LECTURE #6
BIRTH-DEATH PROCESS

204528

Queueing Theory and Applications in Networks

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Outline

- Birth-Death Process
- Markov Process Property
- Continuous Time Birth-Death Markov Chains
- State Transition Diagram
- A Pure Birth System
- A Pure Death System
- A Birth-Death Process
- Equilibrium Solution

Birth-Death Process

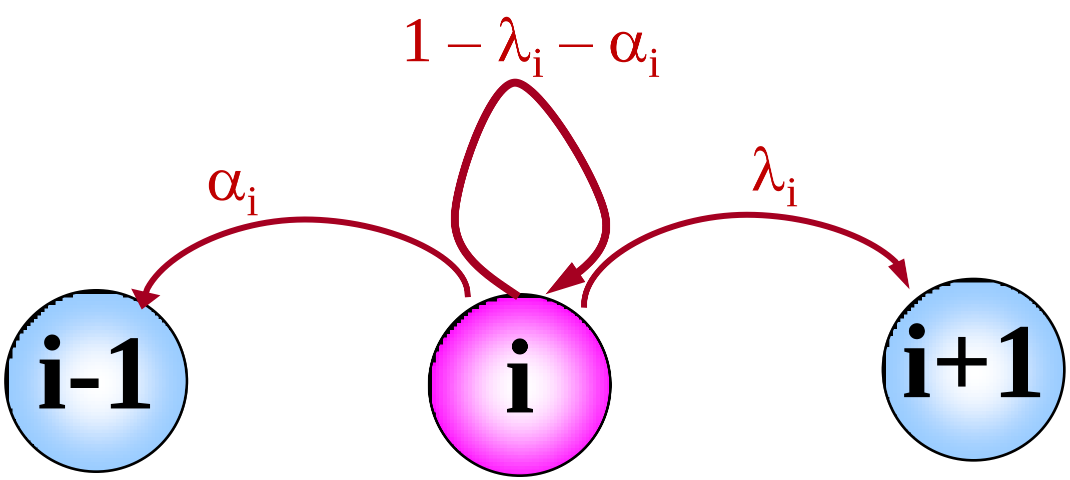
- A Markov Process
- Homogeneous, aperiodic, and irreducible
- Discrete time / Continuous time
- State changes can only happen between neighbors

Birth-Death Process

- Size of population
  - System is in state E_k when consists of $\boldsymbol{k}$ members
  - Changes in population size occur by at most one
  - Size has been increased by one → "***Birth***"
  - Size has been decreased by one → "***Death***"
- Transition probabilities $\boldsymbol{p_{ij}}$ do not change with time

Birth-Death Process

$$p_{ij} = \begin{cases} \alpha_i & j = i - 1 \\ 1 - \lambda_i - \alpha_i & j = i \\ \lambda_i & j = i + 1 \\ 0 & \text{Otherwise} \end{cases}$$



Birth-Death Process

- α_i = death (less one in population size)
- $\alpha_0 = 0$ (no population → no death)
- λ_i = birth (increase one in population)
- $\lambda_i > 0$ (birth is allowed)
- Pure Birth = no decrement, only increment
- Pure Death = no increment, only decrement

Queueing Theory Model

- Population = customers in the queueing system
- Death = a customer departures from the system
- Birth = a customer arrives to the system

Transition matrix

$$P = \begin{bmatrix} 1-\lambda_0 & \lambda_0 & 0 & 0 & 0 & 0 & \cdots \\ \alpha_1 & 1-\lambda_1-\alpha_1 & \lambda_1 & 0 & 0 & 0 & \\ 0 & \alpha_2 & 1-\lambda_2-\alpha_2 & \lambda_2 & & & \\ 0 & & \cdots & & & & \\ 0 & & & & & & \\ & & & \alpha_i & 1-\lambda_i-\alpha_i & \lambda_i & \\ \cdots & & & & & & \end{bmatrix}$$

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Discrete Time Markov Chains

- One can stay in a ***Discrete state (position)*** and is permitted to change state at ***Discrete time***.

Discrete Time Markov Chains

$$P\{X_n = j \mid X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}$$
$$= P\{X_n = j \mid X_{n-1} = i_{n-1}\}$$

Where n = 1,2,3,…

- X_n: The system is in state j at time n
- The system can begin at ***state 0*** with ***initial probability*** $P[X_0 = x]$
- $P\{X_n = j \mid X_{n-1} = i_{n-1}\}$ is the ***one-step transition probability***

Discrete Time Markov Chains

- From ***initial probability*** and ***one-step transition probability***,
- we can find ***probability of being in various states at time n***

Continuous Time Markov Chains

$$P\{X(t_{n+1}) = j \mid X(t_1) = i_1, X(t_2) = i_2, \ldots, X(t_n) = i_n\}$$
$$= P\{X(t_{n+1}) = j \mid X(t_n) = i_n\}$$

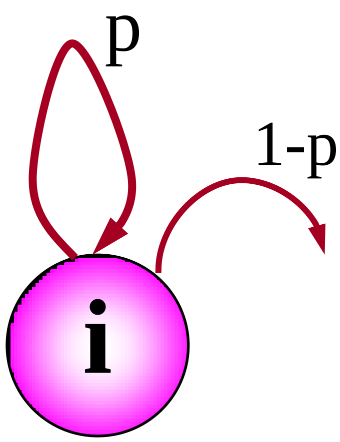
Where $n = 1,2,3,\ldots$ $\quad t_1 < t_2 < \ldots < t_n$

- One can stay in a ***Discrete state (position)*** and is permitted to change state at ***Arbitrary time***

Markov Process Property

- Time that the process spends in any state must be "Memoryless"
- Discrete Time Markov Chains
  - Geometrically distributed state times
- Continuous Time Markov Chains
  - Exponentially distributed state times

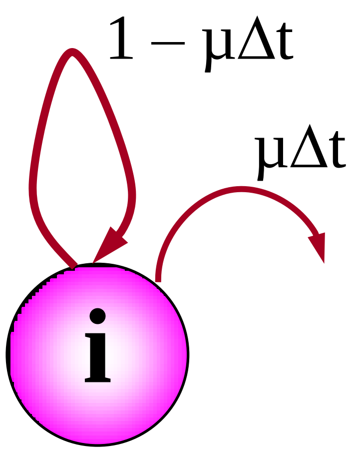
Markov Process Property



For Discrete Time Markov Chain

- P[system in state i for N time units | system in current state i] = p^N
- P[system in state i for N time units before exiting from state i] = $p^N(1-p)$
- Geometrically distributed state times

Markov Process Property



For Continuous Time Markov Chain

- P[system in state i for time T | system in current state i]

  $= (1 - \mu\Delta t)^{T/\Delta t}$

  $= e^{-\mu T}$ where $\Delta t \rightarrow 0$

- Exponentially distributed state times

Continuous Time Birth-Death Markov Chains

- Let λ_i = birth rate in state i

  μ_i = death rate in state i
- Then

  P[state i to state i – 1 in Δt] $= \mu_i \Delta t$

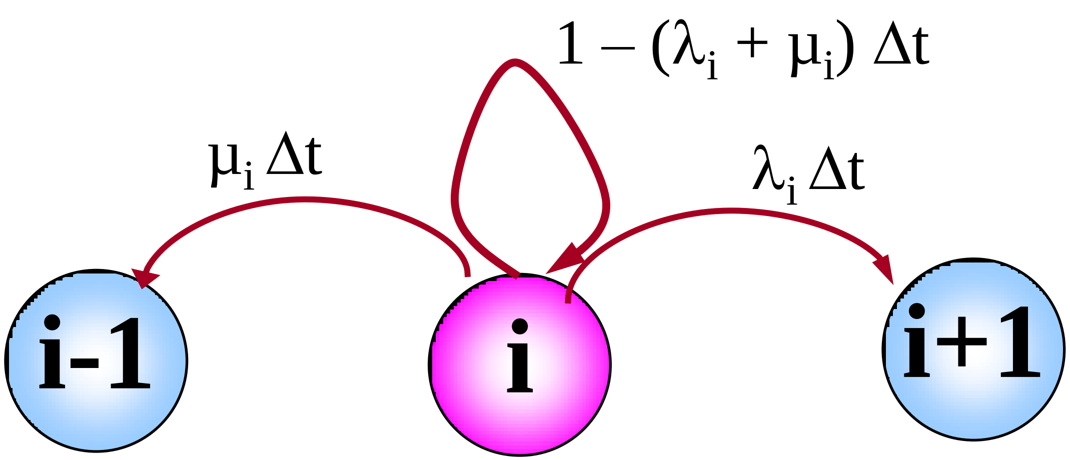
  P[state i to state i + 1 in Δt] $= \lambda_i \Delta t$

  P[state i to state i in Δt] $= 1 - (\lambda_i + \mu_i) \Delta t$

  P[state i to other state in Δt] $= 0$

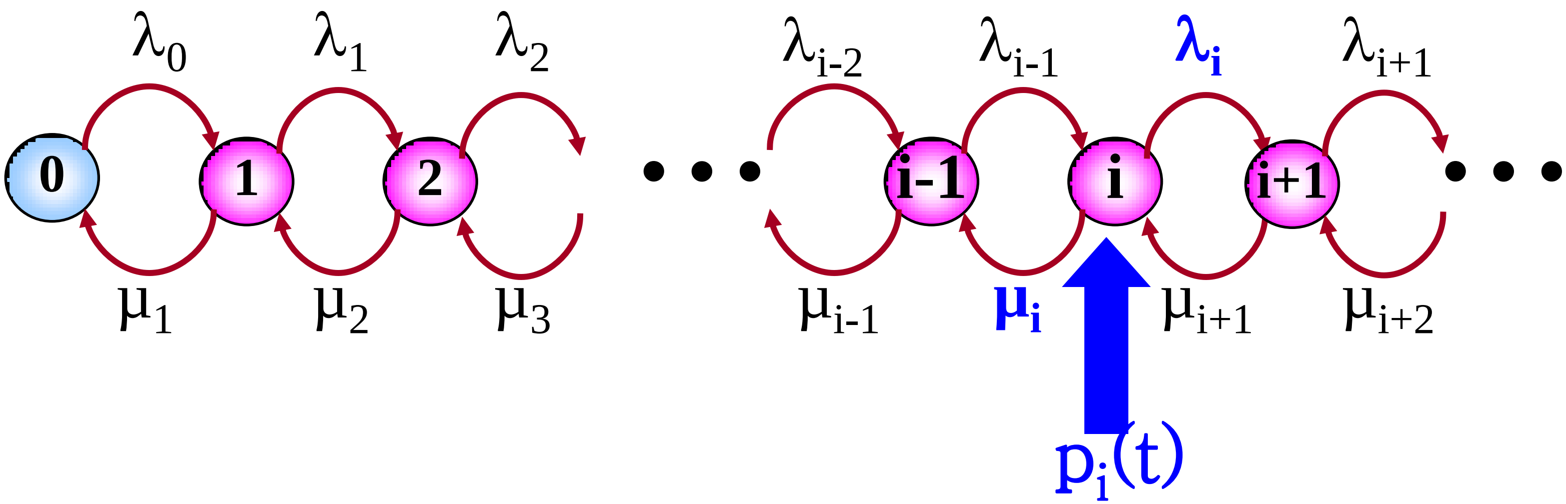
Continuous Time Birth-Death Markov Chains

$$
p_{ij} = \begin{cases} \mu_i \, \Delta t & j = i - 1 \\ 1 - (\lambda_i + \mu_i)\, \Delta t & j = i \\ \lambda_i \, \Delta t & j = i + 1 \\ 0 & \text{Otherwise} \end{cases}
$$



$\Delta t \rightarrow 0$

State Transition Diagram



- $X(t)$ = #customers in the system at time t

  = birth – death in (0,t)
- $p_i(t) = P[\, X(t) = i \,]$

  = Prob. that system is in state i at time t

State Transition Diagram

- From t to $t + \Delta t$

  $p_0(t+\Delta t) = p_0(t)[1 - \lambda_0 \Delta t] + p_1(t)\mu_1 \Delta t$

  $p_i(t+\Delta t) = p_i(t)[1 - (\lambda_i + \mu_i)\Delta t] + p_{i+1}(t)\mu_{i+1}\Delta t + p_{i-1}(t)\, \lambda_{i-1}\Delta t$

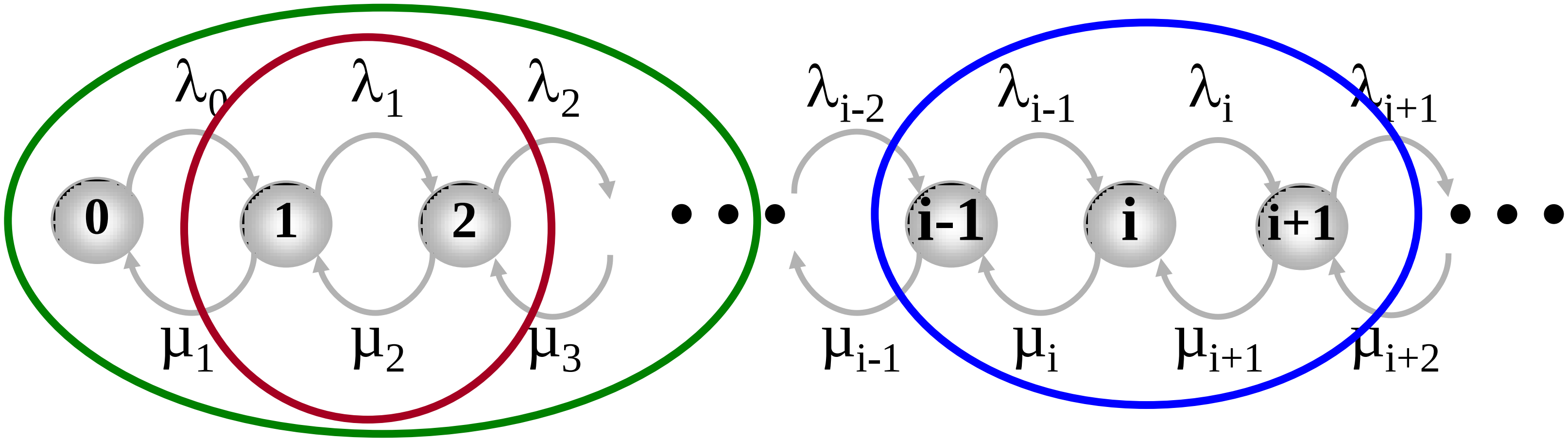
- $\Delta t \rightarrow 0$

  $dp_0(t)/dt = -\lambda_0 p_0(t) + \mu_1 p_1(t)$

  $dp_i(t)/dt = -(\lambda_i + \mu_i)\, p_i(t) + \mu_{i+1} p_{i+1}(t) + \lambda_{i-1} p_{i-1}(t)$

- $\sum_{i=0}^{\infty} p_i(t) = 1$

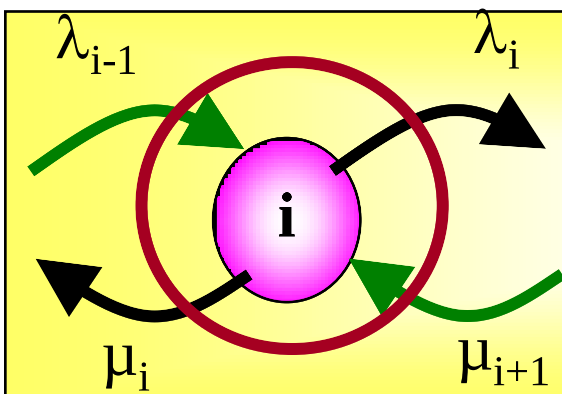
Flow Balance Method



- Draw a closed boundary
- Observe all flows (***In*** and ***Out***) across the boundary

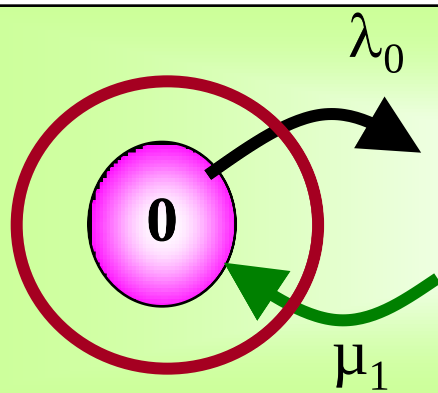
Flow Balance Method

- Flow Out = Flow In



- Draw a closed boundary around state i

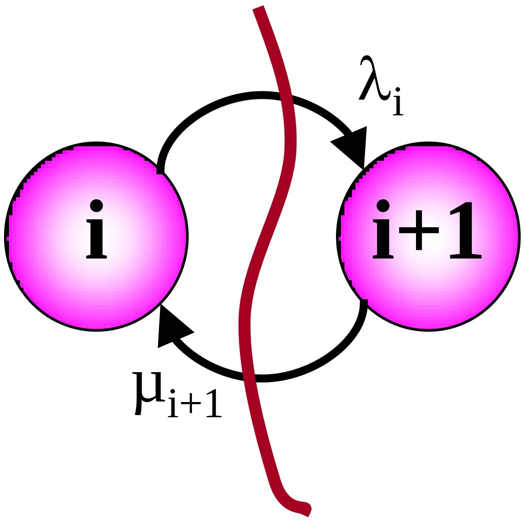
$$(\lambda_i+\mu_i)\, p_i = \mu_{i+1}p_{i+1}+\lambda_{i-1}p_{i-1}$$



- Draw a closed boundary around state 0

$$\lambda_0 p_0 = \mu_1 p_1$$

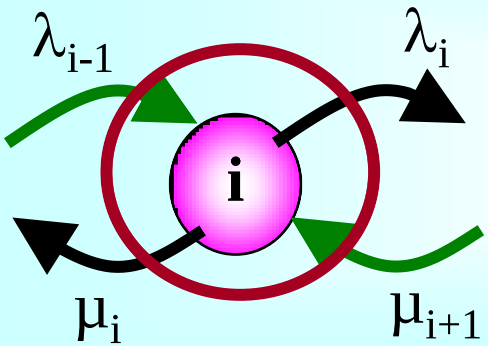
Flow Balance Method



- Draw a closed boundary around state i at infinity

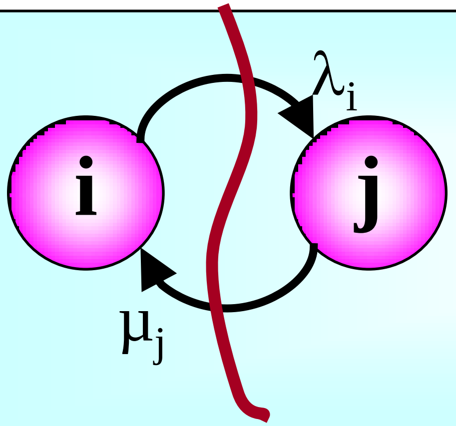
$$\lambda_i p_i = \mu_{i+1} p_{i+1}$$

Flow Balance General Form



- Draw a closed boundary around state i
- ***Global Balance Equation***

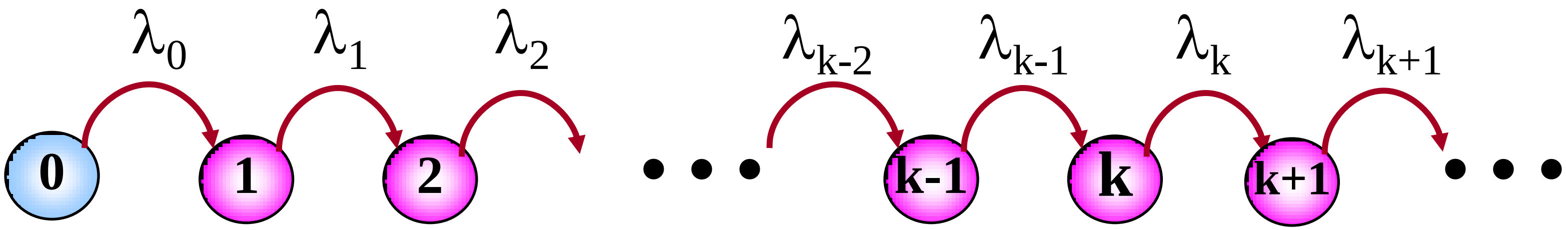
$$\sum_{i \neq j} p_i \, p_{ij} = p_j \sum_{i \neq j} p_{ji}$$



- Draw a closed boundary between state i and j
- ***Detailed Balance Equation***

$$p_i \, p_{ij} = p_j \, p_{ji}$$

A Pure Birth System



- Assumption
  - $\mu_k = 0$ for all k
  - $\lambda_k = \lambda$ for all k
  - The system begins at time t_0 with 0 member

$$p_k(0) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

A Pure Birth System

- $dp_0(t)/dt = -\lambda_0 p_0(t) + \mu_1 p_1(t)$

  ➔ $dp_0(t)/dt = -\lambda p_0(t)$

- $dp_k(t)/dt = -(\lambda_k+\mu_k)\, p_k(t)+\mu_{k+1}p_{k+1}(t)+\lambda_{k-1}p_{k-1}(t)$

  ➔ $dp_k(t)/dt = -\lambda p_k(t) + \lambda p_{k-1}(t)$

- Solution for $p_0(t)$

  ➔ $p_0(t) = e^{-\lambda t}$

$$\frac{d☺}{dt} = -\lambda ☺$$

$$☺ = e^{-\lambda t}$$

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A Pure Birth System

- For $k = 1$

  $\rightarrow$ $dp_1(t)/dt = -\lambda p_1(t) + \lambda p_0(t)$

  $= -\lambda p_1(t) + \lambda e^{-\lambda t}$

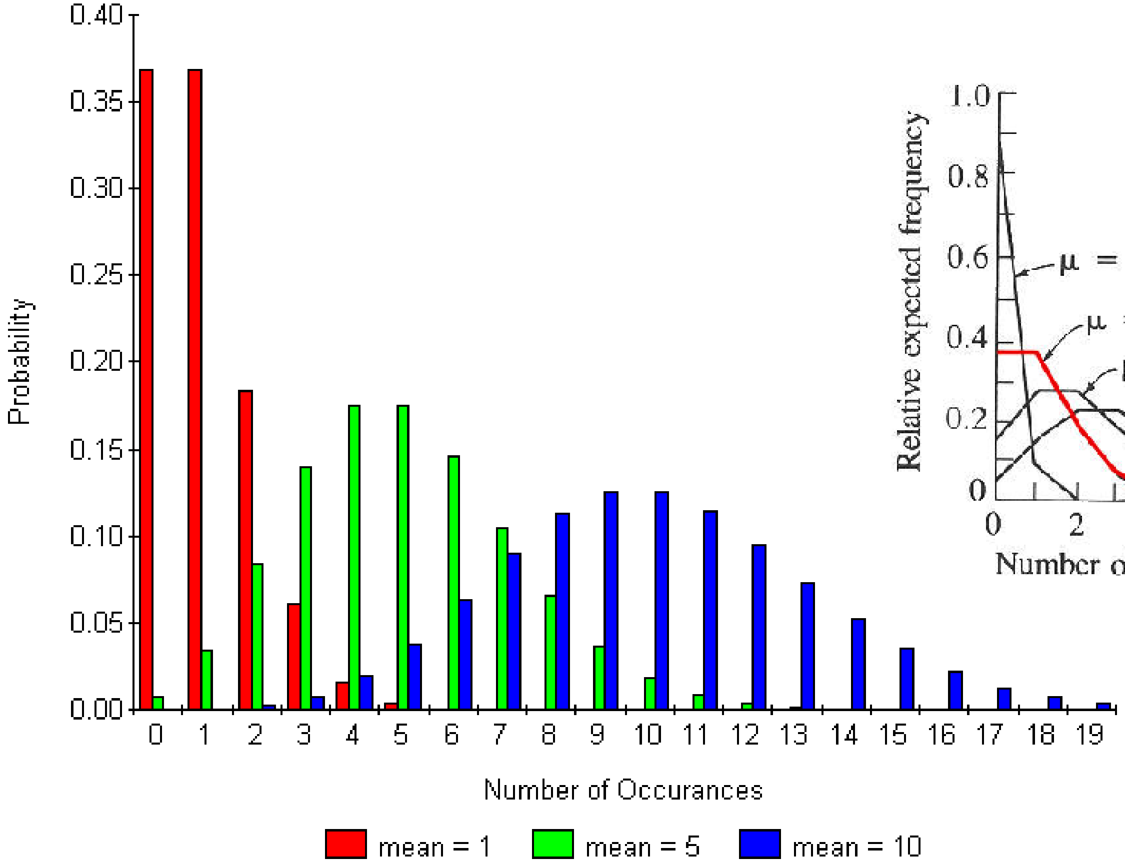
  $\rightarrow$ $p_1(t) = \lambda t e^{-\lambda t}$

- For $k \geq 0,\ t \geq 0$

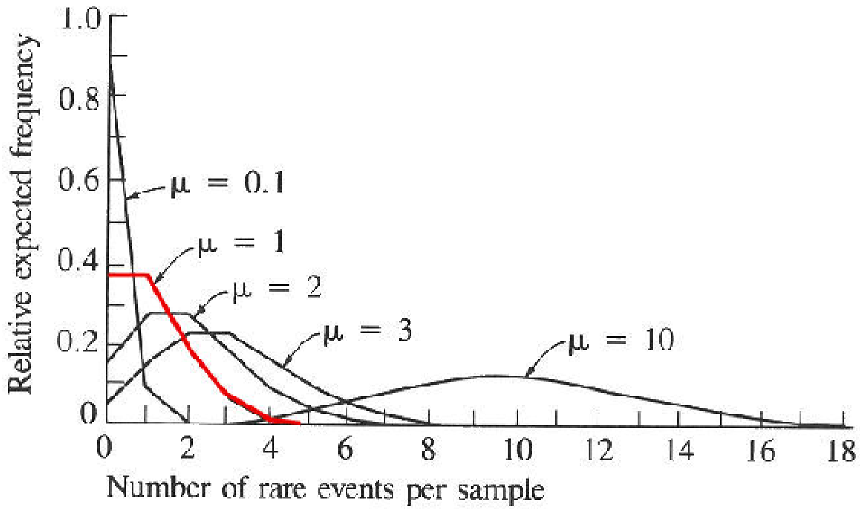
  $$\rightarrow p_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

  Poisson Distribution

Poisson Distribution



http://www.boost.org/doc/libs/1_35_0/libs/math/doc/sf_and_dist/graphs/poisson.png



http://www.mun.ca/biology/scarr/
Poisson_distribution3.jpg

A Poisson Process

- The arrival of customers
- λ = the average rate that customer arrives
- $p_k(t)$ = Prob. that *k arrivals* occur during *(0,t)*
- K = # of arrivals in the interval *t*
- The average # of arrivals in an interval *t*, E[K] = ?

A Poisson Process

$$E[K] = \sum_{k=0}^{\infty} k\ p_k(t) \qquad = e^{-\lambda t} \sum_{k=0}^{\infty} k \frac{(\lambda t)^k}{k!}$$

$$= e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!}$$

$$= e^{-\lambda t}\, \lambda t \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}$$

$$e^{\lambda t}$$

$$= \lambda t$$

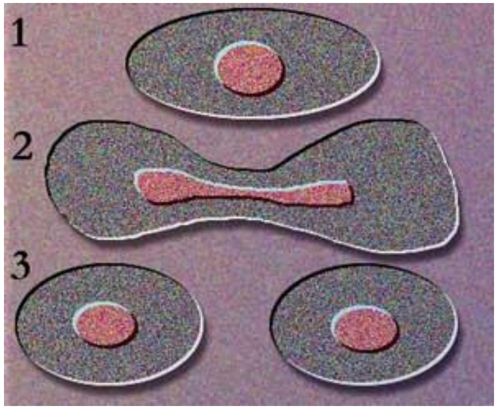
Pure Birth Process Example

- Linear Birth Process
- Yule-Furry Process
- Consider cells which reproduce according to the following rules:
  1) A cell presented at time t has probability $\lambda\Delta t + o(\Delta t)$ of splitting in two in the interval $(t, t + \Delta t)$
  2) This probability is independent of age
  3) Events between different cells are independent

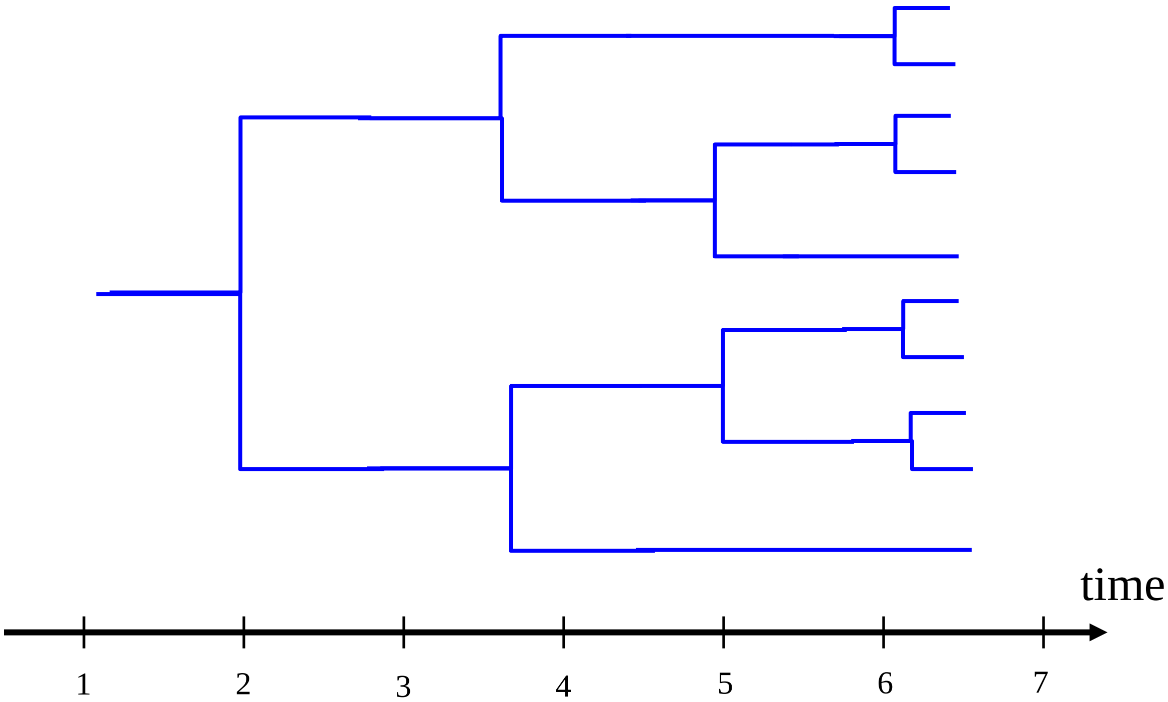
Modified from
1. http://www.bibalex.org/supercourse/supercourseppt/19011-20001/19531.pdf
2. The theory of stochastic processes By D. R. Cox, H. D. Miller

Pure Birth Process Example

Cell Division



http://www.dmturner.org/Teacher/Library/5thText/SimplePart3.html



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Pure Birth Process Example

Non-Probabilistic Analysis

- $n(t)$ = no. of cells at time t
- λ = birth rate per single cell
- $n(t)\lambda\Delta t$ births occur in $(t, t + \Delta t)$

$$n(t + \Delta t) = n(t) + n(t)\lambda\Delta t$$

$$n'(t) = \frac{n(t + \Delta t) - n(t)}{\Delta t} = n(t)\lambda$$

$$\frac{n'(t)}{n(t)} = \frac{d}{dt} \log n(t) = \lambda$$

$$\log n(t) = \lambda t + c$$

$$n(t) = Ke^{\lambda t} \quad , n(0) = n_0$$

$$n(t) = n_0 e^{\lambda t}$$

Pure Birth Process Example
Probabilistic Analysis

- $N(t)$ = no. of cells at time t
- $P\{N(t) = n\} = P_n(t)$
- Prob. of birth in $(t, t + \Delta t)$ if $\{N(t) = n\}$ $= n\Delta t + o(\Delta t)$

$$P_n(t + \Delta t) \quad = P_n(t)(1 - n\,\lambda\Delta t + o(\Delta t)) + P_{n-1}(t)((n-1)\lambda\Delta t + o(\Delta t))$$

$$P_n(t + \Delta t) - P_n(t) = - n\lambda\Delta t P_n(t) + P_{n-1}(t)(n-1)\lambda\Delta t \; + o(\Delta t)$$

$$\frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = - n\lambda P_n(t) + P_{n-1}(t)(n-1)\lambda \; + o(\Delta t) \qquad \text{as } \Delta t \rightarrow 0$$

$$P'_n(t) = - n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)$$

Pure Birth Process Example
Probabilistic Analysis

$$P'_n(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)$$

- Initial condition: $P_{n_0}(0) = P\{n(0) = n_0\} = 1$

$$P_n(t) = \begin{pmatrix} n-1 \\ n-n_0 \end{pmatrix} e^{-\lambda n_0 t}(1 - e^{-\lambda t})^{n-n_0} \qquad n = n_0, n_0+1, \ldots$$

- Solution is negative binomial distribution
  - = Probability of obtaining exactly n_0 successes in n trials

Pure Birth Process Example

Probabilistic Analysis

- Suppose p = prob. of success
  and q = 1 – p = prob. of failure
- Then in the first (n – 1) trials results in (n_0– 1) successes and (n – n_0) failures followed by success on n^{th} trial

$$P_n(t) = \begin{pmatrix} n-1 \\ n_0-1 \end{pmatrix} p^{n_0-1} q^{n-n_0}.p \;=\; \begin{pmatrix} n-1 \\ n-n_0 \end{pmatrix} p^{n_0} q^{n-n_0}$$

$$n = n_0, n_0+1, \ldots$$

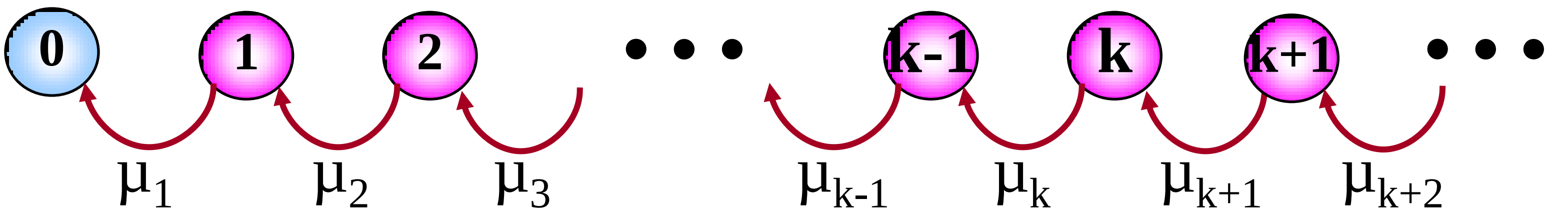
- If $p = e^{-\lambda t}$ and $q = (1 - e^{-\lambda t})$
  - → $P_n(t)$ is as same as previous equation

Yule-Furry Process

- Yule studied this process in connection with theory of evolution
  - i.e. population consists of the species within a genus and creation of new element is due to mutations
  - Neglects probability of species dying out and size of species
- Furry used same model for radioactive transmutations

a **genus** is a low-level taxonomic rank used in the classification of living

A Pure Death System



- Example
  - Microbial (a bacterium that causes disease) risk analysis
- Assumption
  - $\mu_k = \mu \geq 0$ for all k
  - $\lambda_k = 0$ for all k
  - The system begins with N members
  - $k = 1,2,3,\ldots,N$

A Pure Death Process

$$p_k(t) = \frac{(\mu t)^{N-k}}{(N-k)!} e^{-\mu t} \qquad 0 < k \leq N$$

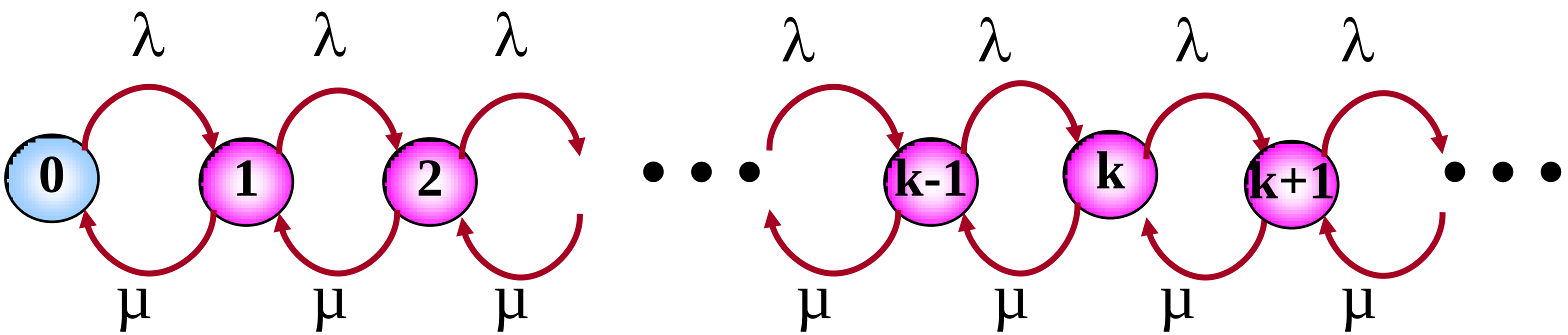
$$\frac{dp_0(t)}{dt} = \frac{\mu(\mu t)^{N-1}}{(N-1)!} e^{-\mu t} \qquad k = 0$$

Erlang Distribution

M/M/1

A Birth-Death Process : M/M/1

- Assumption
  - $\lambda_k = \lambda$ for $k \geq 0$
  - $\mu_k = \mu$ for $k \geq 1$
  - The system begins at time t_0 with 0 member



A Birth-Death Process : M/M/1

- A Birth-Death Process
  - Constant coefficients λ and μ
- Interarrival Time / Service Time / #Servers
- Memoryless / Memoryless / 1 Server
- M/M/1 = A single-server queue with a Poisson arrival and an exponential distribution for service time

A Birth-Death Process : M/M/1

- A(t) = CDF of the arrival time
  - $\lambda_k = \lambda$ for $k \geq 0$
  - $A(t) = 1 - e^{-\lambda t}$
- B(x) = CDF of the service time
  - $\mu_k = \mu$ for $k \geq 1$
  - $B(x) = 1 - e^{-\mu x}$

A Birth-Death Process : M/M/1

- $dp_0(t)/dt = -\lambda p_0(t) + \mu p_1(t)$
- $dp_k(t)/dt = -(\lambda+\mu)p_k(t) + \lambda p_{k-1}(t) + \mu p_{k+1}(t)$
- Now we try to find the solution!
  - Hint: Solved by using *z*-transforms

A Birth-Death Process : M/M/1

$$p_k(t) = e^{-(\lambda+\mu)t}\left(\begin{array}{l} \rho^{(k-i)/2} I_{k-i}(at) \\ + \rho^{(k-i-1)/2} I_{k+i+1}(at) \\ + (1-\rho)\rho^{k} \sum_{j=k+i+2}^{\infty} \rho^{-j/2} I_j(at) \end{array}\right)$$

- Where $\rho = \lambda/\mu$ and $a = 2\mu\rho^{1/2}$
- $I_k(x) = \sum_{m=0}^{\infty} \dfrac{(x/2)^{k+2m}}{(k+m)!m!} \quad k \geq -1$
- Time dependent behavior of the state prob.

Equilibrium Solution

- The time-dependent solution is unmanageable
- $p_k(t) \rightarrow$ not too useful $\rightarrow$ transient
- Let $p_k \equiv$ limiting probability (system = k members)

$$= \lim_{t \rightarrow \infty} p_k(t) \quad = \text{ System in state } E_k$$

- p_k is not time-dependent

Equilibrium Solution

- From

$$dp_0(t)/dt = -\lambda_0 p_0(t) + \mu_1 p_1(t)$$

$$dp_k(t)/dt = -(\lambda_k+\mu_k)p_k(t) + \lambda_{k-1}p_{k-1}(t) + \mu_{k+1}p_{k+1}(t)$$

- If set $\lim_{t \to \infty} dp_k(t)/dt = 0$

- Obtain the result

$$0 = -\lambda_0 p_0 + \mu_1 p_1 \qquad k = 0$$

$$0 = -(\lambda_k+\mu_k)p_k + \lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1} \qquad k \geq 1$$

Equilibrium Solution

- From conservation relation $$\sum_{k=0}^{\infty} p_k(t) = 1$$
- Find the answer for p_0 and p_i

Equilibrium Solution

- From

$$0 = -\lambda_0 p_0 + \mu_1 p_1 \qquad k = 0$$

$$0 = -(\lambda_k + \mu_k) p_k + \lambda_{k-1} p_{k-1} + \mu_{k+1} p_{k+1} \qquad k \geq 1$$

- Yield

$$\lambda_0 p_0 = \mu_1 p_1$$

$$p_1 = (\lambda_0 / \mu_1)\, p_0$$

Equilibrium Solution

- And for k = 1

$$(\lambda_k+\mu_k)p_k = \lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1}$$

$$(\lambda_1+\mu_1)p_1 = \lambda_0 p_0 + \mu_2 p_2$$

$$(\lambda_1+\mu_1)(\lambda_0/\mu_1)p_0 = \lambda_0 p_0 + \mu_2 p_2$$

$$(\lambda_1\lambda_0/\mu_1)p_0 + \lambda_0 p_0 = \lambda_0 p_0 + \mu_2 p_2$$

$$(\lambda_1\lambda_0/\mu_1)p_0 = \mu_2 p_2$$

$$p_2 = \frac{\lambda_1\lambda_0}{\mu_1\mu_2}p_0$$

Equilibrium Solution

$$p_0 = \left(1 + \sum_{i=0}^{\infty} \prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}}\right)^{-1}$$

$$p_i = p_0 \left(\prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}} \right) \qquad \forall i \geq 1$$

Homework

- Please check our class web site for homework assignment