

LECTURE #5

MARKOV PROCESS

204528

Queueing Theory and
Applications in Networks

Assoc. Prof. Anan Phonphoem, Ph.D. (รศ.ดร. อนันต์ พลเพิ่ม)
Computer Engineering Department, Kasetsart University

Outline

2

- Markov Processes
- Discrete Time Markov Chain
- Homogeneous, Irreducible, Transient/Recurrent, Periodic/Aperiodic
- Ergodic
- Stationary Probability
- Transient Behavior
- Birth-Death Process

Markov Processes

3

- $X(t)$ is a Markov Process if it satisfies the “**Markov (Memoryless) Property**”

$$\begin{aligned} P\{X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \dots, X(t_1) = x_1\} \\ = P\{X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n\} \end{aligned}$$

Where $t_1 < t_2 < \dots < t_{n-1} < t_n < t_{n+1}$

- $X(t)$ only depends upon the current state
- The past history is summarized in the current state

Definition

4

- State Space (E)
 - Set of real numbers containing ranges of RV in a stochastic process

From Markov Processes ...

5

- *Discrete Time Markov Process:*
State changes occur at **integer** points
- *Continuous Time Markov Process:*
State changes occur at **arbitrarily** time

From Markov Processes ...

6

- ***Markov Chain:***
Discrete state space Markov Process
- ***Discrete Time Markov Chain:***
State (Discrete State) changes occur at integer points
- ***Continuous Time Markov Chain:***
State (Discrete State) changes occur at arbitrarily time

Markov Chain

7

- The easiest stochastic process
 - (IID) Independent and Identically distributed RV
- A discrete IID process
 - Markov Chain
 - Lack of memory (Future is independent of the past given the present)
- Named for probabilist “A.A.Markov”
 - 1907 finite state Markov Chain
 - 1930 infinite state by A.N.Kolmogorov

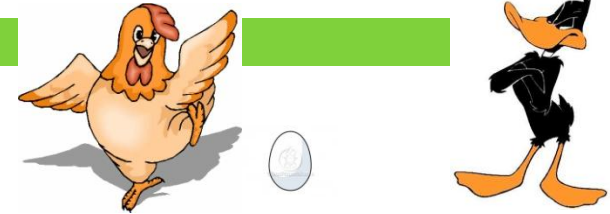


Andrey (Andrei)
Andreyevich Markov
Russia, Born:1856

Markov Chain Examples

8

- Egg game
 - *Kai* has 10 eggs, *Ped* has 10 eggs
 - Toss a die
 - If {1,2,3} occurs → *Ped* gives one egg to *Kai*
 - If {4,5,6} occurs → *Kai* gives one egg to *Ped*
 - Game ends when 20-0 or 0-20 (*Kai* or *Ped* has all eggs)
- Markov chain
 - The fifth play does not depend on the second play



Discrete Time Markov Chains

9

- One can stay in a *Discrete state (position)* and is permitted to change state at *Discrete time*.

Discrete Time Markov Chains

10

$$\begin{aligned} P\{X_n = j \mid X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}\} \\ = P\{X_n = j \mid X_{n-1} = i_{n-1}\} \quad \text{Where } n = 1, 2, 3, \dots \end{aligned}$$

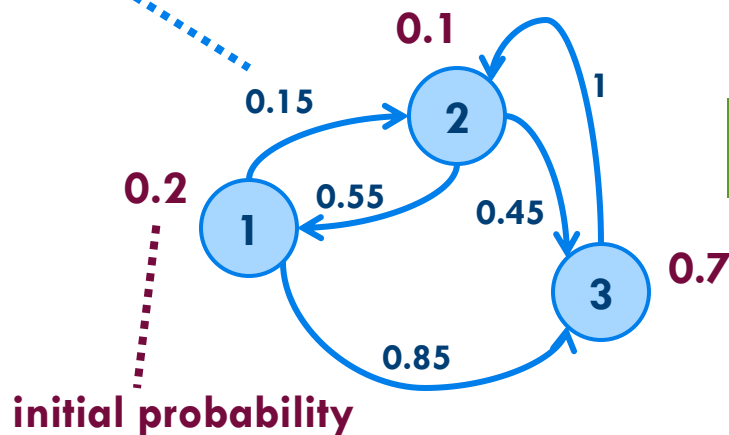
- X_n : The system is in state j at time n
- The system can begin at *state 0* with *initial probability* $P[X_0 = x]$
- $P\{X_n = j \mid X_{n-1} = i_{n-1}\}$ is the *one-step transition probability*

Discrete Time Markov Chains

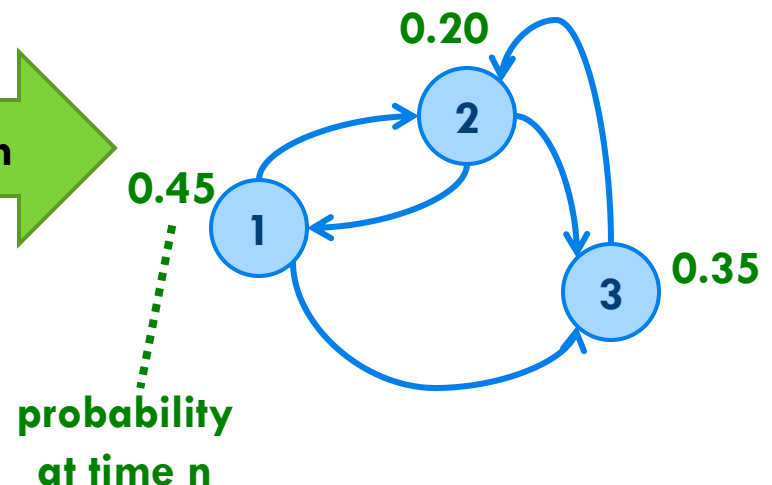
11

- From *initial probability* and *one-step transition probability*,
 - we can find *probability of being in various states at time n*

one-step transition probability



@ time n



Homogeneous Markov Chain

12

- If *transition probabilities* are independent of time n , it is called **Homogeneous Markov Chain**.
- Let
$$p_{ij} \equiv P[X_n = j \mid X_{n-1} = i]$$
- We are in *state i* and going to be in *state j* in the next step
- The state transition prob. will only depend on
 - the *initial probability* and
 - the *transition probability*
 - regardless of transition time

Markov Matrix

13

- A nonnegative square matrix
- Sum value of each row = 1

$$P = \begin{pmatrix} 0 & 0.3 & 0.7 \\ 0.1 & 0.2 & 0.7 \\ 0.8 & 0.2 & 0 \end{pmatrix}$$

Sum = 1
Sum = 1
Sum = 1

Example 1: Jump Game

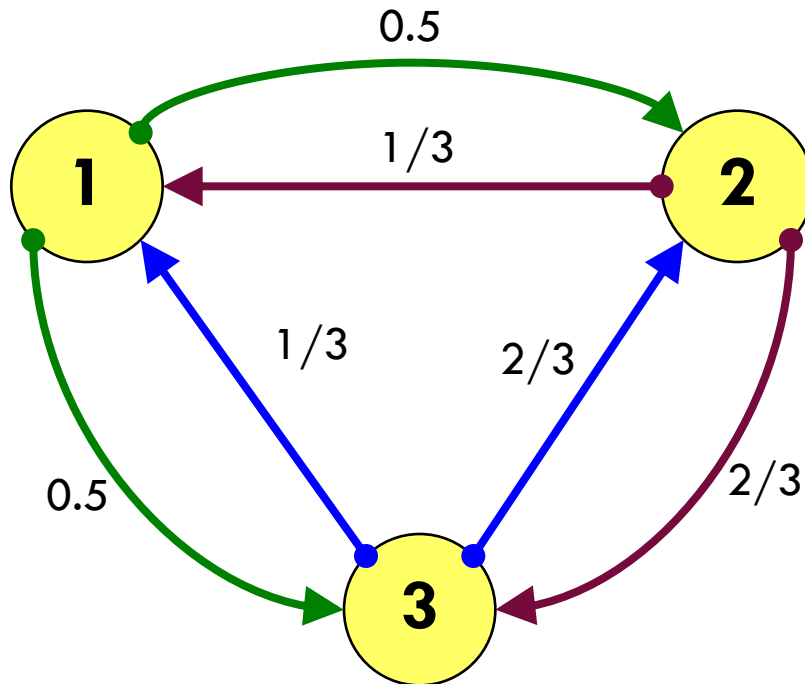
14



- A boy will jump among Plate#1, 2, & 3
- Every time he will change the plate
 - When he is on Plate #1, uses a coin tossing mechanism
 - If “head” \rightarrow he jumps to Plate #2
 - If “tail” \rightarrow he jumps to Plate #3
 - After that, uses a die tossing mechanism
 - If $\{1,2\}$ \rightarrow he will jump back to plate #1
 - Otherwise \rightarrow he will jump to another plate
- Draw a Markov chain

Example 1: Jump Game

15



$$P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 1/3 & 0 & 2/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

Markov Matrix

State space $E = \{1, 2, 3\}$

Random variable $X_n = 1$ or 2 or 3

Example 2: Weather Forecast

16

- Observe the weather in an area
 - Typically longer periods of rainy or dry(sunshine) days
 - **Rain** and **sunshine** are same relative frequency over the entire year
- Sometimes claimed that the best way to predict tomorrow's weather
 - Guess that, tomorrow is as same as today
 - If assume predicting will **be correct in 75%** of the cases (regardless rain or sunshine)

O. Häggström (2002) *Finite Markov Chains and Algorithmic Applications*. CU Press, Cambridge

Example 2: Weather Forecast

17

- The weather can be easily modeled by a **Markov chain**

- The state space consists of the two states

(1 = rain) and (2 = sunshine)

- The transition matrix is

$$\mathbf{P} = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}$$

- In areas where sunshine is much more common than rain such as Bangkok,

- more realistic transition matrix would be

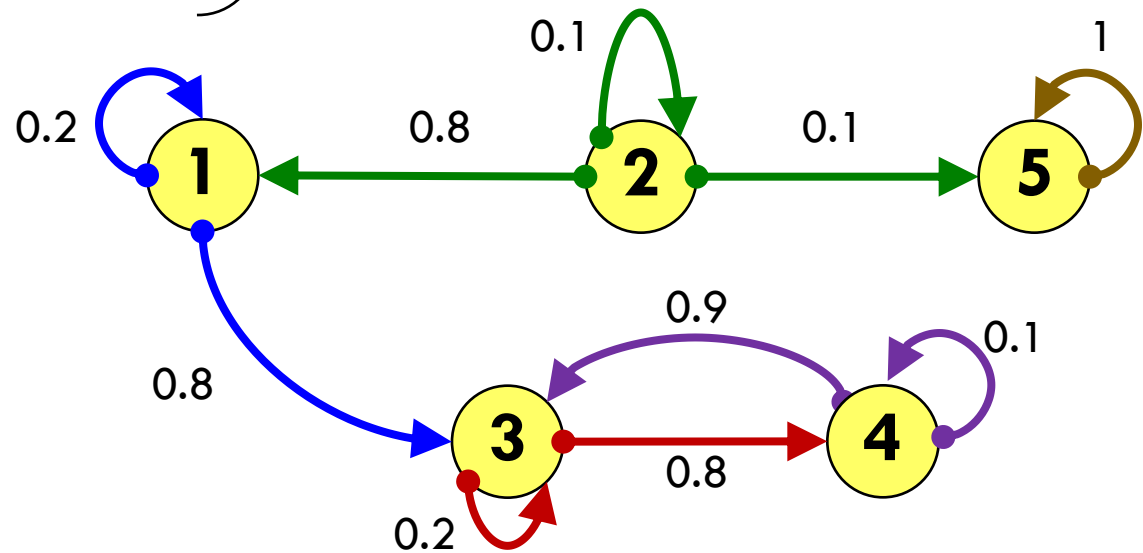
$$\mathbf{P} = \begin{pmatrix} 0.30 & 0.70 \\ 0.15 & 0.85 \end{pmatrix}$$

Example 3

18

$$P = \begin{pmatrix} 0.2 & 0 & 0.8 & 0 & 0 \\ 0.8 & 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Draw a Markov Chain



Multistep Transition

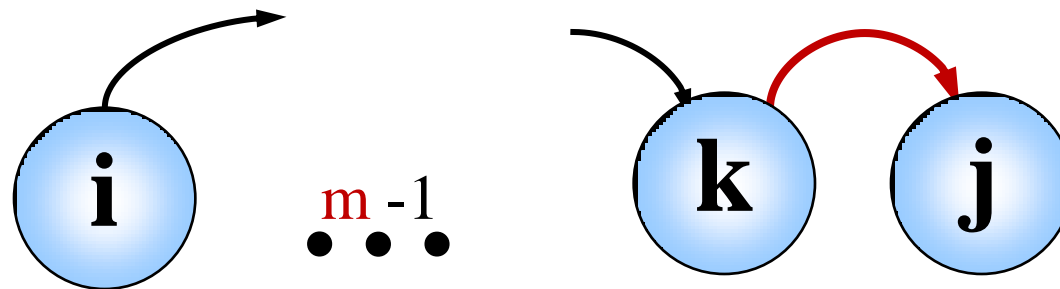
Homogeneous Markov Chain

19

- m -step transition probabilities are

$$p_{ij}^{(m)} \equiv P[X_{n+m} = j \mid X_n = i]$$

$$= \sum_{\forall k} p_{ik}^{(m-1)} p_{kj} \quad m = 2, 3, \dots$$



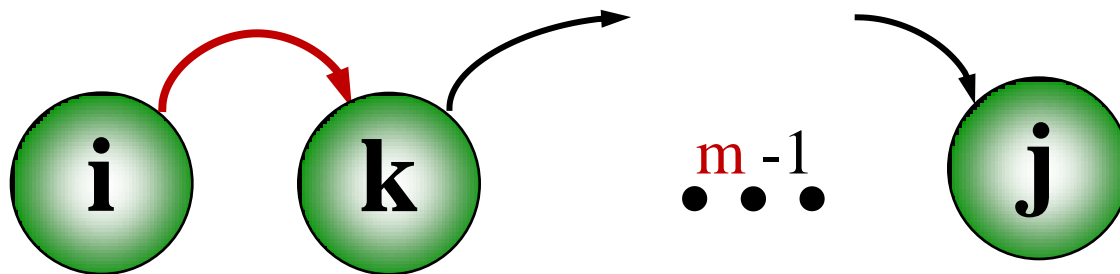
Multistep Transition

Homogeneous Markov Chain

20

$$p_{ij}^{(m)} \equiv P[X_{n+m} = j \mid X_n = i]$$

$$= \sum_{\forall k} p_{ik} p_{kj}^{(m-1)} \quad m = 2, 3, \dots$$



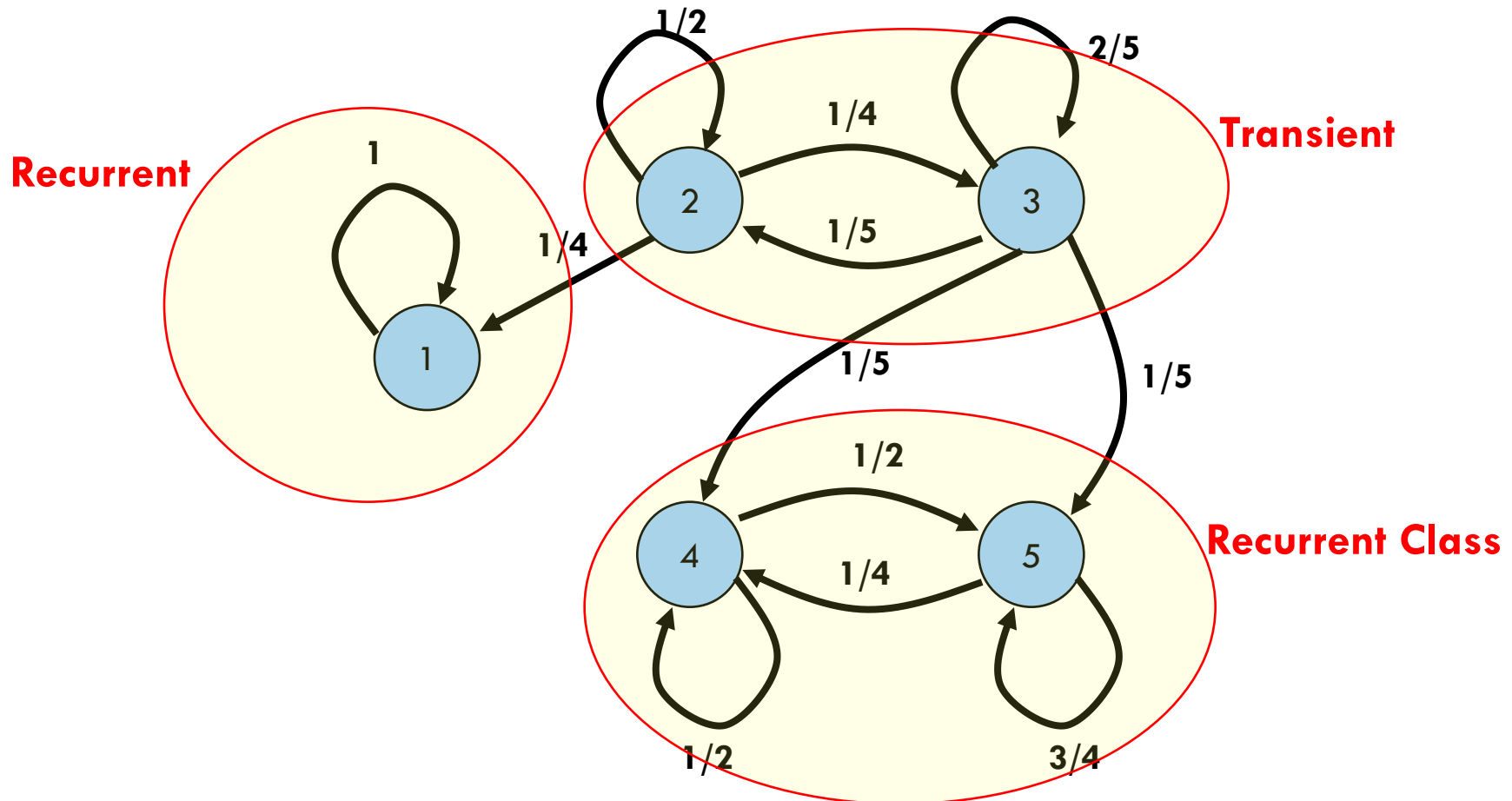
Type of States

21

- Transient State
 - Starting @ state E, it will eventually leave state E and never return
- Recurrent State
 - Starting @ state E, it will **continuously** reoccur (come back to state E)

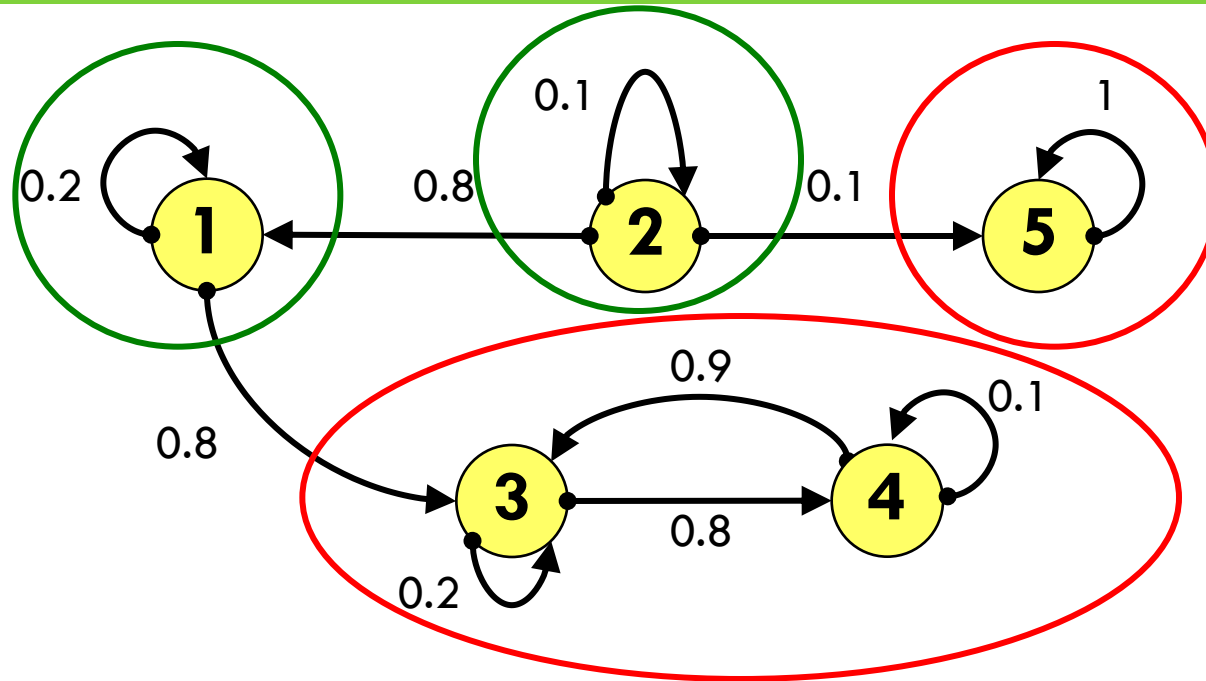
Transient or Recurrent States

22



Example 3 (Revisit)

23



Identify

Transient states = ? {1} , {2}

Recurrent states = ? {5} , {3,4}

Transient or Recurrent States

24

- $f_j^{(n)} = P[\text{the process first returns to state } \mathbf{j} \text{ after leaving state } \mathbf{j} \text{ in } \mathbf{n} \text{ steps}]$
- $f_j = P[\text{the process returns to state } \mathbf{j} \text{ after leaving state } \mathbf{j}]$

$$f_j = \sum_{n=1}^{\infty} f_j^{(n)}$$

- $M_j = \text{Mean recurrence time of state } \mathbf{j}$

$$M_j = \sum_{n=1}^{\infty} n f_j^{(n)}$$

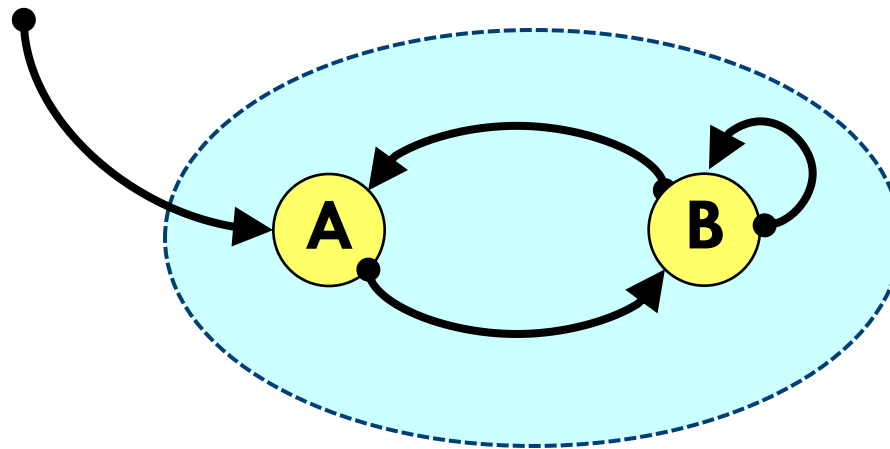
Transient or Recurrent States

25

- If $f_j < 1$
 - State E_j is called “**Transient State**”
- If $f_j = 1$
 - State E_j is called “**Recurrent State**”
 - If $M_j = \infty$
 - State E_j is called “**Recurrent Null State**”
 - If $M_j < \infty$
 - State E_j is called “**Recurrent Nonnull State**”

A Closed Set

- A set that once the Markov chain has entered the set \rightarrow it cannot leave the set

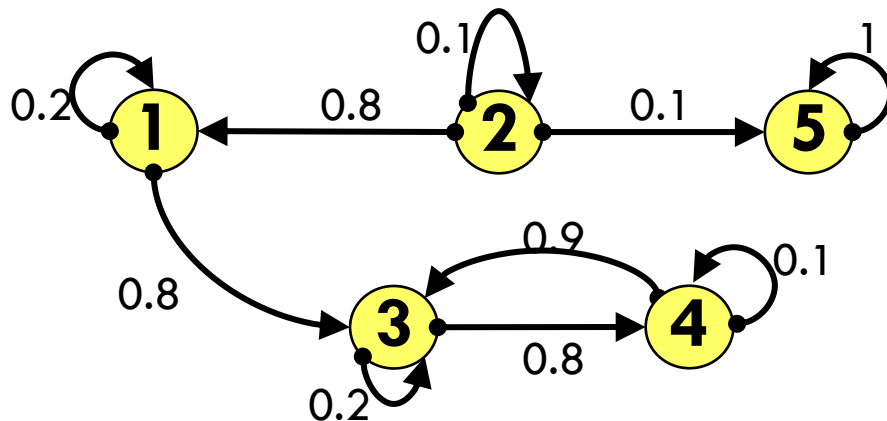


Example 3 (Revisit)

27

$$P = \begin{pmatrix} 0.2 & 0 & 0.8 & 0 & 0 \\ 0.8 & 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Identify is this a closed set ?

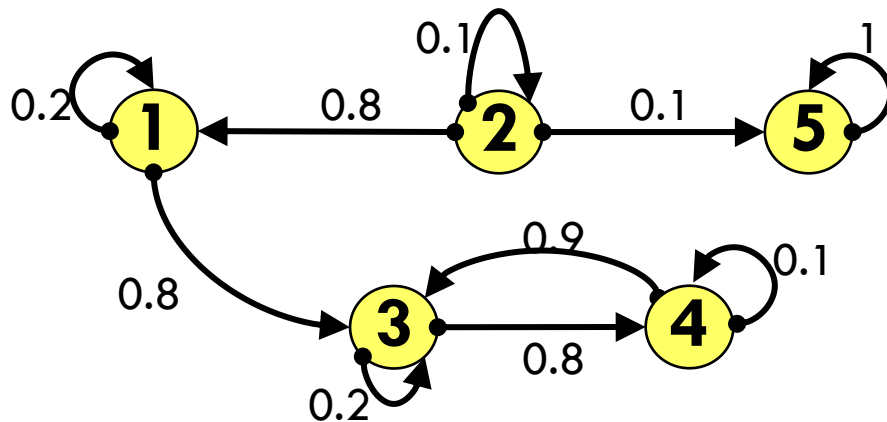


- {1} **Not Closed**
- {1,2} **Not Closed**
- {1,2,3} **Not Closed**
- {2,3} **Not Closed**
- {4,5} **Not Closed**
- {3,4,5}
- {5}
- {1,2,3,4,5}

Example 3 (Revisit)

28

$$P = \begin{pmatrix} 0.2 & 0 & 0.8 & 0 & 0 \\ 0.8 & 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



Identify is this a closed set ?

- {1} **Not Closed**
- {1,2} **Not Closed**
- {1,2,3} **Not Closed**
- {2,3} **Not Closed**
- {4,5} **Not Closed**
- {3,4,5} **Closed**
- {5} **Closed**
- {1,2,3,4,5} **Closed**

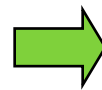
A Closed Set

29

P =

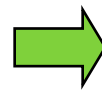
0.2	0	0.8	0	0
0.8	0.1	0	0	0.1
0	0	0.2	0.8	0
0	0	0.9	0.1	0
0	0	0	0	1

$\{1,2,3,4,5\}$ = Closed Set



It can be reduced to a smaller closed set $\{3,4,5\}$

$\{3,4,5\}$ = Closed Set



It can be reduced to a smaller closed set $\{3,4\}$ and $\{5\}$

$\{3,4\}$ = Closed Set
 $\{5\}$ = Closed Set



It can **NOT** be reduced to any smaller closed set

Irreducible Set

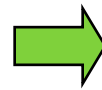
30

P =

0.2	0	0.8	0	0
0.8	0.1	0	0	0.1
0	0	0.2	0.8	0
0	0	0.9	0.1	0
0	0	0	0	1

Reducible Set

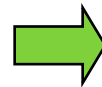
$\{1,2,3,4,5\}$ = Closed Set



It can be reduced to a smaller closed set $\{3,4,5\}$

Reducible Set

$\{3,4,5\}$ = Closed Set



It can be reduced to a smaller closed set $\{3,4\}$ and $\{5\}$

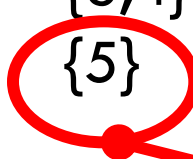
Irreducible Set

$\{3,4\}$ = Closed Set

$\{5\}$ = Closed Set



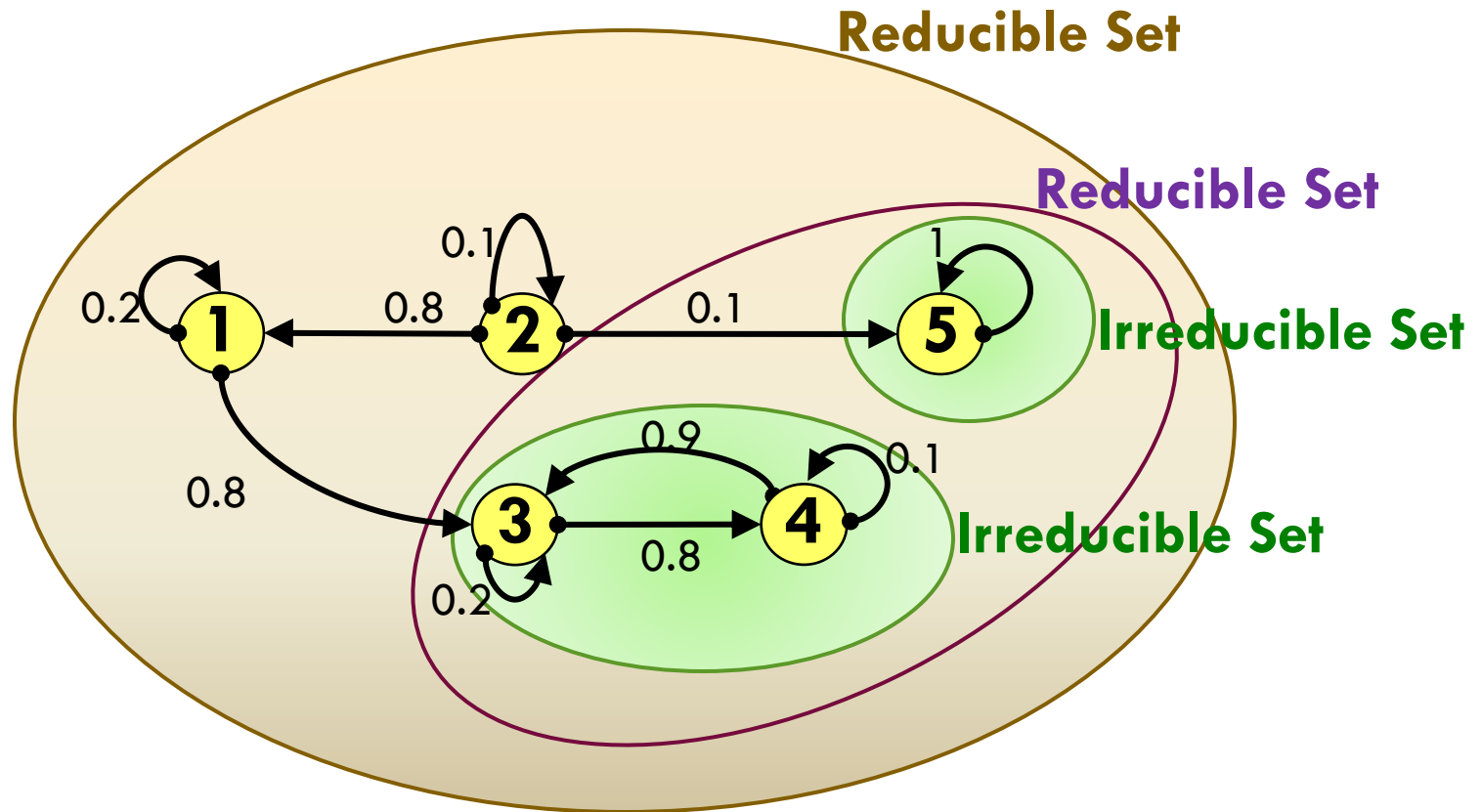
It can NOT be reduced to any smaller closed set



Absorbing State

Irreducible Set

31

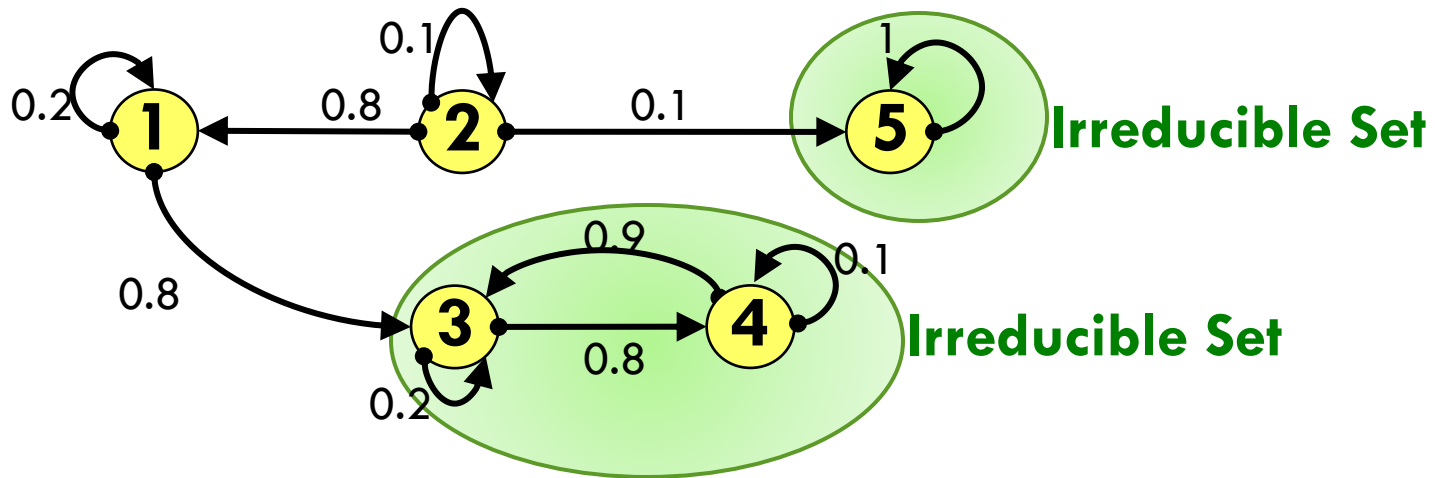


Irreducible Markov Chain

32

- A Markov Chain is *irreducible* if every state can be reached from every other state in a *finite* number of steps.

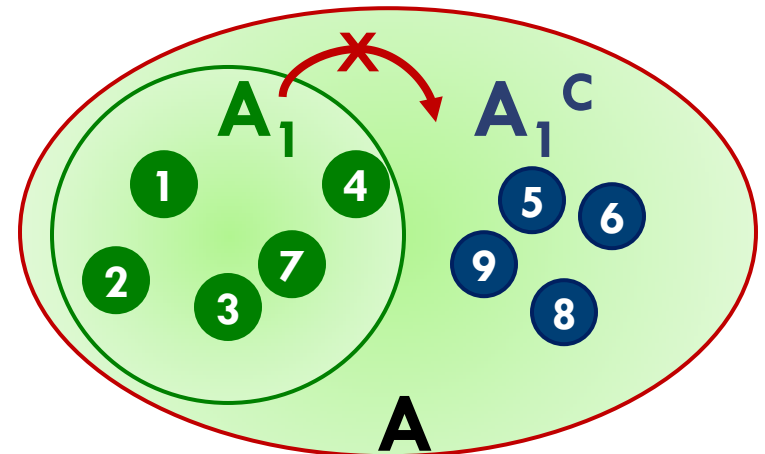
$$p_{ij}^{(m_0)} > 0 \quad \text{for } m_0 = \text{integer}$$



Not Irreducible Markov Chain

33

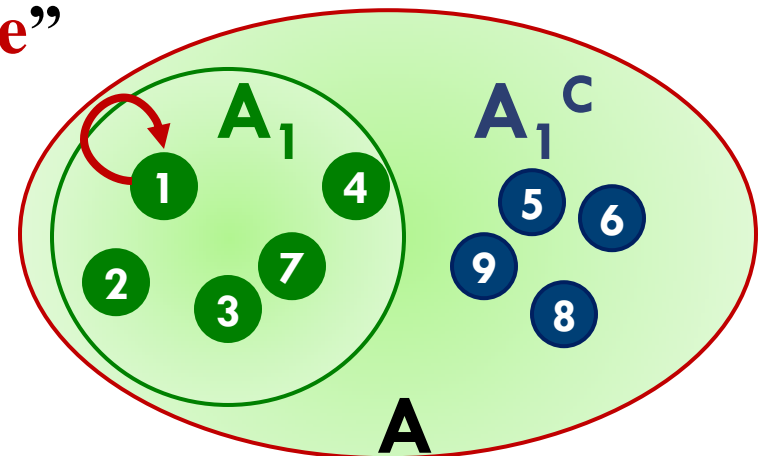
- Case 1
 - For A = set of all states in a Markov chain
 - $A_1 \subset A$
 - If no one-step transition from state A_1 to A_1^c
 - A_1 is defined as “**Closed**”



Not Irreducible Markov Chain

34

- Case 2
 - For A = set of all states in a Markov chain
 - $A_1 \subset A$
 - If A_1 consists of one or more state E_i that once get in state E_i , the process cannot move to any other states
 - E_i is called “**Absorbing State**”
 - $p_{ii} = 1$



Note on Irreducible

35

- All states within an irreducible set are of the same classification
 - If one state is transient \rightarrow all transient
 - If one state is recurrent \rightarrow all recurrent
- Irreducible sets
 - \rightarrow Communication between states
 - \rightarrow Communication must be both ways
 - \rightarrow But does not have to be in one step

Periodic or Aperiodic

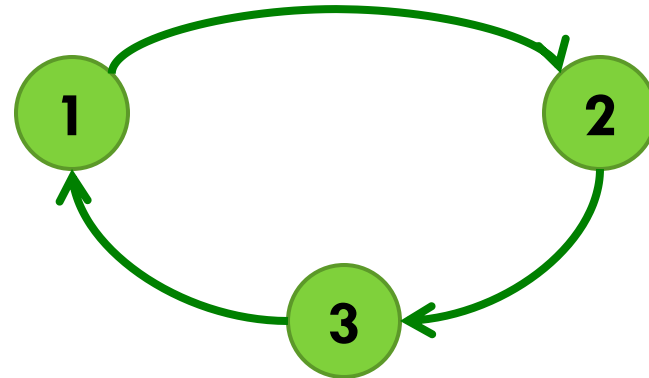
36

- Recurrent State
- Let $\beta =$ integer
- If the **only possible steps** that the process returns to state E_i are $\beta, 2\beta, 3\beta, \dots$
 - If $\beta > 1$ and β is the largest integer
 - State E_i is called “**Periodic**”
 - The **recurrence time** for state E_j has period β
 - If $\beta = 1$
 - State E_i is called “**Aperiodic**”

Example

37

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$



- Starting from state 1
 - It returns to state 1 for every 3 steps
 - Same as other two states.
- The chain is therefore **Periodic**

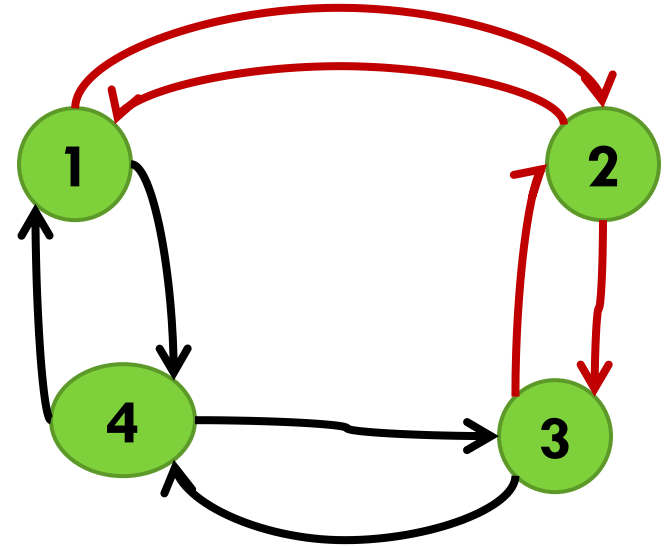
Modified from:

http://www.wikicoursenote.com/wiki/Again_on_Markov_Chain

Example

38

$$\mathbf{P} = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{pmatrix}$$



- Starting from state 1
 - It returns to state 1 for every **multiple of 2 steps**
 - Same as other states.
- The chain is therefore **Periodic**

Modified from:

http://www.wikicoursenote.com/wiki/Again_on_Markov_Chain

Ergodicity

39

- $E_j = \text{Ergodic}$ if
 - $E_j = \text{Aperiodic}$ and *Recurrent Nonnull*
 - (recurrent nonnull)
 - $f_j = 1$, $M_j < \infty$, and $\beta = 1$
 - (recurrent)
 - (Aperiodic)
- A Markov Chain is **ergodic**
 - If **all** states of a Markov Chain are **ergodic**
 - If number of states is **finite** and **all** states of a Markov Chain are **aperiodic**, and **irreducible**

Theorem 1

40

- The states of **an irreducible** Markov Chain are either
 - all transient or
 - all recurrent nonnull or
 - all recurrent null
- If periodic, then all states have the same period β

Definition

41

- Let $\pi_j^{(n)} = P[\text{finding the system in state } E_j \text{ at the } n^{\text{th}} \text{ step}]$
$$\pi_j^{(n)} = P[X_n = j]$$
- Let $\pi_j = \text{Stationary Probability}$
= $P[\text{being in state } j \text{ at arbitrarily time}]$
= $\text{The limiting state probabilities}$

Theorem 2

42

- In an irreducible and aperiodic, homogeneous Markov Chain,
- the limiting state probabilities $[\pi_j]$ always exist and are independent of the initial state probability distribution $[\pi_j^{(0)}]$

$$\pi_j = \lim_{n \rightarrow \infty} \pi_j^{(n)}$$

Theorem 2

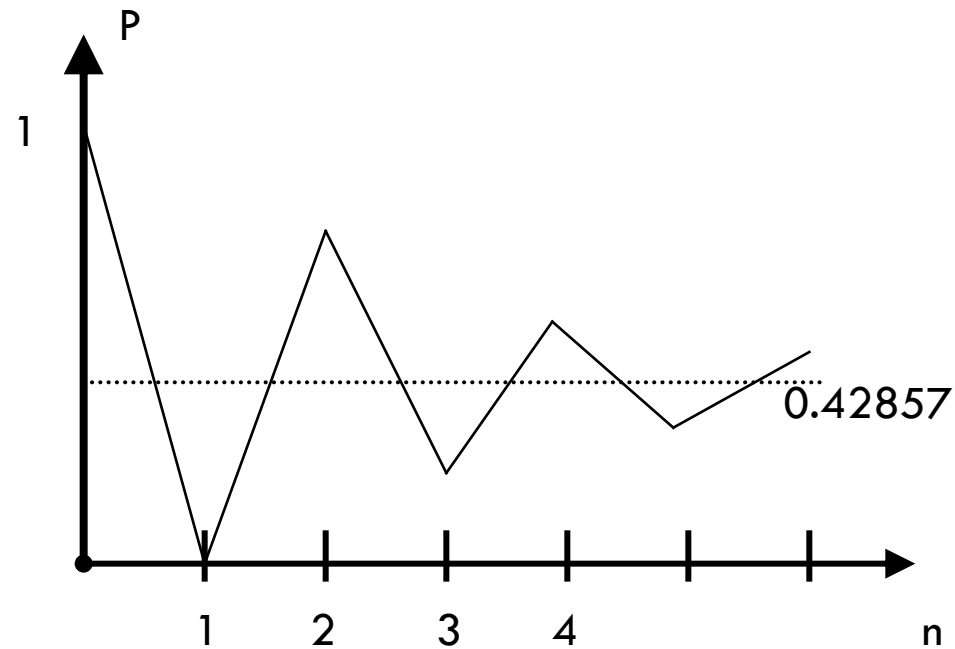
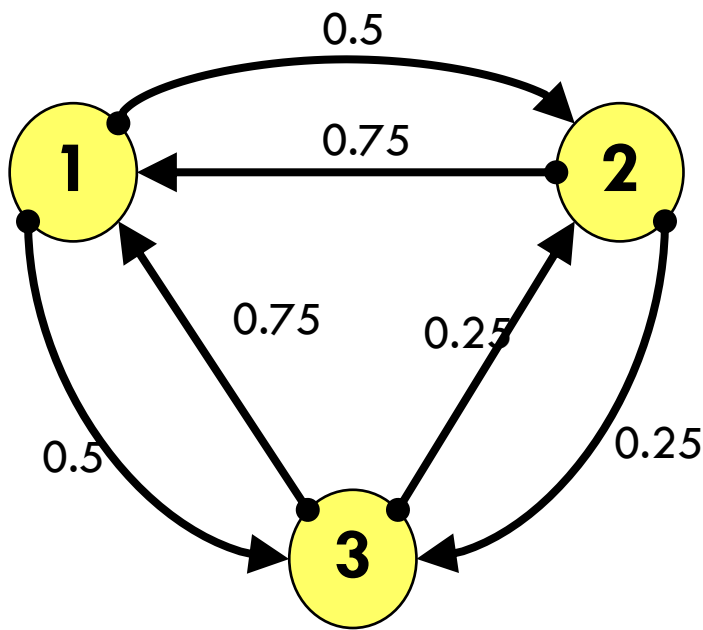
43

- **Either**
- **Case (a)**
 - All states are transient or
 - All states are recurrent null
 - ➔ $\pi_j = 0 \quad \forall j$
 - ➔ No stationary distribution exist
- **Or Case (b)**
 - All states are recurrent nonnull
 - ➔ $\pi_j > 0 \quad \forall j$
 - ➔ Stationary distribution exist ➔ $\pi_j = 1 / M_j$

Steady-State Behavior

(Limiting condition)

$P[\text{be in state 1 @ time } n \mid \text{in state 1 @ } t = 0]$



$$\lim_{n \rightarrow \infty} \Pr\{ X_n = 1 \mid X_0 = 1 \} = 0.42857$$

$$\lim_{n \rightarrow \infty} \Pr\{ X_n = 1 \mid X_0 = 2 \} = 0.42857$$

Same as
Anan Phonphoem

To solve for π_j

45

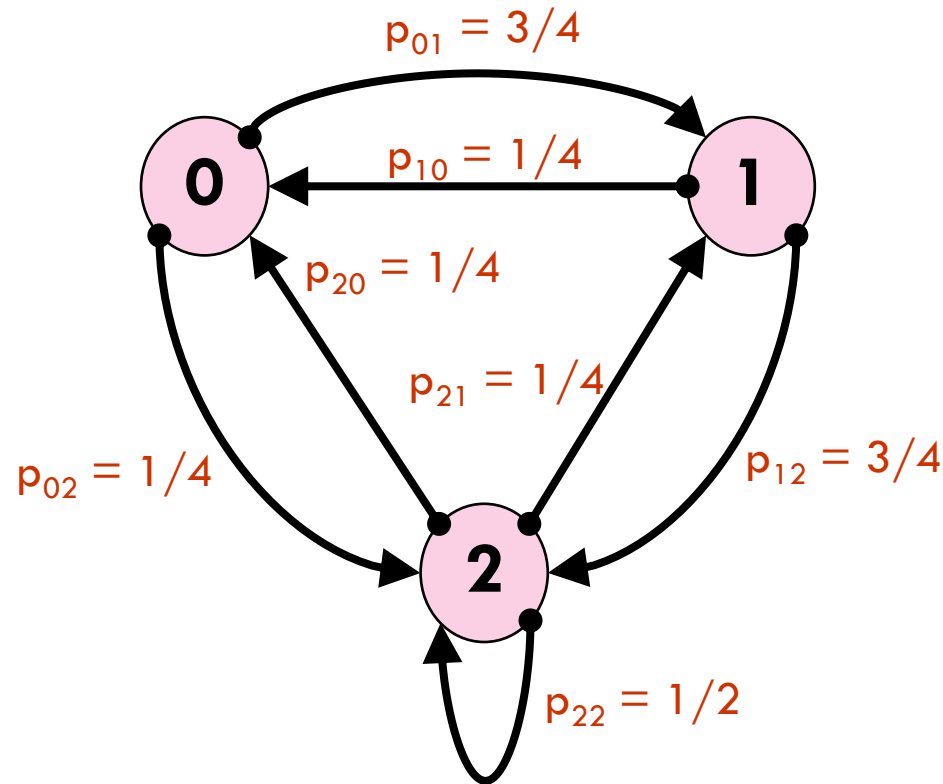
Balance Equations: $\pi_j = \sum_i \pi_i p_{ij}$
(Linear dependency)

Normalization condition: $1 = \sum_i \pi_i$

Markov Chain Example

46

- Driving from town to town



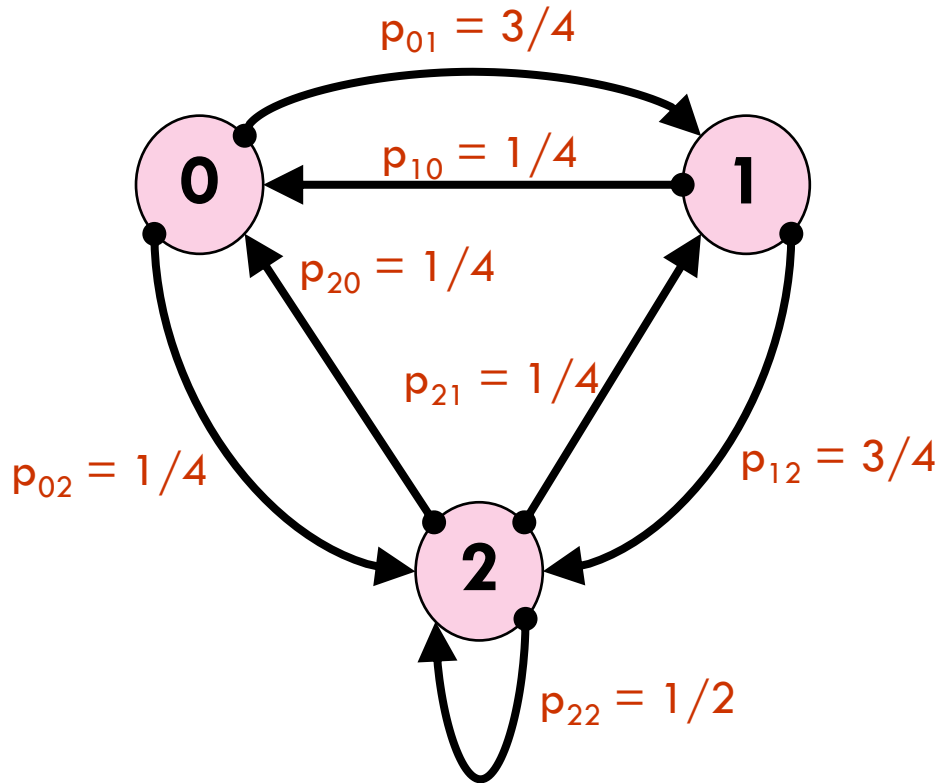
Markov Chain Example

47

- Let $P =$ Transition probability matrix
 $= [p_{ij}]$
- Let $\pi = [\pi_0, \pi_1, \pi_2, \dots]$
- From Balance equation
$$\pi = \pi P$$

Markov Chain Example

48



$$P = \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/4 & 0 & 3/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

Markov Chain Example

49

$$\pi = \pi \mathbf{P}$$
$$[\pi_0, \pi_1, \pi_2] = [\pi_0, \pi_1, \pi_2] \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/4 & 0 & 3/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$\pi_0 = 0 \quad \pi_0 + 1/4 \pi_1 + 1/4 \pi_2$$

$$\pi_1 = 3/4 \pi_0 + 0 \quad \pi_1 + 1/4 \pi_2$$

$$\pi_2 = 1/4 \pi_0 + 3/4 \pi_1 + 1/2 \pi_2$$

$$\mathbf{1} = \pi_0 + \pi_1 + \pi_2$$

Markov Chain Example

50

Solution:

$$\pi_0 = 0.20$$

$$\pi_1 = 0.28$$

$$\pi_2 = 0.52$$

- This is the stationary (equilibrium) state probability
- This is the ergodic Markov Chain
 - Finite number of states
 - Irreducible (recurrent)

Transient Behavior

51

- We want to know the probability of finding the process in state E_j at time n
- $\pi^{(n)} = [\pi_0^{(n)}, \pi_1^{(n)}, \pi_2^{(n)}, \dots]$
- From Transition Probability \mathbf{P}
 - We can calculate:
$$\pi^{(1)} = \pi^{(0)}\mathbf{P}$$
$$\pi^{(n)} = \pi^{(n-1)}\mathbf{P}$$
 - By recursive:
$$\pi^{(n)} = \pi^{(0)}\mathbf{P}^n$$

Transient Behavior

52

- From stationary probability: $\pi = \lim_{n \rightarrow \infty} \pi^{(n)}$

- From $\pi^{(n)} = \pi^{(n-1)}\mathbf{P}$

$$\lim_{n \rightarrow \infty} \pi^{(n)} = \lim_{n \rightarrow \infty} \pi^{(n-1)} \mathbf{P}$$

$$\pi = \pi \mathbf{P}$$

- Note: The solution π is independent of $\pi^{(0)}$

Transient Behavior

53

$$\pi^{(0)} = [1, 0, 0]$$

n	0	1	2	3	∞
$\pi_0^{(n)}$	1	0	0.25	0.187	0.20
$\pi_1^{(n)}$	0	0.75	0.062	0.359	0.28
$\pi_2^{(n)}$	0	0.25	0.688	0.454	0.52

$$\pi^{(0)} = [0, 0, 1]$$

n	0	1	2	3	∞
$\pi_0^{(n)}$	0	0.25	0.187	0.203	0.20
$\pi_1^{(n)}$	0	0.25	0.313	0.266	0.28
$\pi_2^{(n)}$	1	0.50	0.500	0.531	0.52

Birth-Death Process

54

- A Markov Process
- Homogeneous, aperiodic, and irreducible
- Discrete time / Continuous time
- State changes can only happen between neighbors

Birth-Death Process

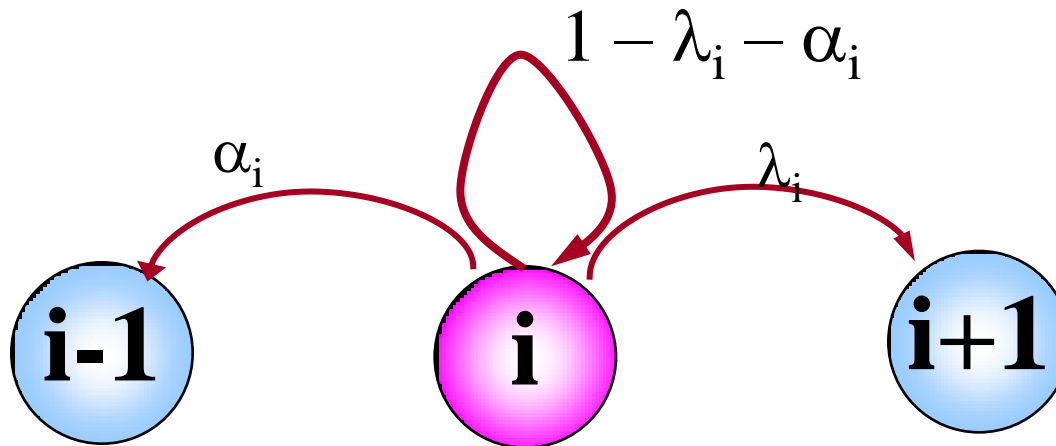
55

- Size of population
 - System is in state E_k when consists of k members
 - Changes in population size occurred by at most one
 - Size increased by one \rightarrow “*Birth*”
 - Size decreased by one \rightarrow “*Death*”
- Transition probabilities p_{ij} do not change with time

Birth-Death Process

56

$$p_{ij} = \begin{cases} \alpha_i & j = i - 1 \\ 1 - \lambda_i - \alpha_i & j = i \\ \lambda_i & j = i + 1 \\ 0 & \text{Otherwise} \end{cases}$$



Birth-Death Process

57

- $\alpha_i =$ death (less one in population size)
- $\alpha_0 = 0$ (no population \rightarrow no death)
- $\lambda_i =$ birth (increase one in population)
- $\lambda_i > 0$ (birth is allowed)
- Pure Birth = no decrement, only increment
- Pure Death = no increment, only decrement

Queueing Theory Model

58

- **Population** = customers in the queueing system
- **Death** = a customer departure from the system
- **Birth** = a customer arrival to the system

Transition matrix

59

$$P = \begin{bmatrix} 1 - \lambda_0 & \lambda_0 & 0 & 0 & 0 & 0 & \dots \\ \alpha_1 & 1 - \lambda_1 - \alpha_1 & \lambda_1 & 0 & 0 & 0 & \\ 0 & \alpha_2 & 1 - \lambda_2 - \alpha_2 & \lambda_2 & & & \\ 0 & & \dots & & & & \\ 0 & & & & & & \\ & & & \alpha_i & 1 - \lambda_i - \alpha_i & \lambda_i & \\ \dots & & & & & & \end{bmatrix}$$