

# LECTURE #6

## BIRTH-DEATH PROCESS

204528

Queueing Theory and  
Applications in Networks

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# Outline

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- Birth-Death Process
- Markov Process Property
- Continuous Time Birth-Death Markov Chains
- State Transition Diagram
- A Pure Birth System
- A Pure Death System
- A Birth-Death Process
- Equilibrium Solution

# Birth-Death Process

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- A Markov Process
- Homogeneous, aperiodic, and irreducible
- Discrete time / Continuous time
- State changes can only happen between neighbors

# Birth-Death Process

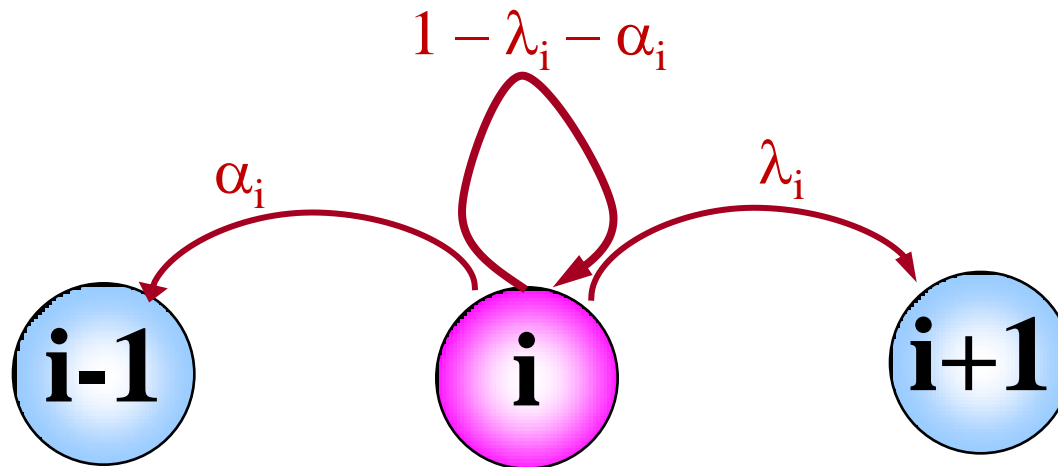
4

- Size of population
  - System is in state  $E_k$  when consists of  $k$  members
  - Changes in population size occur by at most one
  - Size has been increased by one  $\rightarrow$  “*Birth*”
  - Size has been decreased by one  $\rightarrow$  “*Death*”
- Transition probabilities  $p_{ij}$  do not change with time

# Birth-Death Process

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$$p_{ij} = \begin{cases} \alpha_i & j = i - 1 \\ 1 - \lambda_i - \alpha_i & j = i \\ \lambda_i & j = i + 1 \\ 0 & \text{Otherwise} \end{cases}$$



# Birth-Death Process

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- $\alpha_i =$  death (less one in population size)
- $\alpha_0 = 0$  (no population  $\rightarrow$  no death)
- $\lambda_i =$  birth (increase one in population)
- $\lambda_i > 0$  (birth is allowed)
- Pure Birth = no decrement, only increment
- Pure Death = no increment, only decrement

# Queueing Theory Model

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- **Population** = customers in the queueing system
- **Death** = a customer departures from the system
- **Birth** = a customer arrives to the system

# Transition matrix

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$$P = \begin{bmatrix} 1 - \lambda_0 & \lambda_0 & 0 & 0 & 0 & 0 & \dots \\ \alpha_1 & 1 - \lambda_1 - \alpha_1 & \lambda_1 & 0 & 0 & 0 & \\ 0 & \alpha_2 & 1 - \lambda_2 - \alpha_2 & \lambda_2 & & & \\ 0 & & \dots & & & & \\ 0 & & & & & & \\ \dots & & & \alpha_i & 1 - \lambda_i - \alpha_i & \lambda_i & \end{bmatrix}$$



# Discrete Time Markov Chains

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- One can stay in a *Discrete state (position)* and is permitted to change state at *Discrete time*.

# Discrete Time Markov Chains

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$$\begin{aligned} P\{X_n = j \mid X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\} \\ = P\{X_n = j \mid X_{n-1} = i_{n-1}\} \quad \text{Where } n = 1, 2, 3, \dots \end{aligned}$$

- $X_n$ : The system is in state  $j$  at time  $n$
- The system can begin at *state 0* with *initial probability*  $P[X_0 = x]$
- $P\{X_n = j \mid X_{n-1} = i_{n-1}\}$  is the *one-step transition probability*

# Discrete Time Markov Chains

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- From *initial probability* and *one-step transition probability*,
- we can find *probability of being in various states at time  $n$*

# Continuous Time Markov Chains

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$$\begin{aligned} P\{X(t_{n+1}) = j \mid X(t_1) = i_1, X(t_2) = i_2, \dots, X(t_n) = i_n\} \\ = P\{X(t_{n+1}) = j \mid X(t_n) = i_n\} \end{aligned}$$

Where  $n = 1, 2, 3, \dots$        $t_1 < t_2 < \dots < t_n$

- One can stay in a *Discrete state (position)* and is permitted to change state at *Arbitrary time*

# Markov Process Property

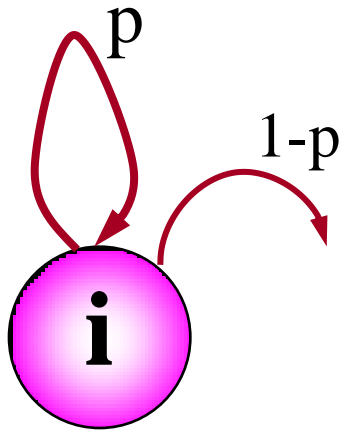
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- Time that the process spends in any state must be “Memoryless”
- Discrete Time Markov Chains
  - Geometrically distributed state times
- Continuous Time Markov Chains
  - Exponentially distributed state times

# Markov Process Property

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## *For Discrete Time Markov Chain*

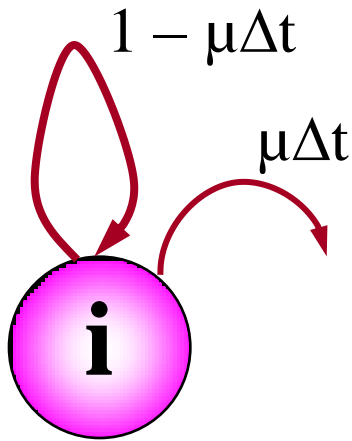


- $P[\text{system in state } i \text{ for } N \text{ time units} \mid \text{system in current state } i] = p^N$
- $P[\text{system in state } i \text{ for } N \text{ time units before exiting from state } i] = p^N (1-p)$
- Geometrically distributed state times

# Markov Process Property

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## *For Continuous Time Markov Chain*



- $P[\text{system in state } i \text{ for time } T \mid \text{system in current state } i]$   
 $= (1 - \mu\Delta t)^{T/\Delta t}$   
 $= e^{-\mu T}$  where  $\Delta t \rightarrow 0$
- Exponentially distributed state times

# Continuous Time Birth-Death Markov Chains

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- Let  $\lambda_i =$  birth rate in state  $i$   
 $\mu_i =$  death rate in state  $i$

- Then

$$P[\text{state } i \text{ to state } i - 1 \text{ in } \Delta t] = \mu_i \Delta t$$

$$P[\text{state } i \text{ to state } i + 1 \text{ in } \Delta t] = \lambda_i \Delta t$$

$$P[\text{state } i \text{ to state } i \text{ in } \Delta t] = 1 - (\lambda_i + \mu_i) \Delta t$$

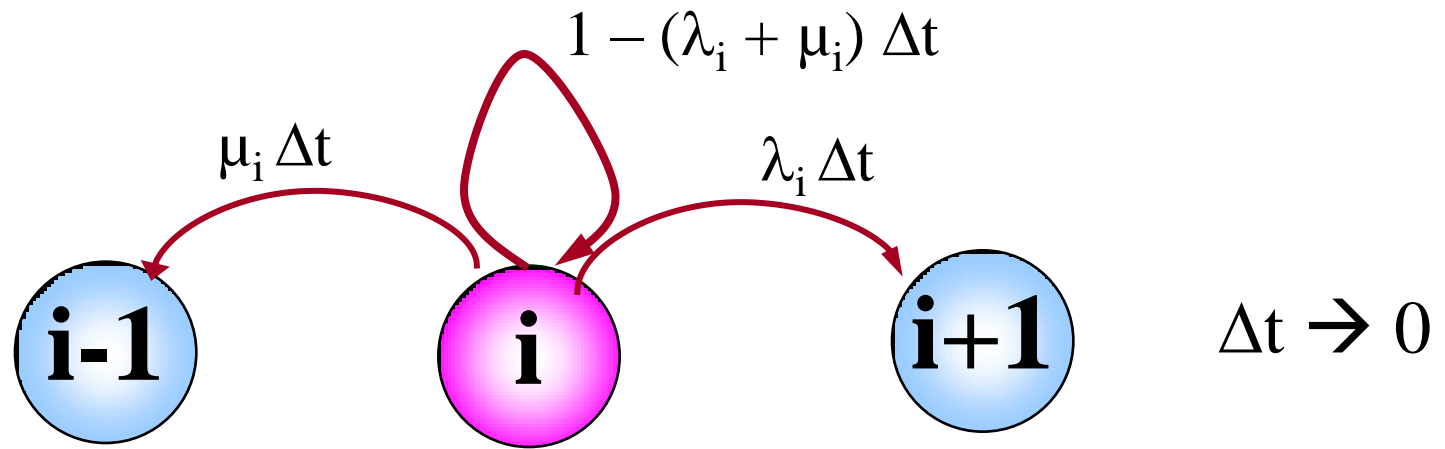
$$P[\text{state } i \text{ to other state in } \Delta t] = 0$$



# Continuous Time Birth-Death Markov Chains

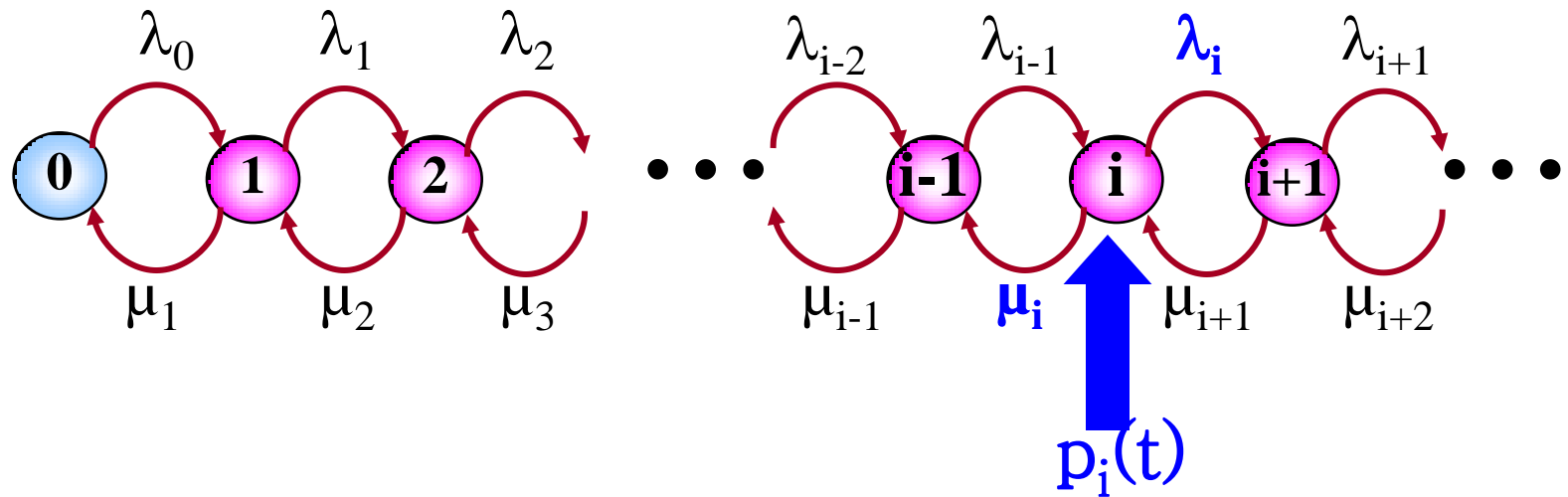
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$$p_{ij} = \begin{cases} \mu_i \Delta t & j = i - 1 \\ 1 - (\lambda_i + \mu_i) \Delta t & j = i \\ \lambda_i \Delta t & j = i + 1 \\ 0 & \text{Otherwise} \end{cases}$$



# State Transition Diagram

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- $X(t)$  = #customers in the system at time  $t$   
= birth – death in  $(0, t)$
- $p_i(t) = P[ X(t) = i ]$   
= Prob. that system is in state  $i$  at time  $t$

# State Transition Diagram

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- From  $t$  to  $t + \Delta t$

$$p_0(t+\Delta t) = p_0(t)[1 - \lambda_0\Delta t] + p_1(t)\mu_1\Delta t$$

$$p_i(t+\Delta t) = p_i(t)[1 - (\lambda_i+\mu_i)\Delta t] + p_{i+1}(t)\mu_{i+1}\Delta t + p_{i-1}(t)\lambda_{i-1}\Delta t$$

- $\Delta t \rightarrow 0$

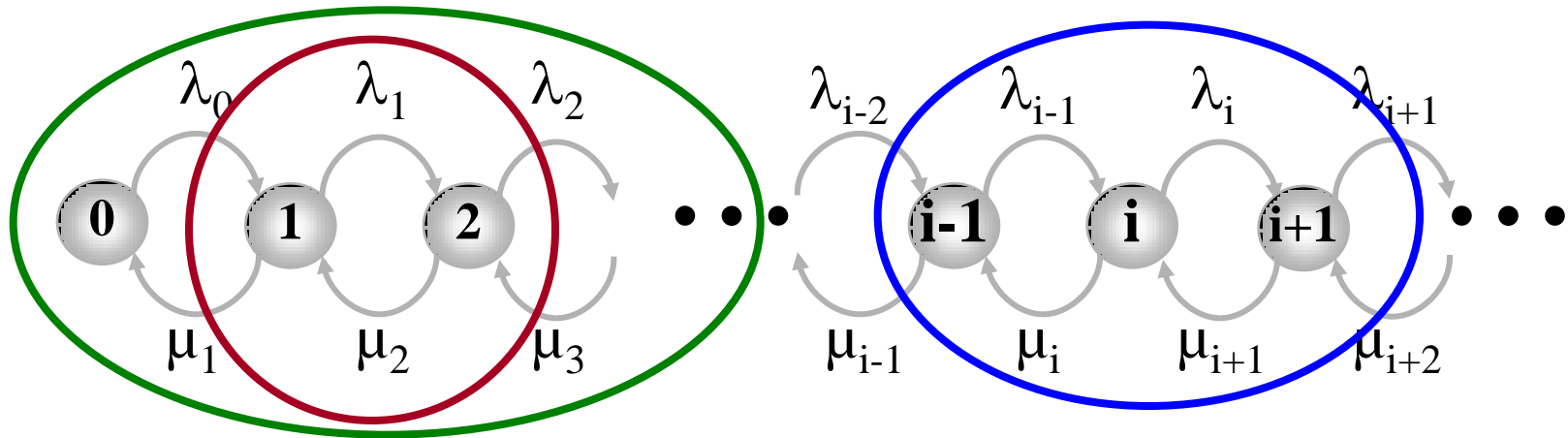
$$dp_0(t)/dt = -\lambda_0p_0(t) + \mu_1p_1(t)$$

$$dp_i(t)/dt = -(\lambda_i+\mu_i)p_i(t) + \mu_{i+1}p_{i+1}(t) + \lambda_{i-1}p_{i-1}(t)$$

- $\sum_{i=0}^{\infty} p_i(t) = 1$

# Flow Balance Method

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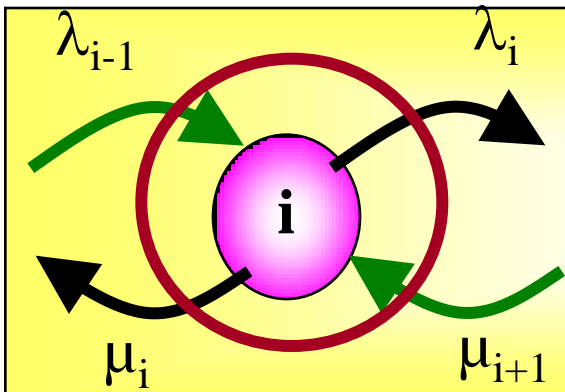


- Draw a closed boundary
- Observe all flows (*In* and *Out*) across the boundary

# Flow Balance Method

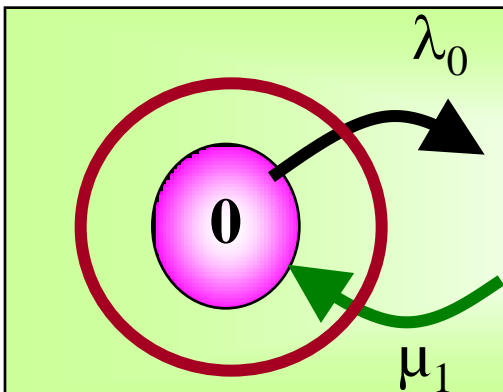
21

- Flow Out = Flow In



- Draw a closed boundary around state  $i$

$$(\lambda_i + \mu_i) p_i = \mu_{i+1} p_{i+1} + \lambda_{i-1} p_{i-1}$$

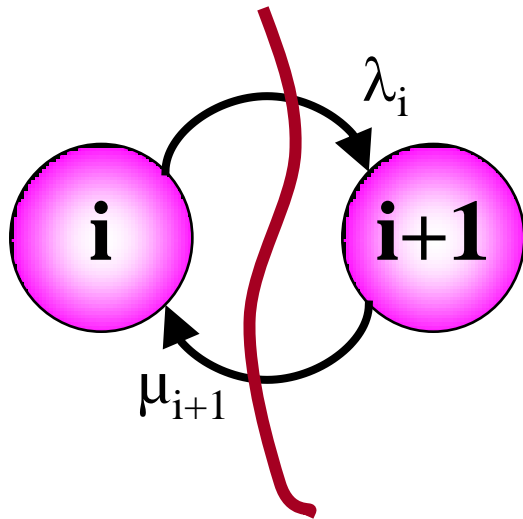


- Draw a closed boundary around state  $0$

$$\lambda_0 p_0 = \mu_1 p_1$$

# Flow Balance Method

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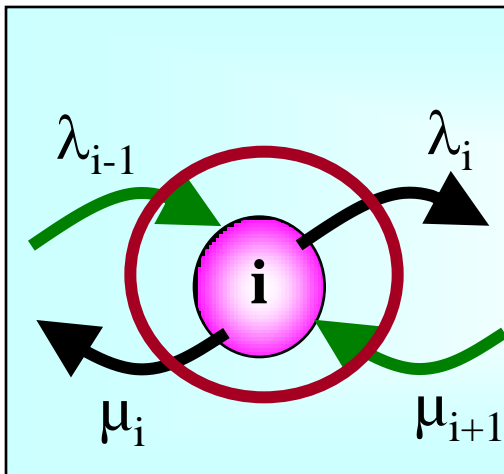


- Draw a closed boundary around state  $i$  at infinity

$$\lambda_i p_i = \mu_{i+1} p_{i+1}$$

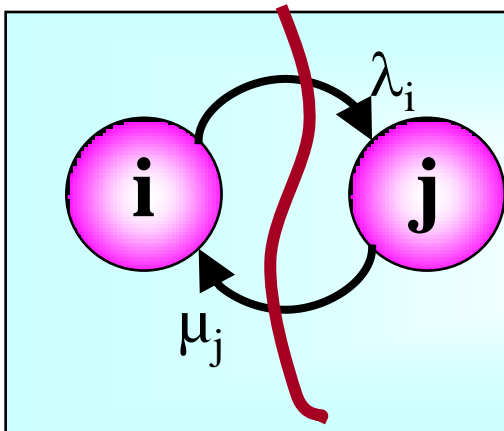
# Flow Balance General Form

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- Draw a closed boundary around state  $i$
- ***Global Balance Equation***

$$\sum_{i \neq j} p_i p_{ij} = p_j \sum_{i \neq j} p_{ji}$$

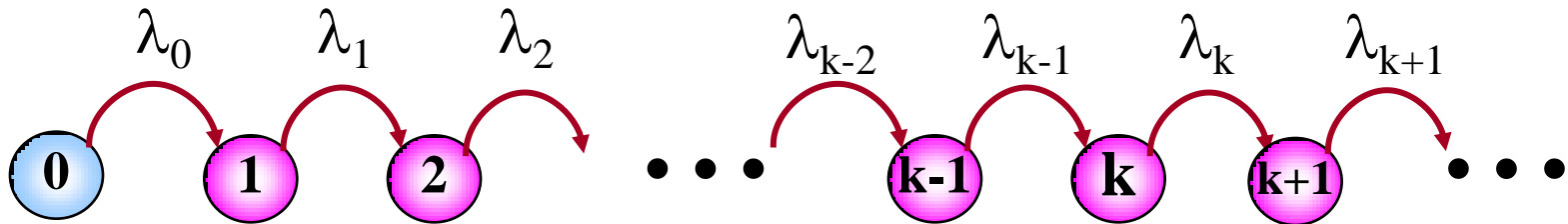


- Draw a closed boundary between state  $i$  and  $j$
- ***Detailed Balance Equation***

$$p_i p_{ij} = p_j p_{ji}$$

# A Pure Birth System

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- Assumption

- $\mu_k = 0$  for all  $k$
- $\lambda_k = \lambda$  for all  $k$
- The system begins at time  $t_0$  with 0 member

$$p_k(0) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$



# A Pure Birth System

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- $dp_0(t)/dt = -\lambda_0 p_0(t) + \mu_1 p_1(t)$   
→  $dp_0(t)/dt = -\lambda p_0(t)$
- $dp_k(t)/dt = -(\lambda_k + \mu_k) p_k(t) + \mu_{k+1} p_{k+1}(t) + \lambda_{k-1} p_{k-1}(t)$   
→  $dp_k(t)/dt = -\lambda p_k(t) + \lambda p_{k-1}(t)$
- Solution for  $p_0(t)$   
→  $p_0(t) = e^{-\lambda t}$

$$\frac{d\text{☺}}{dt} = -\lambda \text{☺}$$

$$\text{☺} = e^{-\lambda t}$$

# A Pure Birth System

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- For  $k = 1$

$$\begin{aligned}\rightarrow dp_1(t)/dt &= -\lambda p_1(t) + \lambda p_0(t) \\ &= -\lambda p_1(t) + \lambda e^{-\lambda t}\end{aligned}$$

$$\rightarrow p_1(t) = \lambda t e^{-\lambda t}$$

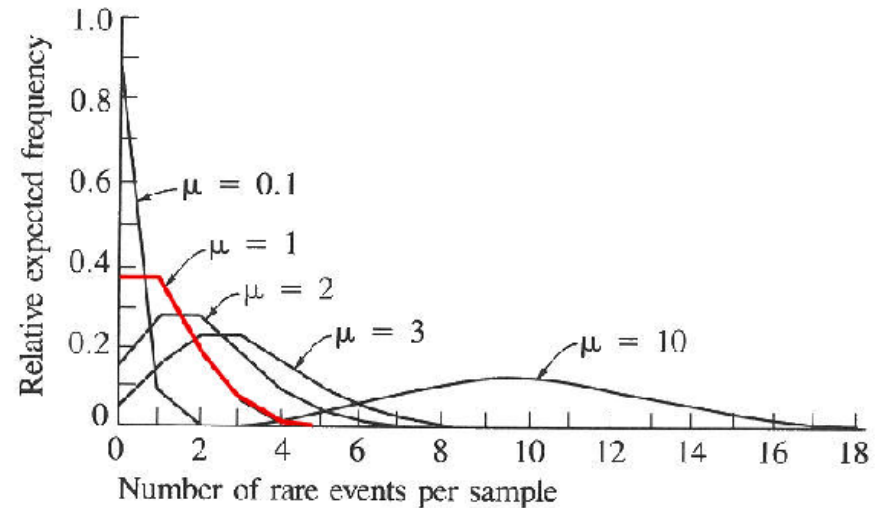
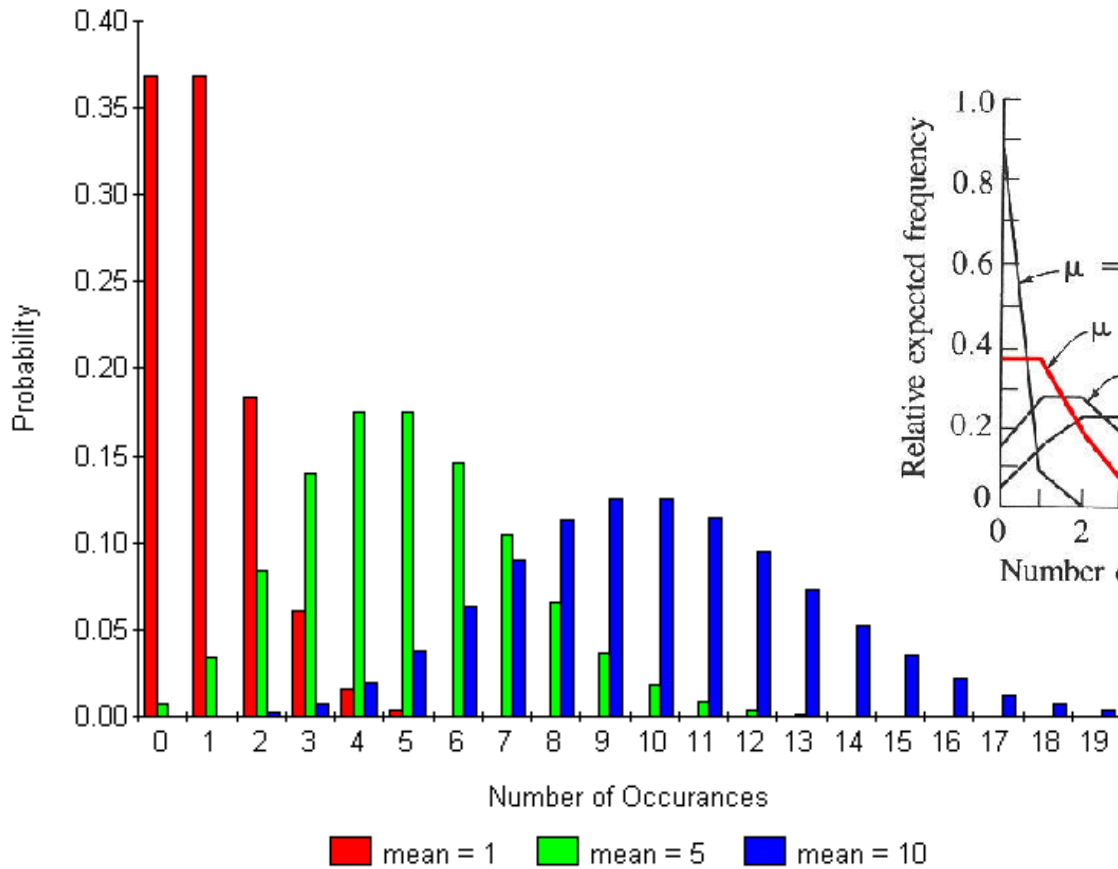
- For  $k \geq 0, t \geq 0$

$$\rightarrow p_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Poisson Distribution

# Poisson Distribution

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[http://www.mun.ca/biology/scarr/Poisson\\_distribution3.jpg](http://www.mun.ca/biology/scarr/Poisson_distribution3.jpg)

[http://www.boost.org/doc/libs/1\\_35\\_0/libs/math/doc/sf\\_and\\_dist/graphs/poisson.png](http://www.boost.org/doc/libs/1_35_0/libs/math/doc/sf_and_dist/graphs/poisson.png)

# A Poisson Process

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- The arrival of customers
- $\lambda$  = the average rate that customer arrives
- $p_k(t)$  = Prob. that  $k$  arrivals occur during  $(0, t)$
- $K$  = # of arrivals in the interval  $t$
- The average # of arrivals in an interval  $t$ ,  $E[K] = ?$

# A Poisson Process

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$$\begin{aligned} E[K] &= \sum_{k=0}^{\infty} k p_k(t) &= e^{-\lambda t} \sum_{k=0}^{\infty} k \frac{(\lambda t)^k}{k!} \\ & &= e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!} \\ & &= e^{-\lambda t} \lambda t \left( \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \right) && \nearrow e^{\lambda t} \\ & &= \lambda t \end{aligned}$$

# Pure Birth Process Example

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- Linear Birth Process
- Yule-Furry Process
- Consider **cells** which **reproduce** according to the following rules:
  - 1) A cell presented at time  $t$  has probability  $\lambda\Delta t + o(\Delta t)$  of splitting in two in the interval  $(t, t + \Delta t)$
  - 2) This probability is independent of age
  - 3) Events between different cells are independent

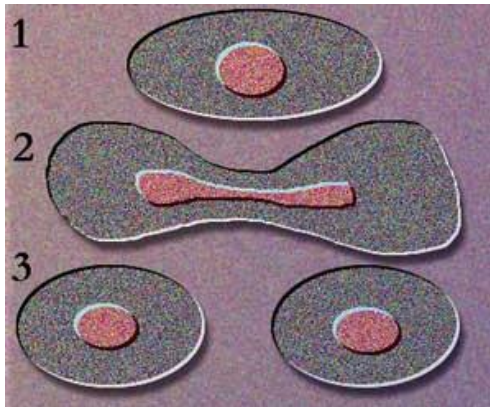
Modified from

1. <http://www.bibalex.org/supercourse/supercourseppt/19011-20001/19531.pdf>
2. The theory of stochastic processes By D. R. Cox, H. D. Miller

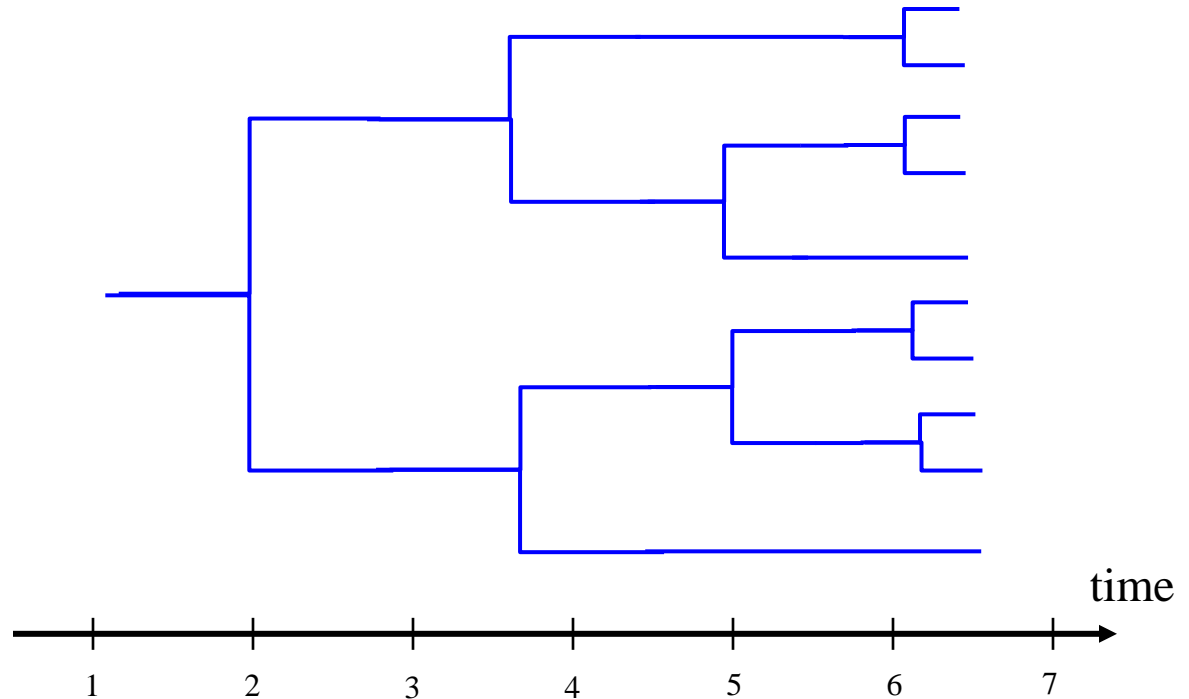
# Pure Birth Process Example

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## Cell Division



<http://www.dmturner.org/Teacher/Library/5thText/SimplePart3.html>



# Pure Birth Process Example

## Non-Probabilistic Analysis

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- $n(t)$  = no. of cells at time  $t$
- $\lambda$  = birth rate per single cell
- $n(t)\lambda\Delta t$  births occur in  $(t, t + \Delta t)$

$$n(t + \Delta t) = n(t) + n(t)\lambda\Delta t$$

$$n'(t) = \frac{n(t + \Delta t) - n(t)}{\Delta t} = n(t)\lambda$$
$$\frac{n'(t)}{n(t)} = \frac{d}{dt} \log n(t) = \lambda$$

$$\log n(t) = \lambda t + c$$

$$n(t) = Ke^{\lambda t}, \quad n(0) = n_0$$

$$n(t) = n_0 e^{\lambda t}$$



# Pure Birth Process Example

## Probabilistic Analysis

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- $N(t)$  = no. of cells at time  $t$
- $P\{N(t) = n\} = P_n(t)$
- Prob. of birth in  $(t, t + \Delta t)$  if  $\{N(t) = n\} = n\Delta t + o(\Delta t)$

$$P_n(t + \Delta t) = P_n(t)(1 - n\lambda\Delta t + o(\Delta t)) + P_{n-1}(t)((n-1)\lambda\Delta t + o(\Delta t))$$

$$P_n(t + \Delta t) - P_n(t) = -n\lambda\Delta t P_n(t) + P_{n-1}(t)(n-1)\lambda\Delta t + o(\Delta t)$$

$$\frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = -n\lambda P_n(t) + P_{n-1}(t)(n-1)\lambda + o(\Delta t) \quad \text{as } \Delta t \rightarrow 0$$

$$P'_n(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)$$

# Pure Birth Process Example

## Probabilistic Analysis

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$$P'_n(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)$$

- Initial condition:  $P_{n_0}(0) = P\{n(0) = n_0\} = 1$

$$P_n(t) = \binom{n-1}{n-n_0} e^{-\lambda n_0 t} (1 - e^{-\lambda t})^{n-n_0} \quad n = n_0, n_0 + 1, \dots$$

- Solution is negative binomial distribution
  - = Probability of obtaining exactly  $n_0$  successes in  $n$  trials

# Pure Birth Process Example

## Probabilistic Analysis

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- Suppose  $p = \text{prob. of success}$   
and  $q = 1 - p = \text{prob. of failure}$
- Then in the first  $(n - 1)$  trials results in  $(n_0 - 1)$  successes and  $(n - n_0)$  failures followed by success on  $n^{\text{th}}$  trial

$$P_n(t) = \binom{n-1}{n_0-1} p^{n_0-1} q^{n-n_0} \cdot p = \binom{n-1}{n-n_0} p^{n_0} q^{n-n_0}$$

- If  $p = e^{-\lambda t}$  and  $q = (1 - e^{-\lambda t})$   $n = n_0, n_0 + 1, \dots$ 
  - $\rightarrow P_n(t)$  is as same as previous equation

# Yule-Furry Process

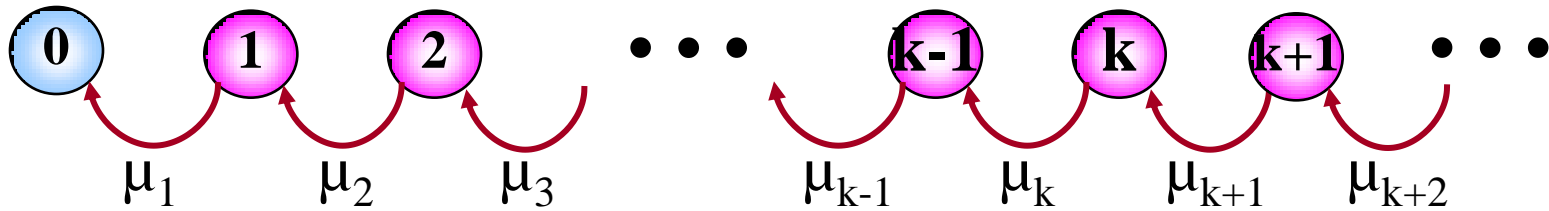
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- Yule studied this process in connection with theory of evolution
  - i.e. population consists of the species within a genus and creation of new element is due to mutations
  - Neglects probability of species dying out and size of species
- Furry used same model for radioactive transmutations

a **genus** is a low-level taxonomic rank used in the classification of living

# A Pure Death System

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- Example
  - Microbial (a bacterium that causes disease) risk analysis
- Assumption
  - $\mu_k = \mu \geq 0$  for all  $k$
  - $\lambda_k = 0$  for all  $k$
  - The system begins with  $N$  members
  - $k = 1, 2, 3, \dots, N$

# A Pure Death Process

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$$p_k(t) = \frac{(\mu t)^{N-k}}{(N-k)!} e^{-\mu t} \quad 0 < k \leq N$$

$$\frac{dp_0(t)}{dt} = \frac{\mu(\mu t)^{N-1}}{(N-1)!} e^{-\mu t} \quad k = 0$$

Erlang Distribution

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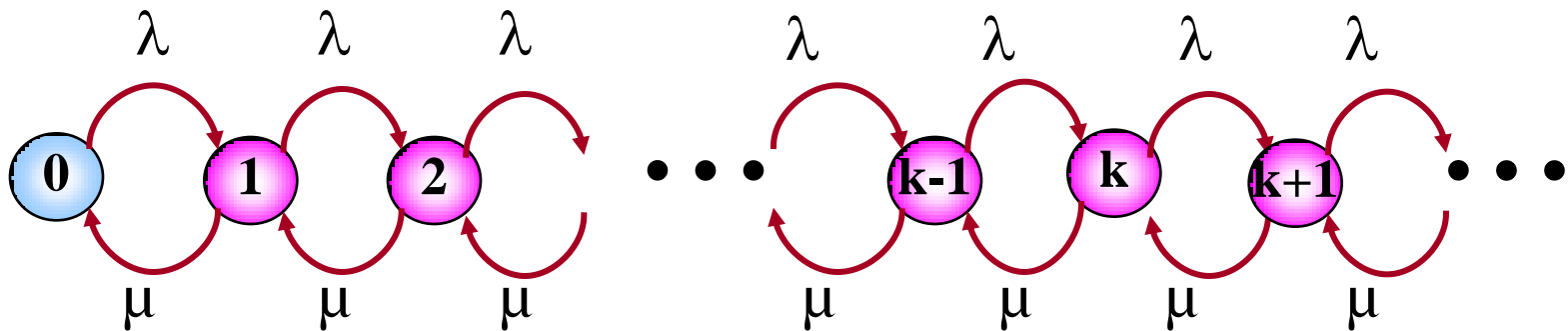
**M/M/1**

# A Birth-Death Process : M/M/1

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- Assumption

- $\lambda_k = \lambda$  for  $k \geq 0$
- $\mu_k = \mu$  for  $k \geq 1$
- The system begins at time  $t_0$  with 0 member





# A Birth-Death Process : M/M/1

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- A Birth-Death Process
  - Constant coefficients  $\lambda$  and  $\mu$
- Interarrival Time / Service Time / #Servers
- Memoryless / Memoryless / 1 Server
- M/M/1 = A single-server queue with a Poisson arrival and an exponential distribution for service time

# A Birth-Death Process : M/M/1

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- $A(t)$  = CDF of the arrival time
  - $\lambda_k = \lambda$  for  $k \geq 0$
  - $A(t) = 1 - e^{-\lambda t}$
- $B(x)$  = CDF of the service time
  - $\mu_k = \mu$  for  $k \geq 1$
  - $B(x) = 1 - e^{-\mu x}$

# A Birth-Death Process : M/M/1

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- $dp_0(t)/dt = -\lambda p_0(t) + \mu p_1(t)$
- $dp_k(t)/dt = -(\lambda + \mu)p_k(t) + \lambda p_{k-1}(t) + \mu p_{k+1}(t)$
- Now we try to find the solution!
  - Hint: Solved by using  $z$ -transforms

# A Birth-Death Process : M/M/1

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$$p_k(t) = e^{-(\lambda+\mu)t} \left( \begin{array}{l} \rho^{(k-i)/2} I_{k-i}(at) \\ + \rho^{(k-i-1)/2} I_{k+i+1}(at) \\ + (1-\rho)\rho^k \sum_{j=k+i+2}^{\infty} \rho^{-j/2} I_j(at) \end{array} \right)$$

- Where  $\rho = \lambda/\mu$  and  $a = 2\mu\rho^{1/2}$
- $I_k(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{k+2m}}{(k+m)!m!} \quad k \geq -1$
- Time dependent behavior of the state prob.

# Equilibrium Solution

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- The time-dependent solution is unmanageable
- $p_k(t) \rightarrow$  not too useful  $\rightarrow$  transient
- Let  $p_k \equiv$  limiting probability (system = k members)  
$$= \lim_{t \rightarrow \infty} p_k(t) = \text{System in state } E_k$$
- $p_k$  is not time-dependent

# Equilibrium Solution

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- From

$$dp_0(t)/dt = -\lambda_0 p_0(t) + \mu_1 p_1(t)$$

$$dp_k(t)/dt = -(\lambda_k + \mu_k) p_k(t) + \lambda_{k-1} p_{k-1}(t) + \mu_{k+1} p_{k+1}(t)$$

- If set  $\lim_{t \rightarrow \infty} dp_k(t)/dt = 0$

- Obtain the result

$$0 = -\lambda_0 p_0 + \mu_1 p_1 \quad k = 0$$

$$0 = -(\lambda_k + \mu_k) p_k + \lambda_{k-1} p_{k-1} + \mu_{k+1} p_{k+1} \quad k \geq 1$$

# Equilibrium Solution

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- From conservation relation  $\sum_{k=0}^{\infty} p_k(t) = 1$
- Find the answer for  $p_0$  and  $p_i$

# Equilibrium Solution

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- From

$$0 = -\lambda_0 p_0 + \mu_1 p_1 \quad k = 0$$

$$0 = -(\lambda_k + \mu_k) p_k + \lambda_{k-1} p_{k-1} + \mu_{k+1} p_{k+1} \quad k \geq 1$$

- Yield

$$\lambda_0 p_0 = \mu_1 p_1$$

$$p_1 = (\lambda_0 / \mu_1) p_0$$



# Equilibrium Solution

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- And for  $k = 1$

$$(\lambda_k + \mu_k)p_k = \lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1}$$

$$(\lambda_1 + \mu_1)p_1 = \lambda_0p_0 + \mu_2p_2$$

$$(\lambda_1 + \mu_1)(\lambda_0/\mu_1)p_0 = \lambda_0p_0 + \mu_2p_2$$

$$(\lambda_1\lambda_0/\mu_1)p_0 + \lambda_0p_0 = \lambda_0p_0 + \mu_2p_2$$

$$(\lambda_1\lambda_0/\mu_1)p_0 = \mu_2p_2$$

$$p_2 = \frac{\lambda_1\lambda_0}{\mu_1\mu_2} p_0$$

# Equilibrium Solution

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$$p_0 = \left( 1 + \sum_{i=0}^{\infty} \prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}} \right)^{-1}$$

$$p_i = p_0 \left( \prod_{k=0}^{i-1} \frac{\lambda_k}{\mu_{k+1}} \right) \quad \forall i \geq 1$$

# Homework

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- Please check our class web site for homework assignment