  - Upon departure of customer, a new customer is allowed to enter the service facility
  - After finish stage 1 → enter stage 2
  - Only one customer is allowed in the service facility

A series of two exponential servers

- What is the distribution of the time spent in the service facility ?
  - Sum of the two independent random variables
  - The convolution of the density function of the two random variables

Convolution of two RVs

- $Y = X_1 + X_2$
- $F_Y(y) = P[Y \le y] = P[X_1 + X_2 \le y]$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{y - x_2} f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{y - x_2} f_{X_1}(x_1)\, dx_1 \right] f_{X_2}(x_2)\, dx_2$$

$$= \int_{-\infty}^{\infty} F_{X_1}(y - x_2)\, f_{X_2}(x_2)\, dx_2$$

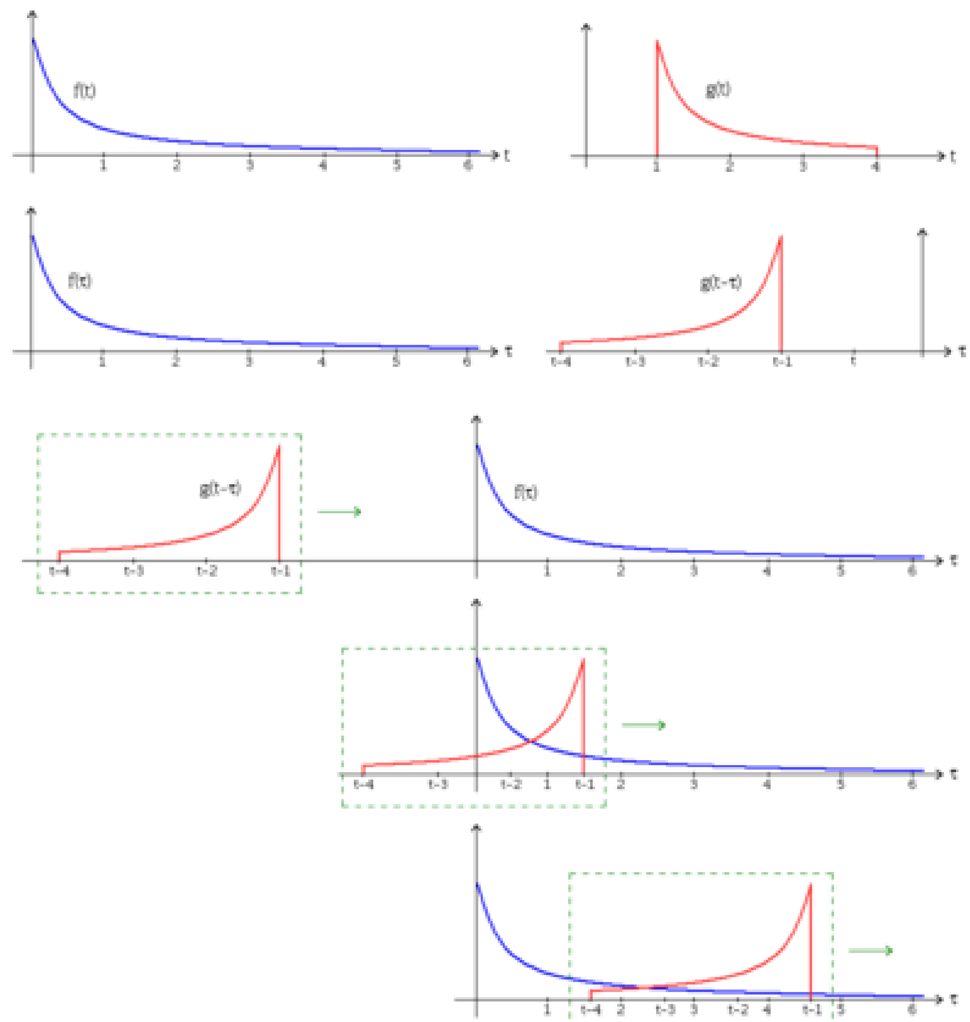
Convolution of two RVs

- $F_Y(y) = \int_{-\infty}^{\infty} F_{X_1}(y - x_2)\, f_{X_2}(x_2)\, dx_2$
- $f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(y - x_2)\, f_{X_2}(x_2)\, dx_2$
- $f_Y(y) = f_{X_1}(y) \otimes f_{X_2}(y)$

Convolution of two RVs

$$(f * g)(t) \stackrel{\mathrm{def}}{=} \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

- Express each function in terms of a dummy variable τ.
- Reflect one of the functions: $g(\tau) \rightarrow g(-\tau)$.
- Add a time-offset, t, which allows $g(t-\tau)$ to slide along the τ-axis.
- Start t at $-\infty$ and slide it all the way to $+\infty$. Wherever the two functions intersect, find the integral of their product.



http://en.wikipedia.org/wiki/Convolution

Laplace transform in probability

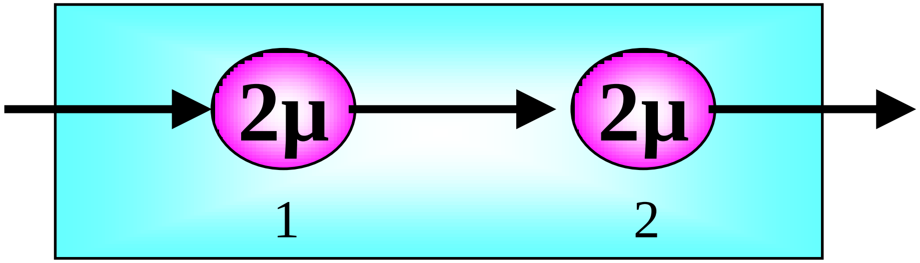
- In probability, the Laplace transform is defined by means of an expectation value.
- If X is a random variable with probability density function f,
  - then the Laplace transform of f is given by the expectation

$$(\mathcal{L}f)(s) = E\left[e^{-sX}\right].$$

$$= \int_0^\infty e^{-sx} \; f_X(x)\, dx$$

http://en.wikipedia.org/wiki/Laplace_transform

Laplace transform of pdf



Service Facility

- $B^*(s) \triangleq \int_0^\infty e^{-sx}\, b(x)\, dx$
- $H^*(s) \triangleq \int_0^\infty e^{-sy}\, h(y)\, dy$
- $B^*(s) = H^*(s)\, H^*(s) = [H^*(s)]^2$

2-stage Erlangian Server E_2

- $h(y) = 2\mu e^{-2\mu y}$
- $H^*(s) \triangleq \int_0^\infty e^{-sy}\, h(y)\, dy$

$$= \frac{2\mu}{s + 2\mu}$$

- $B^*(s) = [H^*(s)]^2$

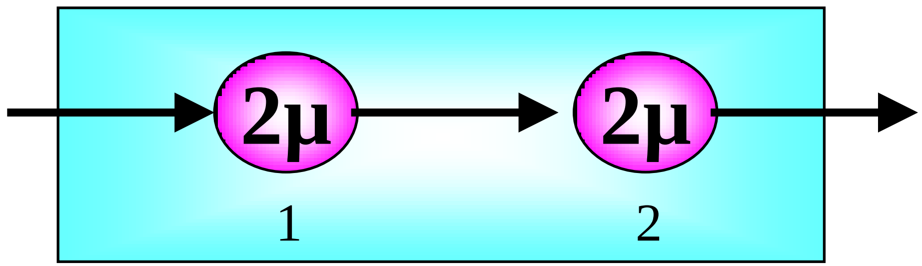
$$= \left(\frac{2\mu}{s + 2\mu}\right)^2$$

Inverse

$$b(x) = 2\mu(2\mu x)\, e^{-2\mu x}$$

For $x \geq 0$

A series of two exponential servers



Service Facility

- For Mean and Variance of the service time

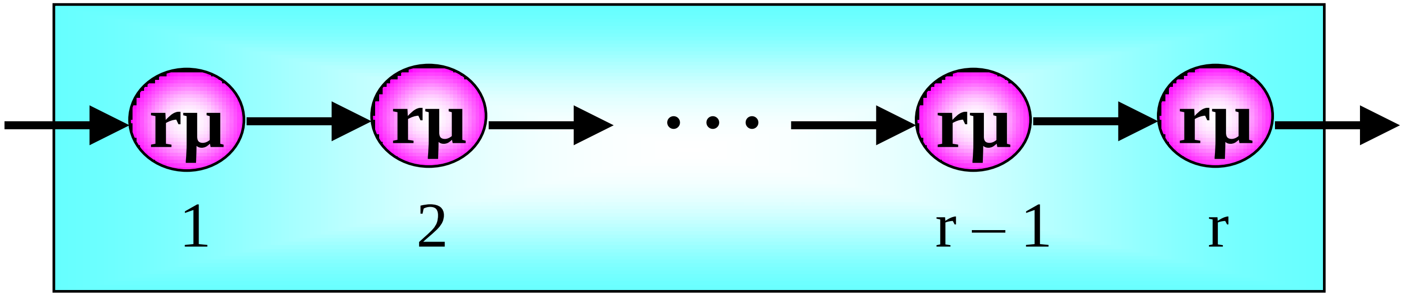
$$E[x] = 2E[y] = \frac{1}{\mu}$$

$$\sigma_b^2 = \sigma_h^2 + \sigma_h^2 = \frac{1}{2\mu^2}$$

A series of two exponential servers

- Mean time for single and two-stage are the same
  - By speed up the internal service 2 times
- Variance of the two-stage is one-half of the single server

r-stage Erlangian server E_r



Service Facility

- $h(y) = r\mu e^{-r\mu y} \qquad y \geq 0$
- For Mean and Variance of each stage

$$E[y] = \frac{1}{r\mu}$$

$$\sigma_h^2 = \left(\frac{1}{r\mu}\right)^2$$

r-stage Erlangian server Er

- For Mean and Variance of the service time

$$E[x] = rE[y] = \frac{1}{\mu}$$

$$\sigma_b^2 = r\left(\frac{1}{r\mu}\right)^2 = \frac{1}{r\mu^2}$$

r-stage Erlangian server E_r

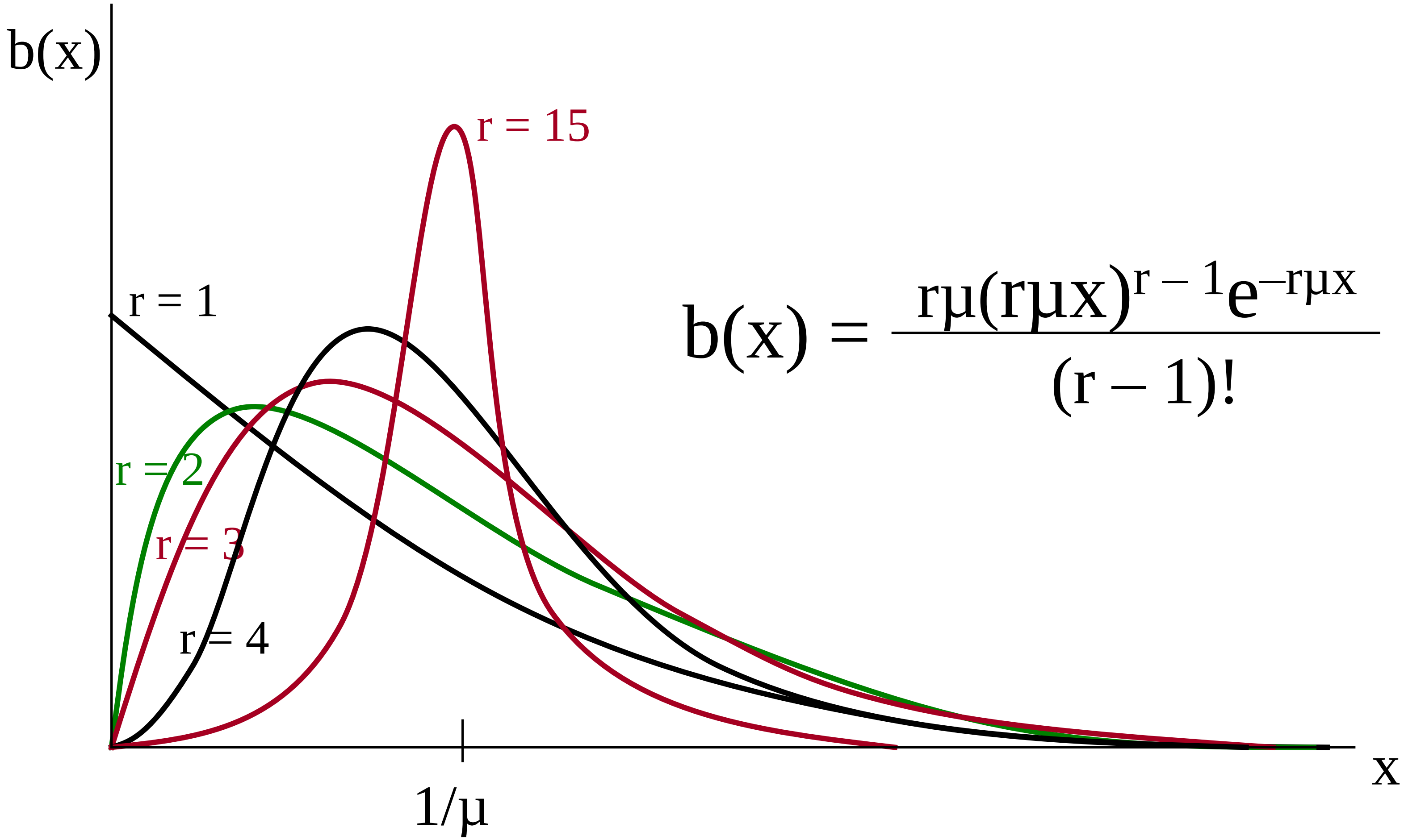
- Solve for the pdf of the service time

$$B^*(s) = \left(\frac{r\mu}{s + r\mu}\right)^r$$

$$b(x) = \frac{r\mu(r\mu x)^{r-1} e^{-r\mu x}}{(r-1)!} \qquad x \geq 0$$

Erlang Distribution

r-stage Erlangian distribution E_r



$$b(x) = \frac{r\mu(r\mu x)^{r-1}e^{-r\mu x}}{(r-1)!}$$

r-stage Erlangian distribution E_r

- Note
  - $r \uparrow$ ➔ pdf approaches Gaussian Distribution
  - Mean is constant
  - The standard deviation (width) shrinks by $1/\sqrt{r}$
  - If r ➔ ∞ then
    pdf ➔ unit impulse function at $x = 1/\mu$

$M / E_r / 1$

- Memoryless / Erlangian/1 server
- Define

$$a(t) = \lambda e^{-\lambda t} \qquad t \geq 0$$

$$b(x) = \frac{r\mu(r\mu x)^{r-1} e^{-r\mu x}}{(r-1)!} \qquad x \geq 0$$

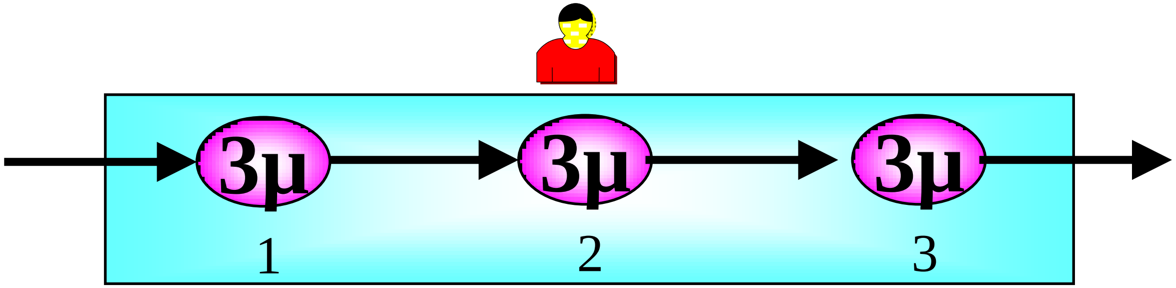
$M / E_r / 1$

- State variable = total number of service stage yet to be completed
- k = # of customers
- Now we are in the i^{th} stage of service (1, r)
- j = number of stages left in the total system

$$= (k - 1)r + (r - i + 1)$$

$$= rk - i + 1$$

of stages left to be completed



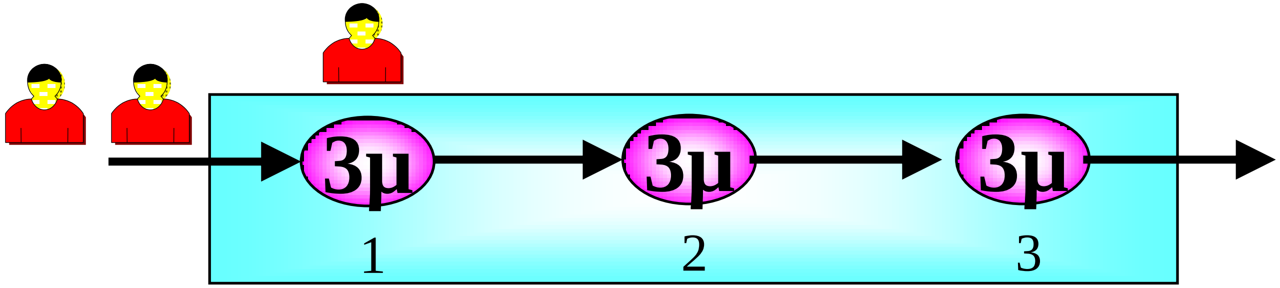
Service Facility

- $j = rk - i + 1$
- For $r = 3$ stages, 1 customer, now in stage 2

  $j = 3*1 - 2 + 1 = 2$

  = " there are **2 stages** left to be completed"

of stages left to be completed



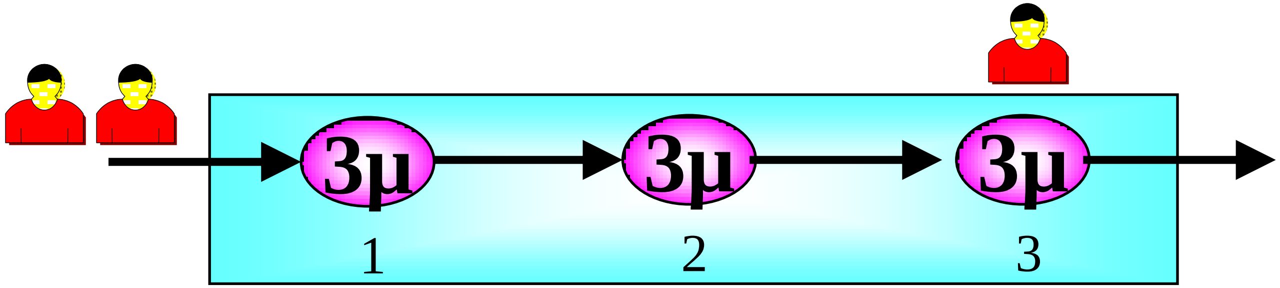
Service Facility

- $j = rk - i + 1$
- For $r = 3$ stages, 3 customers, now in stage 1

  $j = 3*3 - 1 + 1 = 9$

  = "there are **9 stages** left to be completed"

of stages left to be completed



Service Facility

- $j = rk - i + 1$
- For r = 3 stages, 3 customers, now in stage 3

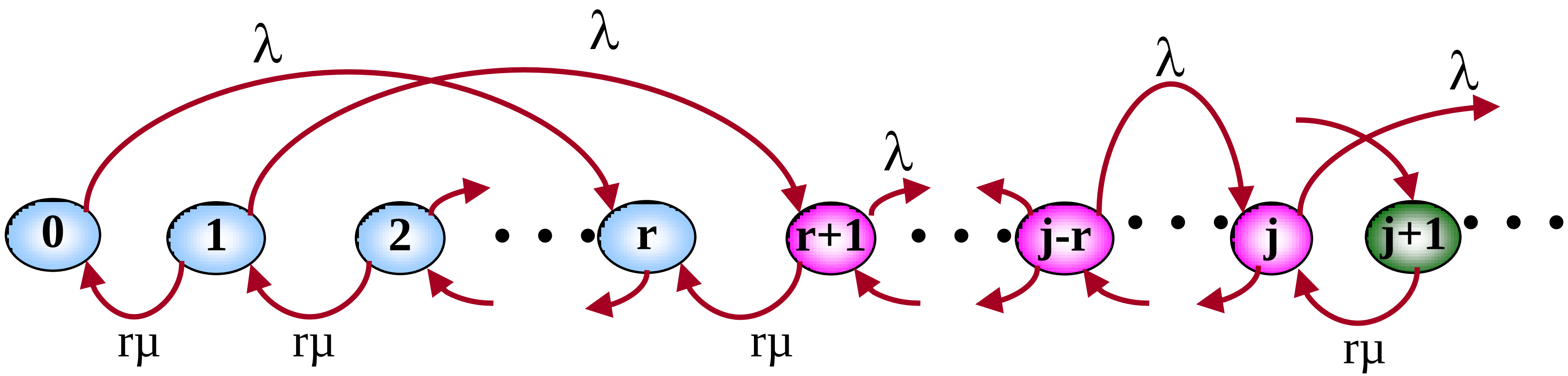
  $j = 3*3 - 3 + 1 = 7$

  = "there are **7 stages** left to be completed"

M / E_r / 1

- p_k = equilibrium prob. for the # of customers in the system
- P_j = P[j stages in the system]
- $p_k = \sum_{j=(k-1)r+1}^{kr} P_j \qquad k = 1, 2, 3, \ldots$

M / E_r / 1 State diagram



- $P_j = 0 \qquad\qquad j < 0$
- $\lambda P_0 = r\mu P_1$
- $(\lambda + r\mu)P_j = \lambda P_{j-r} + r\mu P_{j+1} \qquad j = 1,2,\ldots$

$M / E_r / 1$

- Solved by using *z*-transform

$$P(z) = \sum_{j=0}^{\infty} P_j\, z^j$$

- Multiplied by z^j and sum

$$\sum_{j=1}^{\infty}(\lambda + r\mu)P_j\, z^j = \sum_{j=1}^{\infty} \lambda P_{j-r} z^j + \sum_{j=1}^{\infty} r\mu P_{j+1} z^j$$

$$(\lambda + r\mu)\left(\sum_{j=0}^{\infty} P_j\, z^j - P_0\right) = \lambda z^r \sum_{j=1}^{\infty} P_{j-r} z^{j-r} + (r\mu/z) \sum_{j=1}^{\infty} P_{j+1} z^{j+1}$$

$M / E_r / 1$

$$(\lambda + r\mu)\,[P(z) - P_0] = \lambda z^r P(z) + (r\mu/z)\,[P(z) - P_0 - P_1 z]$$

$$P(z) = \frac{r\mu P_0(1 - z)}{r\mu + \lambda z^{r+1} - (\lambda + r\mu)z}$$

Finding P_0

$$P(1) = 1 = \frac{r\mu P_0}{r\mu - \lambda r}$$

$$P_0 = 1 - (\lambda/\mu) \longrightarrow \rho$$

$M / E_r / 1$

$$P(z) = \frac{r\mu(1-\rho)(1-z)}{r\mu + \lambda z^{r+1} - (\lambda + r\mu)z}$$

- Then invert the *z*-transform
  ➔ Distribution of # of the stages in the system

M / E_r / 1 (For r = 1)

- For case r = 1

$$P(z) = \frac{\mu(1-\rho)(1-z)}{\mu + \lambda z^2 - (\lambda + \mu)z}$$

$$= \frac{(1-\rho)(1-z)}{1 + \rho z^2 - (1+\rho)z}$$

$$= \frac{(1-\rho)}{(1-\rho z)}$$

$M / E_r / 1$ (For $r = 1$)

- Invert z-transform

$$\boxed{P_k = (1 - \rho)\,\rho^k \qquad k = 0, 1, 2, \ldots}$$

- $p_k = \sum_{j=(k-1)r+1}^{kr} P_j \qquad k = 1, 2, 3, \ldots$

- For $r = 1 \qquad p_k = P_k \Rightarrow \quad M / M / 1$

M / E_r / 1 (For general r)

$$P(z) = \frac{r\mu(1-\rho)(1-z)}{r\mu + \lambda z^{r+1} - (\lambda + r\mu)z}$$

$$= \frac{(1-\rho)}{(1-z/z_1)\,(1-z/z_2)\,\ldots\,(1-z/z_r)}$$

$$= (1-\rho)\sum_{i=1}^{r} \frac{A_i}{(1-z/z_i)}$$

$$A_i = \prod_{\substack{n=1 \\ n \neq i}}^{r} \frac{1}{(1-z_i/z_n)}$$

$M / E_r / 1$ (For general r)

- Invert z-transform

$$P_j = (1 - \rho) \sum_{i=1}^{r} A_i (z_i)^{-j} \qquad j = 1, 2, \ldots, r$$

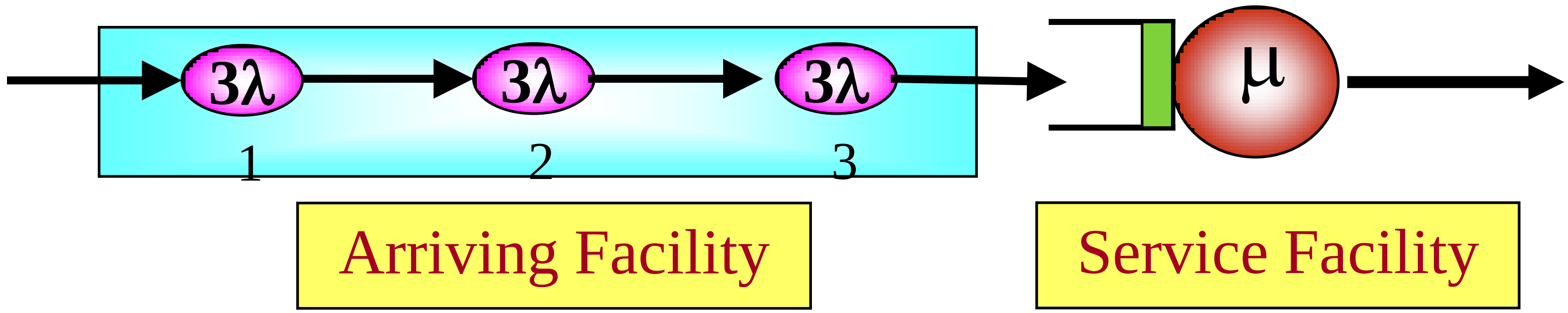
E_r / M / 1

- Erlangian / Memoryless /1 server
- Define

$$a(t) = \frac{r\lambda(r\lambda t)^{r-1}e^{-r\lambda t}}{(r-1)!} \qquad t \geq 0$$

$$b(x) = \mu e^{-\mu x} \qquad x \geq 0$$

E_r / M / 1



Arriving : $$a(t) = \frac{r\lambda(r\lambda t)^{r-1}e^{-r\lambda t}}{(r-1)!} \qquad t \geq 0$$

Service : $$b(x) = \mu e^{-\mu x} \qquad x \geq 0$$

E_r / M / 1

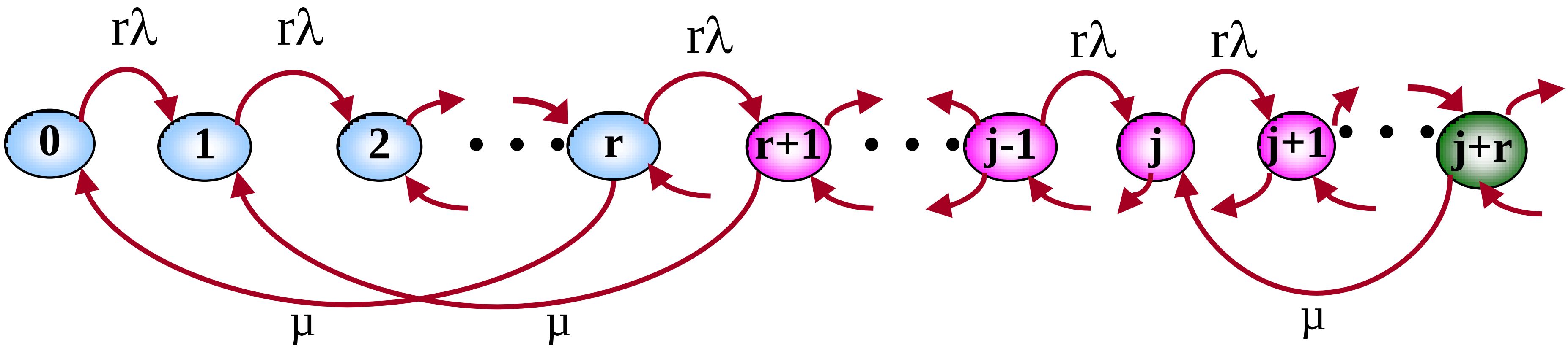
- State variable = total number of service stage yet to be completed
- k = # of customers
- Now the arriving customer is in the i^{th} stage of service [1, r]
- j = total number of stages of arrival in the system

  $= rk + i - 1$

E_r / M / 1

- p_k = equilibrium prob. for the # of customers in the system
- P_j = # of arrival stages in the system
- $p_k = \sum_{j=rk}^{r(k+1)-1} P_j$

E_r / M / 1 State diagram



- Each departure = remove "r" stages of arrival
- $r\lambda P_0 = \mu P_r$
- $r\lambda P_0 = r\lambda P_{j-1} + \mu P_{j+r}$ $\quad 1 \le j \le r-1$
- $(r\lambda + \mu)P_j = r\lambda P_{j-1} + \mu P_{j+r}$ $\quad r \le j$

E_r / M / 1

- Solved by using *z*-transform

$$P(z) = \sum_{j=0}^{\infty} P_j z^j$$

- Multiplied by z^j and sum
- For $j \geq 1$

$$\sum_{j=1}^{\infty}(\mu + r\lambda)P_j z^j - \sum_{j=1}^{r-1}\mu P_j z^j = \sum_{j=1}^{\infty} r\lambda P_{j-1} z^j + \sum_{j=1}^{\infty}\mu P_{j+r} z^j$$

$E_r / M / 1$

$$P(z) = \frac{(1 - z^r) \sum_{j=1}^{r-1} P_j z^j}{r \rho z^{r+1} - (1 + r\rho) z^r + 1}$$

$$P(z) = \frac{(1 - z^r)(1 - 1/z_0)}{r(1 - z)(1 - z/z_0)}$$

$$P_j = \begin{cases} \frac{1}{r}(1 - z_0^{-j-1}) & 0 \le j < r \\ \rho(z_0 - 1) z_0^{r-j-1} & j \ge r \end{cases}$$

$E_r / M / 1$

$$p_k = \begin{cases} 1 - \rho & k = 0 \\ \rho(z_0^{\,r} - 1) z_0^{-rk} & k > 0 \end{cases}$$