LECTURE #4
STOCHASTIC PROCESS

204528

Queueing Theory and Applications in Networks

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Outline

- Stochastic Process
- Counting Process
- Poisson Process
- Brownian Motion Process
- Autocovarience and Autocorrelation
- Random Sequence
- Stationary Process
- Wide-sense Stationary Process

Definition

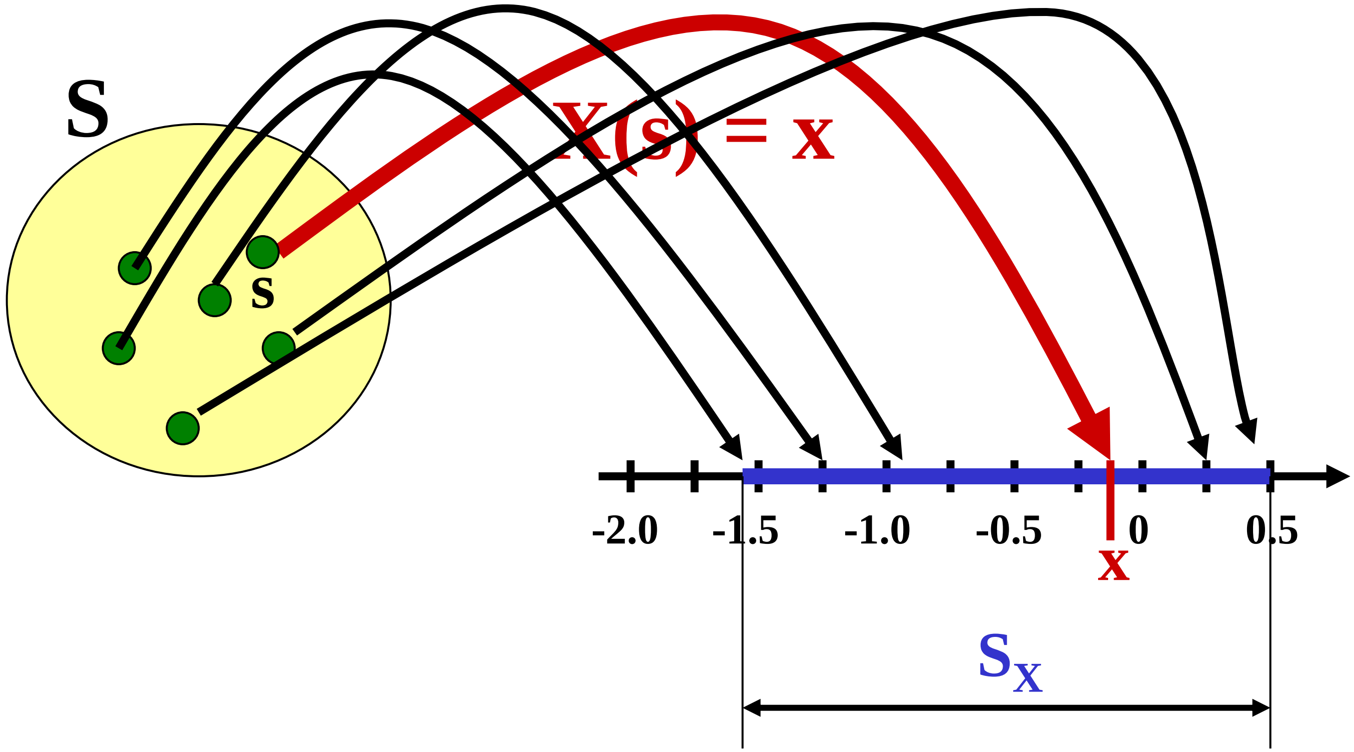
Stochastic Process

Random — **Function of Time**

Random Process

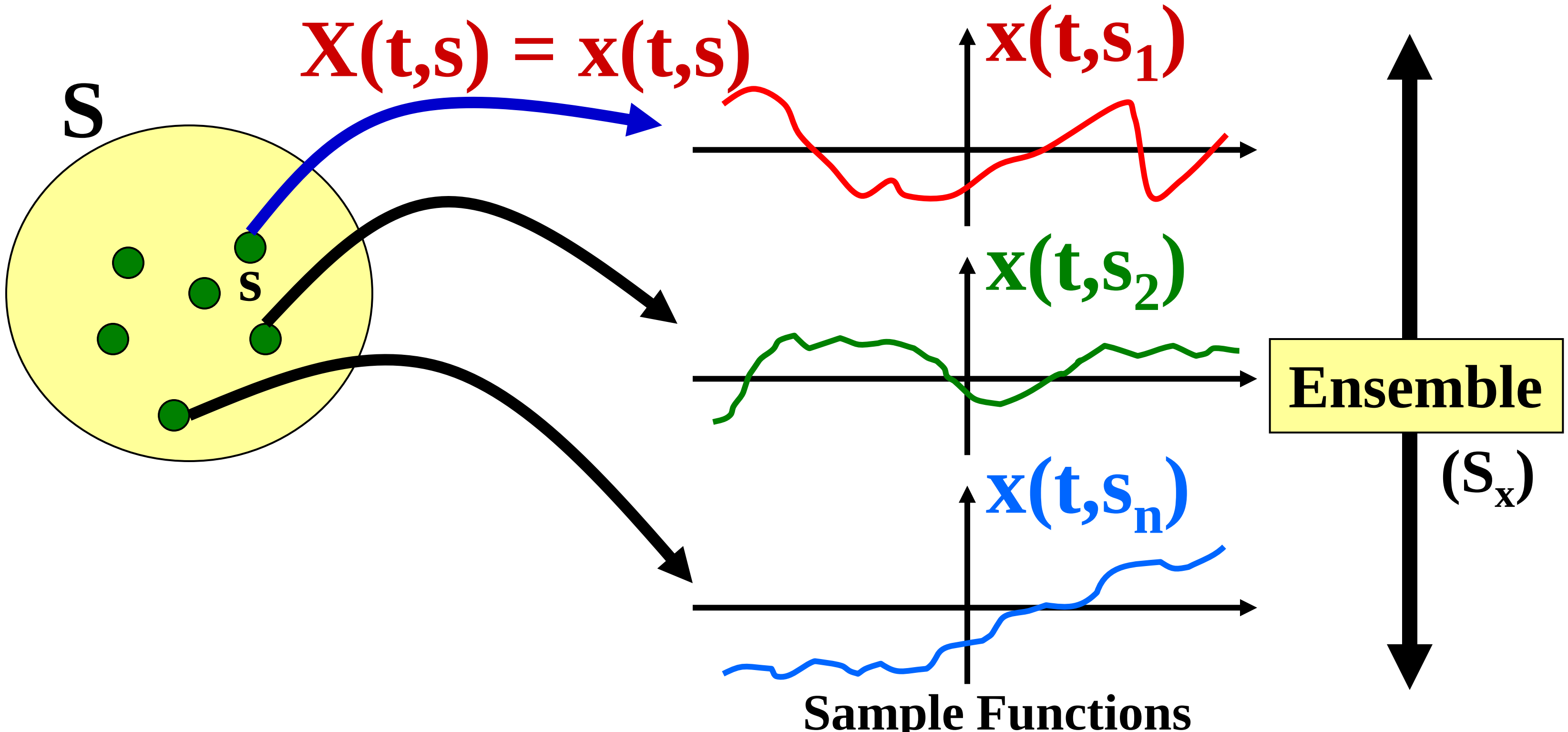
Random Variable

X is a function that maps each outcome, s, in S to a real number X(s), x



Random Process

X(t) is a function that maps each outcome, s, in S to a time function x(t,s)

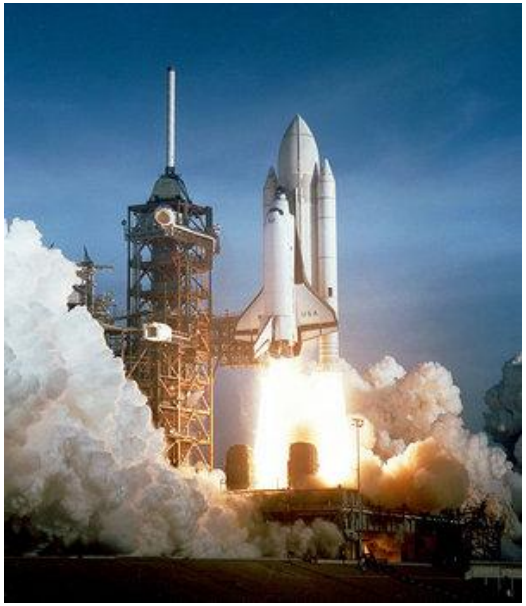


Example 1

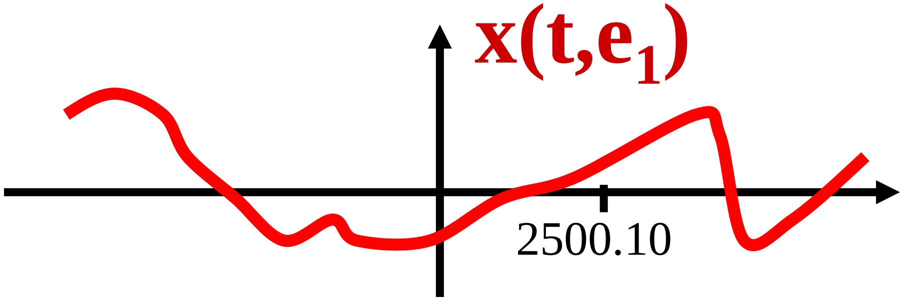
- Taking temperature at the surface of a space shuttle
- Starting at launch time t = 0
- X(t) = temp in degree Celsius on the surface
- Each launch, record x(t,s)

http://spaceflight.nasa.gov/history/shuttle-mir/spacecraft/s-orb-sscomponents-main.htm

Example 1



http://sites.indianriverschools.org/srhs/tea
chers/nyberg/main.htm

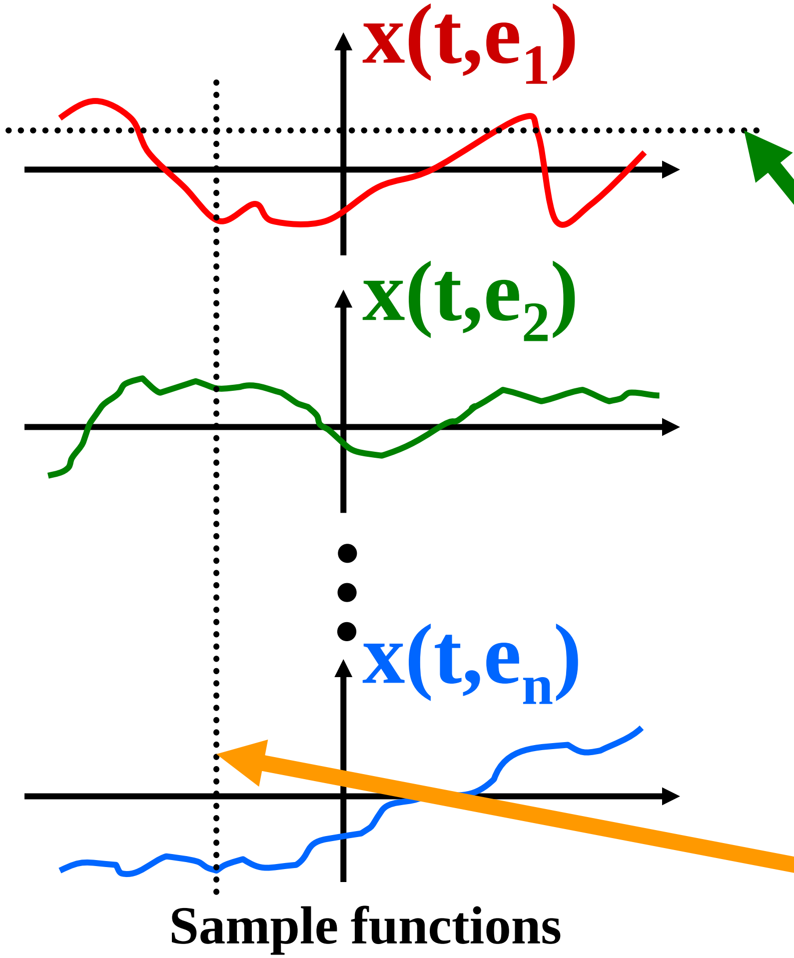


At time **t = 2500.10 sec**

$x(2500.10, e_1) = 1200$ °C

e_1 = 1st launch measuring

Example 1



Average temp at $\mathbf{e_1}$
for completed mission is
5000 C ➔ **E[**x(t, e_1)**]**

Time Average

Average temp of engine
at t = 2500.10 sec is
1320 C ➔ **E[**X(2500.10)**]**

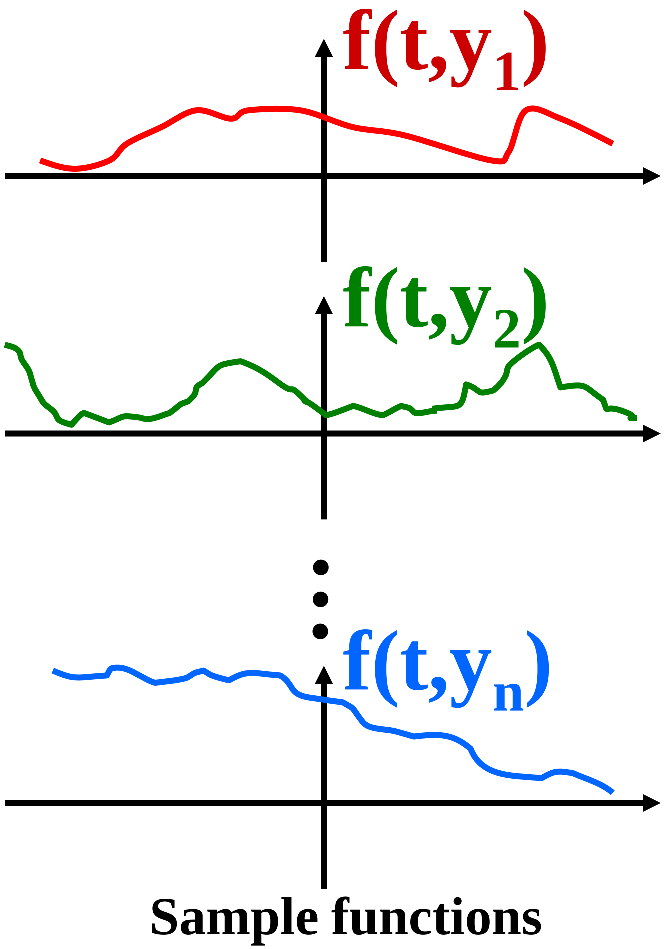
Ensemble Average

Example 2

- Measure the rain fall in a day @Bangkhen every day
- Let F(t) = random process
- f(t,y) = a sample function for measuring at day “t” of the year “y”

http://snap.newsadvance.com/lna/snap/media_view/94

Example 2

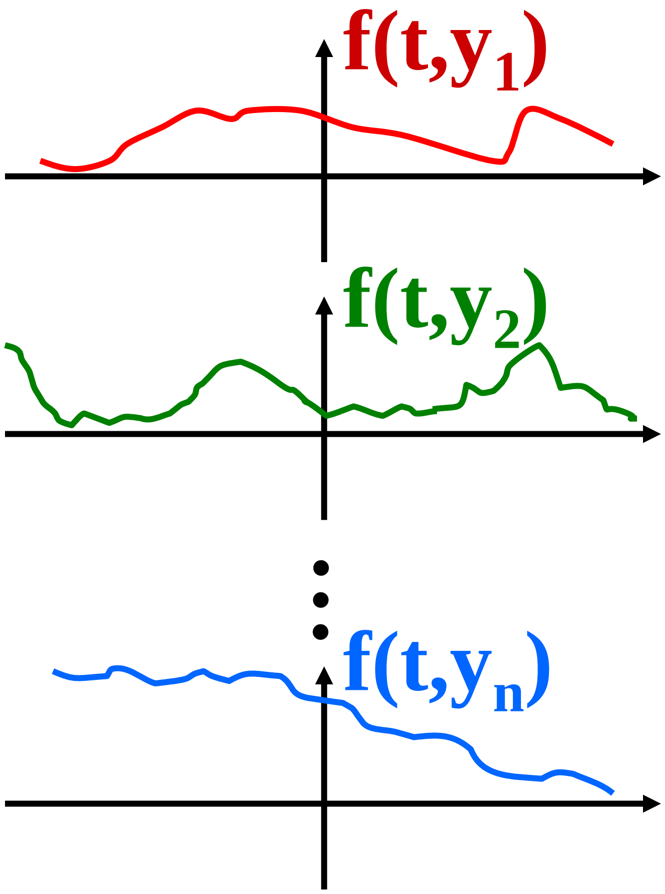


A sample function of rain fall in y_1 = year 2002 ($1 \leq t \leq 365$)

A sample function of rain fall in y_2 = year 2003 ($1 \leq t \leq 365$)

A sample function of rain fall in y_n = year 2009 ($1 \leq t \leq 365$)

Example 2



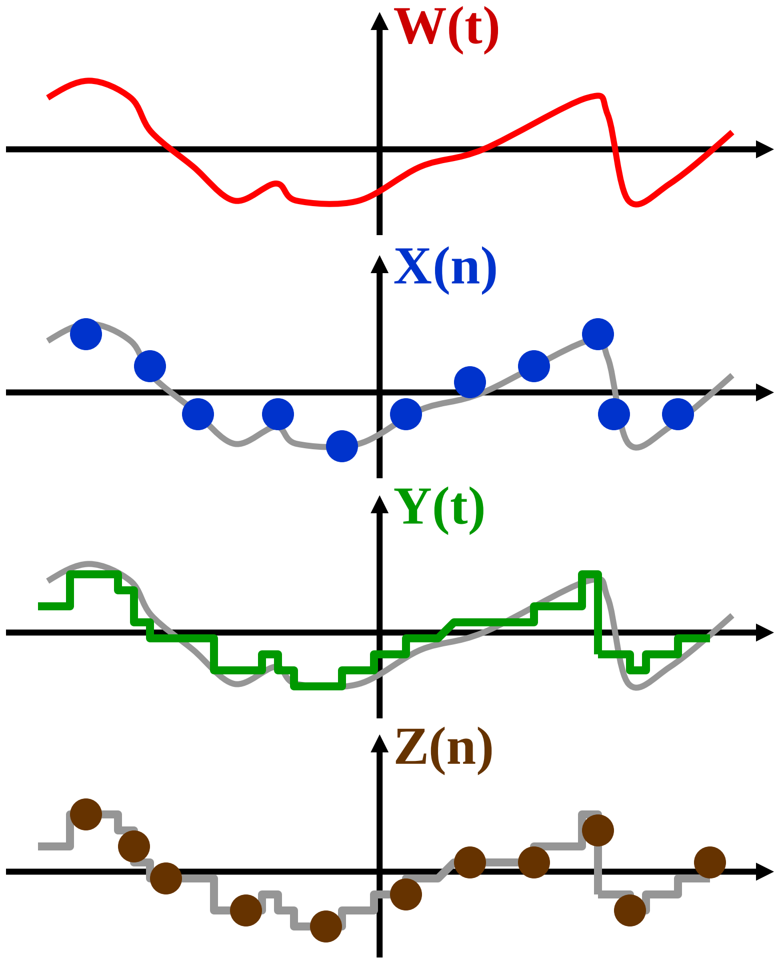
Sample functions

Therefore,

we might want to know

- The average rain fall in year 2007
- The average rain fall for June 29

Types of Stochastic Process



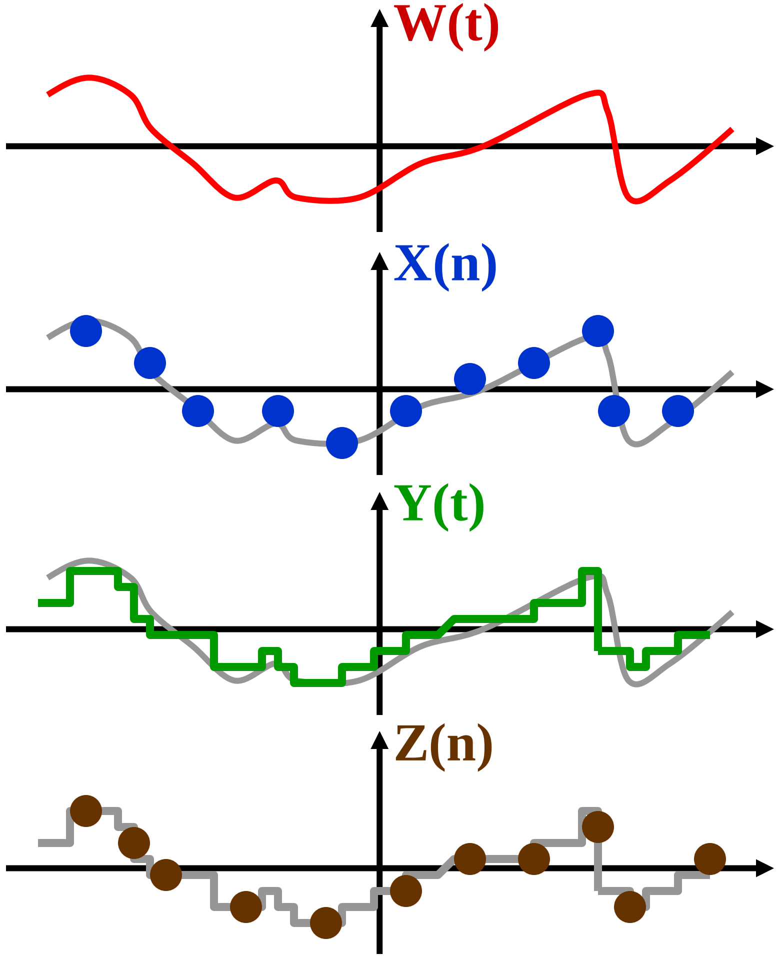
Continuous Time,
Continuous value Process

Discrete Time,
Continuous value Process

Continuous Time,
Discrete value Process

Discrete Time,
Discrete value Process

Stochastic Process Examples



Record temperature as a continuous time

Record temperature every T seconds

Record round(temperature) as a continuous time

Record round(temperature) every T seconds

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IID Random Sequence

- Independent, Identically Distributed Random Sequence
- Independent trials of an experiment at a constant rate
- Discrete / Continuous

Theorem:

$$P_{X_{n_1} \ldots X_{n_k}}(x_1, \ldots, x_k) = P_X(x_1) \ldots P_X(x_k) = \prod_{i=1}^{k} P_X(x_i)$$

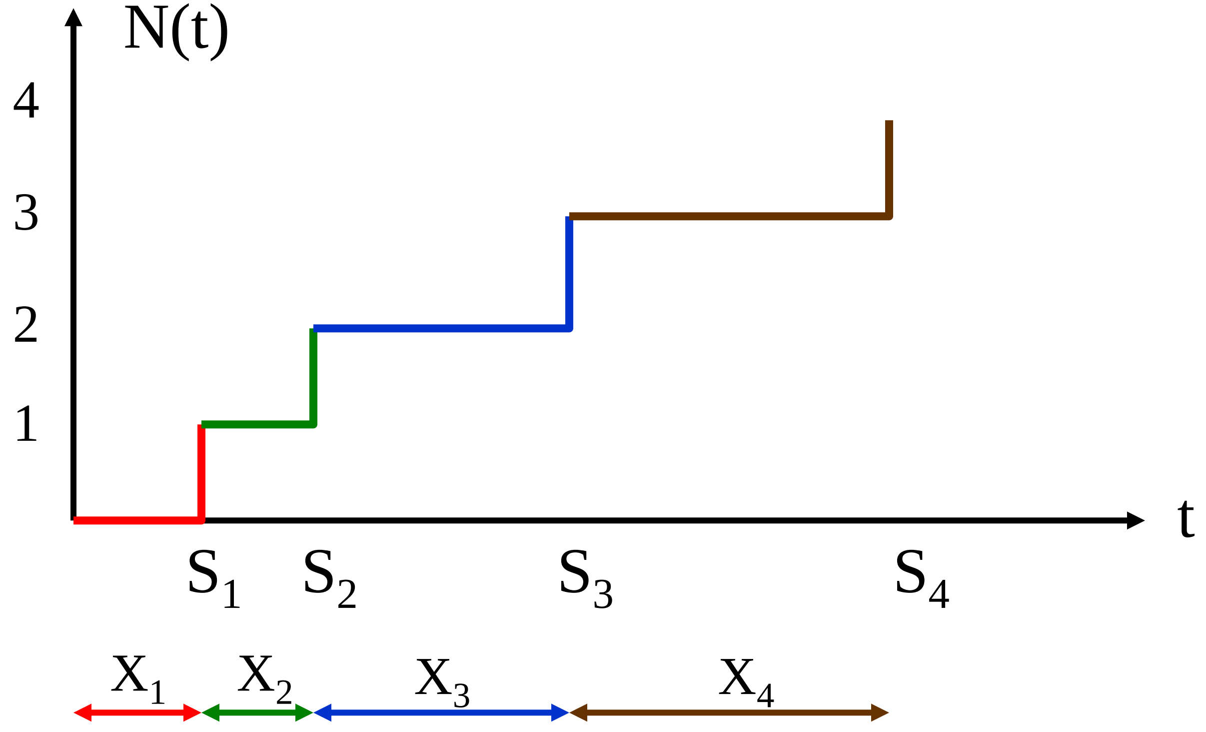
Counting Process

Definition: A Stochastic Process is a Counting Process N(t) if

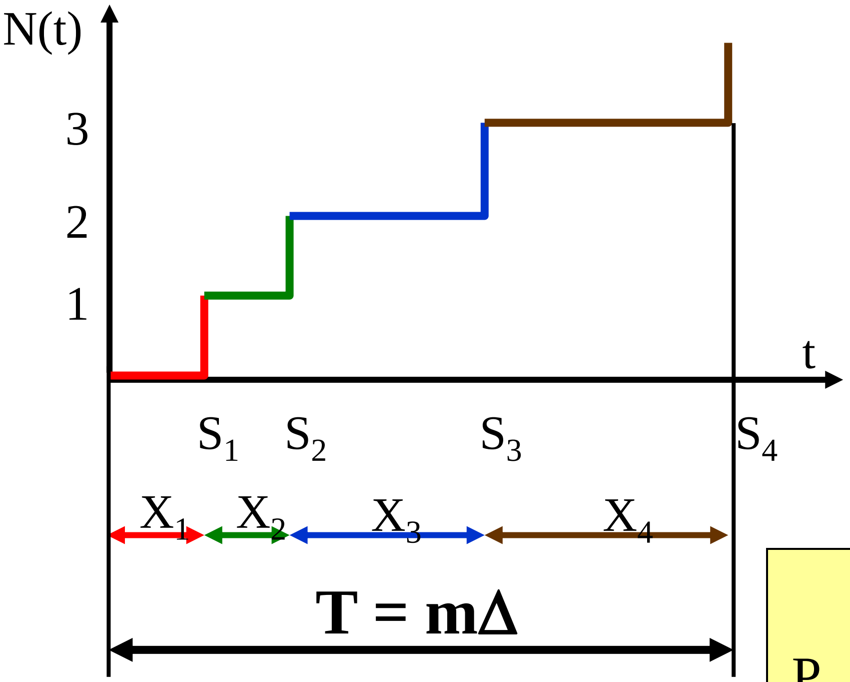
- n(t,s) = 0   for t < 0
- n(t,s) = integer valued and non-decreasing

Counting Process

# of customers **arrive** at (0,t]



Counting Process



- For a small step **Δ**, only one arrival ($X_n = 1$)
- Success Prob. of $X_n = \lambda\,\Delta$ $= \lambda\, T/m$

Binomial PMF

$$P_{N_m}(n) = \begin{cases} \binom{m}{n} (\lambda T/m)^n (1-\lambda T/m)^{m-n} & n = 0,1,2,.. \\ 0 & \text{Otherwise} \end{cases}$$

Counting Process

Binomial Process

$$P_{Nm}(n) = \begin{cases} \binom{m}{n} (\lambda T/m)^n (1- \lambda T/m)^{m-n} & n = 0,1,2,.. \\ 0 & \text{Otherwise} \end{cases}$$

$m \rightarrow \infty$ or $\Delta \rightarrow 0$

Poisson Process

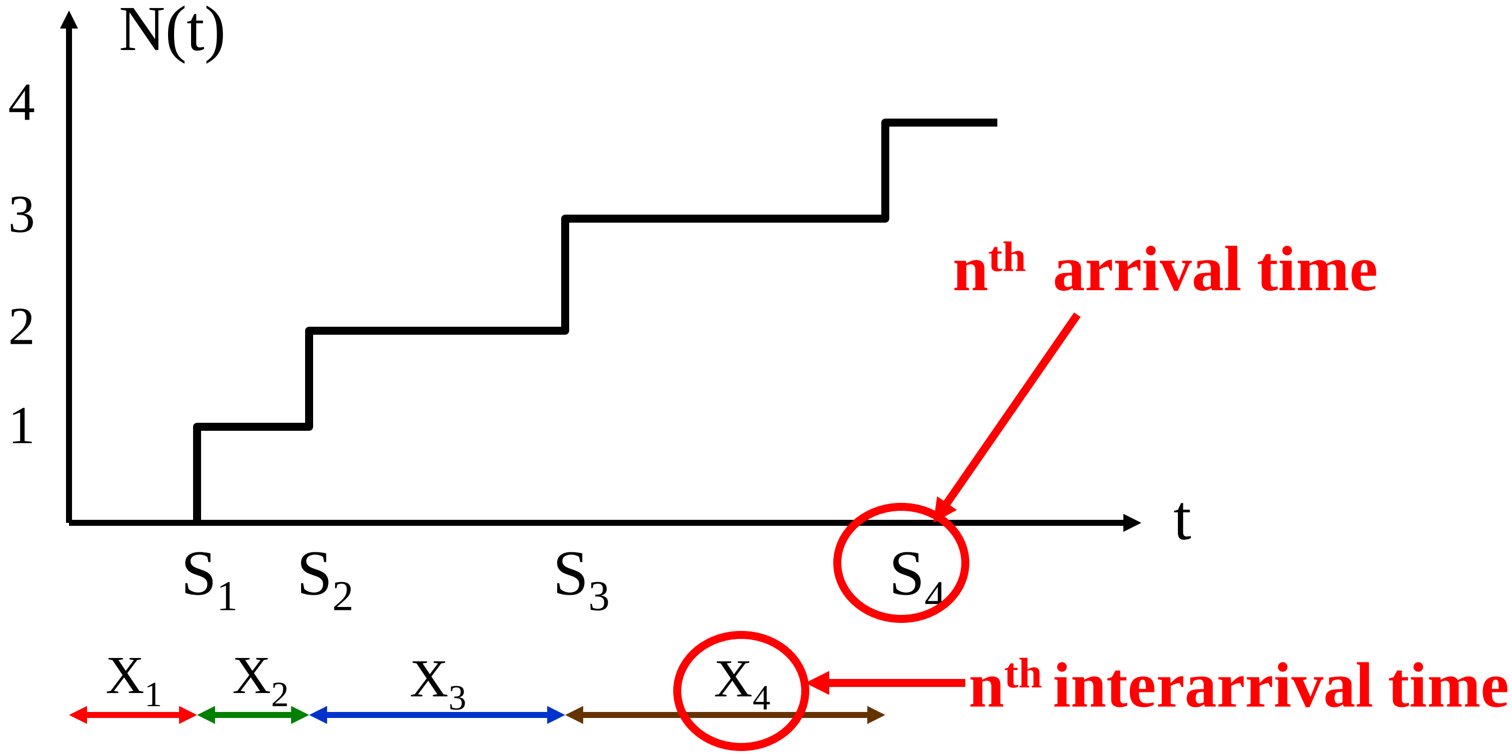
$$P_{N(t)}(n) = \begin{cases} \dfrac{(\lambda T)^n e^{-\lambda T}}{n!} & n = 0,1,2,.. \\ 0 & \text{Otherwise} \end{cases}$$

Poisson Process

- Poisson Process is

  a Counting Process that the # of **Arrival** during any interval is Poisson RV
- An arrival during any instant is **independent** of the past history of the process → **Memoryless**
- X_n is called **Interarrival Time**

Poisson Process



Poisson Process

Definition:

A Counting Process N(t) is a Poisson Process N(t) if

- # of arrivals in $(t_0, t_1]$, $N(t_1) - N(t_0)$, is a Poisson RV with expected value $\lambda(t_1 - t_0)$
- # of arrivals in each interval are independent random variable

Poisson Process

- Process rate (λ) = E[N(t)] / t
- $M = N(t_1) - N(t_0)$ = Poisson RV

$$P_M(m) = \begin{cases} \dfrac{[\lambda(t_1-t_0)]^m}{m!} e^{-\lambda(t_1-t_0)} & m = 0,1,2,.. \\ 0 & \text{Otherwise} \end{cases}$$

Joint PMF

Theorem: Poisson Process $N(t)$ of rate λ,

Joint PMF of $N(t_1),\ldots,N(t_k)$, $t_1 <\ldots< t_k$

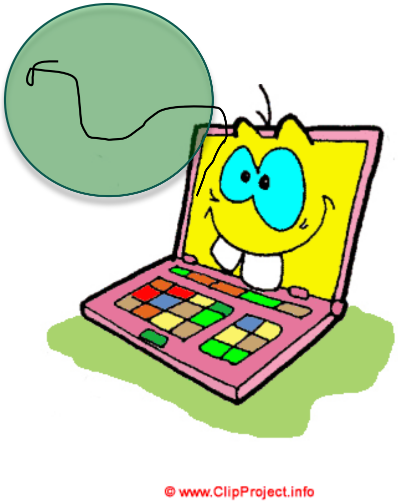
$$P_{N(t_1),\ldots,N(t_2)}(n_1,\ldots,n_k)$$

$$= \begin{cases} \dfrac{\alpha_1^{n_1} e^{-\alpha_1}}{n_1!} \dfrac{\alpha_2^{(n_2-n_1)} e^{-\alpha_2}}{(n_2 - n_1)!} \cdots \dfrac{\alpha_k^{(n_k-n_{k-1})} e^{-\alpha_k}}{(n_k - n_{k-1})!} & 0 \le n_1 \le \ldots \le n_k \\ 0 & \text{Otherwise} \end{cases}$$

$$\alpha_i = \lambda(t_i - t_{i-1})$$

Example

- A mobile station transmits data packet as Poisson process with rate 12 packets/sec
  - Find # of packets transmitted in the k^{th} hour
  - Find Joint PMF of # of packets transmitted in the k^{th} hour and z^{th} hour



Example

- Let N_k = # of packets transmitted in k^{th} hour
- # packets in each hour is IID

$$P_{N_i}(n) = \begin{cases} \dfrac{[12(3600-0)]^n \, e^{-12(3600-0)}}{n!} & n = 0,1,2,.. \\ 0 & \text{Otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{[43200]^n \, e^{-43200}}{n!} & n = 0,1,2,.. \\ 0 & \text{Otherwise} \end{cases}$$

Example

- Joint PMF of # of packets transmitted in the k[th] hour and z[th] hour

$$P_{N_k,N_z}(n_k,n_z) = \begin{cases} \dfrac{\alpha_k^{n_k} e^{-\alpha_k}}{n_k!} \dfrac{\alpha_z^{n_z} e^{-\alpha_z}}{n_z!} & n_k = 0,1,\ldots \\ & n_z = 0,1,\ldots \\ 0 & \text{Otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{\alpha^{(n_k+n_z)}}{n_k!\,n_z!} e^{-2\alpha} & n_k = 0,1,\ldots \\ & n_z = 0,1,\ldots \\ 0 & \text{Otherwise} \end{cases}$$

$$\alpha = \alpha_k = \alpha_Z = \lambda T = [12(3600-0)] = 43200$$

Interarrival Time

Theorem:

Poisson Process of rate λ, the interarrival times X_1, X_2, … are an iid random sequence with **Exponential PDF**

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

Interarrival Time

Theorem:

A Counting Process with ***independent exponential interarrival time*** $X_1, X_2, \ldots$ with $E[X_i] = 1/\lambda$ is a ***Poisson Process*** of rate λ

Brownian Motion Process

- A continuous time, continuous value random process

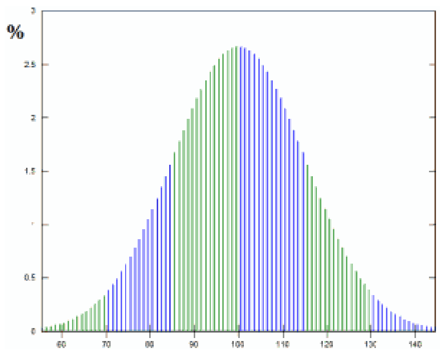
Definition: A Brownian Motion Process $X(t)$ has the property that

- $X(0) = 0$
- For $\tau > 0$, $X(t + \tau) - X(t)$ = Gaussian RV with **$E[X(t)] = 0$** and **$Var[X(t)] = \alpha\tau$** that is independent of $X(t')$ for $t' \leq t$

Brownian Motion Process

Gaussian RV

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x - \mu)^2}{2\sigma^2}}$$



http://www.damninteresting.net/content/gaussian_iq_distribution.gif

$E[X(t)] = 0$ and $Var[X(t)] = \alpha\tau$

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi\alpha\tau}} e^{\frac{-x^2}{2\,\alpha\tau}}$$

Joint PDF

- **Theorem:** For the Brownian motion process $X(t)$, joint PDF of $X(t_1),\ldots,X(t_k)$

$$f_{X(t_1),\ldots,X(t_k)}(x_1,\ldots,x_k) = \prod_{n=1}^{k} \frac{1}{\sqrt{2\pi\alpha(t_n - t_{n-1})}}\, e^{\frac{-(x_n - x_{n-1})^2}{2\alpha(t_n - t_{n-1})}}$$

Expected Value

X(t): $X(t_1) \rightarrow f_{X(t_1)}(x) \rightarrow E[X(t_1)]$

Definition: The expected value of a stochastic process X(t) is the deterministic function

$$\mu_X(t) = E[X(t)]$$

Covariance of X and Y

Definition: $Cov[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$

$$\begin{aligned} Cov[X,Y] &= E[XY - \mu_X Y - \mu_Y X + \mu_x \mu_y] \\ &= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_x \mu_y \\ &= E[XY] - \mu_x \mu_y - \mu_x \mu_y + \mu_x \mu_y \end{aligned}$$

Theorem: $Cov[X,Y] = E[XY] - \mu_x \mu_y$

- High covariance = an observation of X provides an accurate indication of Y

Autocovariance

- Auto = self = same process
- For a same process X(t), in 2 different times t_1 and $t_2 = t + \tau$
- For high covariance = a sample function, is unlikely to change in τ interval
- For near zero covariance = rapid change

How much the sample function
is likely to change in the τ interval after t

Autocovariance

Definition:

The autocovariance function of a stochastic process X(t) is

$$C_X(t,\tau) = Cov[\ X(t), X(t+\tau)\ ]$$

Note

- For $\tau = 0$ ➔ $C_X(t,t) = Var[X(t)]$

Correlation

Definition: The correlation of X and Y is $r_{X,Y}$

$$r_{X,Y} = E[XY]$$

Theorem: $Cov[X,Y] = r_{X,Y} - \mu_x \mu_y$

Autocorrelation

- **Definition**: The autocorrelation function of a stochastic process X(t) is

$$R_X(t,\tau) = E\,[\,X(t)\,X(t+\tau)\,]$$

Autocovariance & Autocorrelation

Theorem: The autocovariance and autocorrelation functions of a process X(t) satisfy

$$C_X(t,\tau) = R_X(t,\tau) - \mu_x(t)\,\mu_x(t+\tau)$$

<u>Note:</u>

Autocovariance → use X(t) to predict a future X(t+τ)

Autocorrelation → describe the power of a random signal

Random Sequence

- For a discrete time process, the sample function is described by the ordered sequence of random variable $X_n = X(nT)$

Definition: A random sequence X_n is an ordered sequence of random variable $X_0, X_1, \ldots$

Autocovariance & Autocorrelation of a Random Sequence

Definition: The autocovariance of a random sequence X_n is

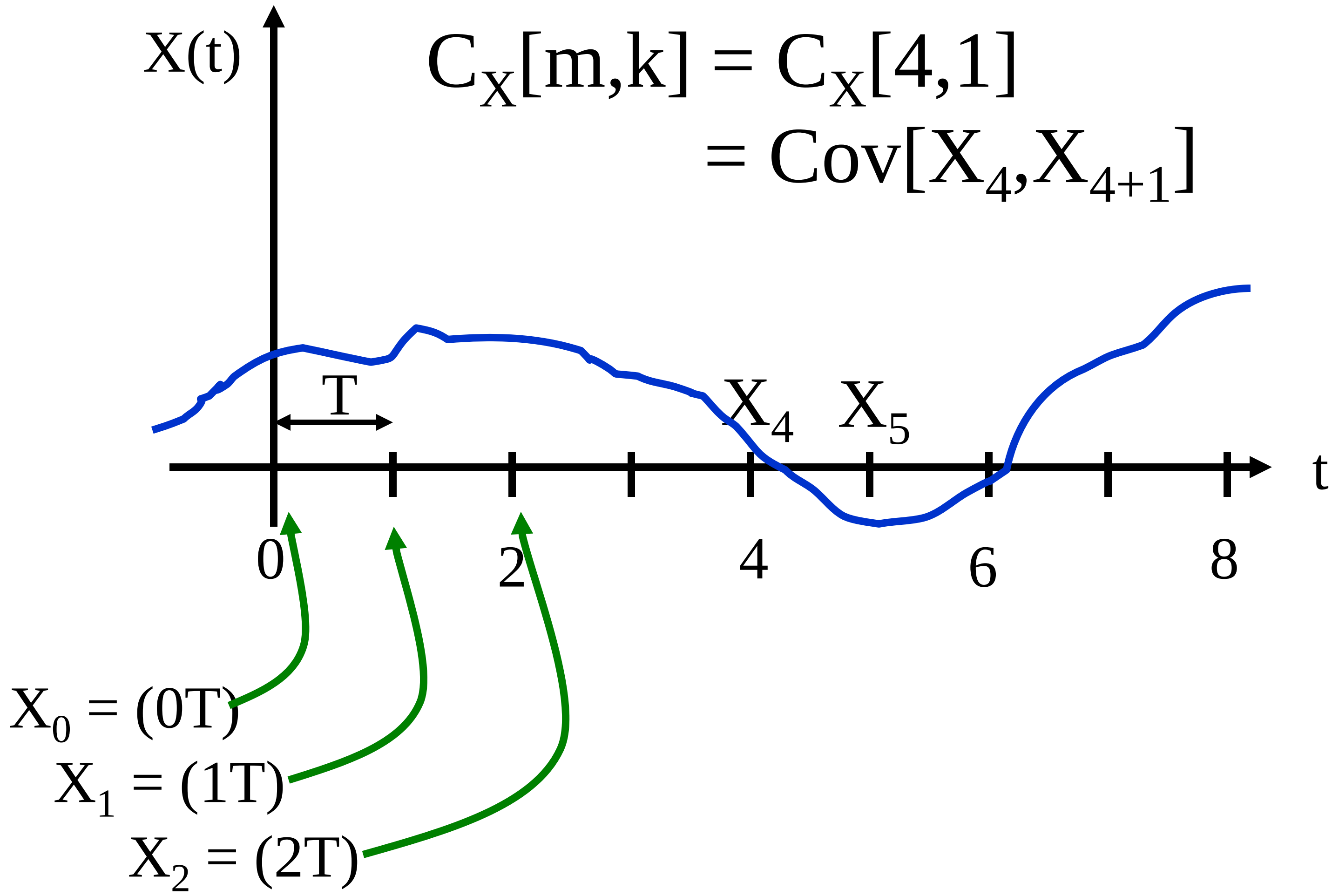
$$C_X[m,k] = Cov[X_m, X_{m+k}]$$

Definition: The autocorrelation of a random sequence X_n is

$$R_X[m,k] = E[X_m X_{m+k}]$$

m and k are integers

Autocovariance & Autocorrelation of a Random Sequence



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STATIONARY PROCESS

Stationary Process

- For a random process X(t), normally,

  at t_1: $X(t_1)$ has pdf = $f_{X(t_1)}(x)$ [depends on t_1]

- For a random process X(t),

  at t_1: $X(t_1)$ has pdf = $f_{X(t_1)}(x)$ [not depend on t_1]

Stationary Process

= same random variable at all time

= no statistical properties change with time

$$f_{X(t_1)}(x) = f_{X(t_1+\tau)}(x) = f_X(x)$$

Stationary Process

Definition: A stochastic process X(t) is stationary iif for all sets of time $t_1,\ldots,t_m$ and any time different τ,

$$f_{X(t_1),\ldots,X(t_m)}(x_1,\ldots,x_m) = f_{X(t_1+\tau),\ldots,X(t_m+\tau)}(x_1,\ldots,x_m)$$

Stationary Random Sequence

Definition: A random sequence X_n is stationary iif for any finite sets of time instants $n_1,\ldots,n_m$ and any time different k,

$$f_{X(n_1),\ldots,X(n_m)}(x_1,\ldots,x_m) = f_{X(n_1+k),\ldots,X(n_m+k)}(x_1,\ldots,x_m)$$

Stationary Process

Theorem: A stationary process X(t),

$$\mu_X(t) = \mu_X$$

$$R_X(t,\tau) = R_X(0,\tau) = R_X(\tau)$$

$$C_X(t,\tau) = R_X(\tau) - \mu^2_X = C_X(\tau)$$

Stationary Random Sequence

Theorem: A stationary random sequence X_n, for all m

$$E[X_m] = \mu_X$$

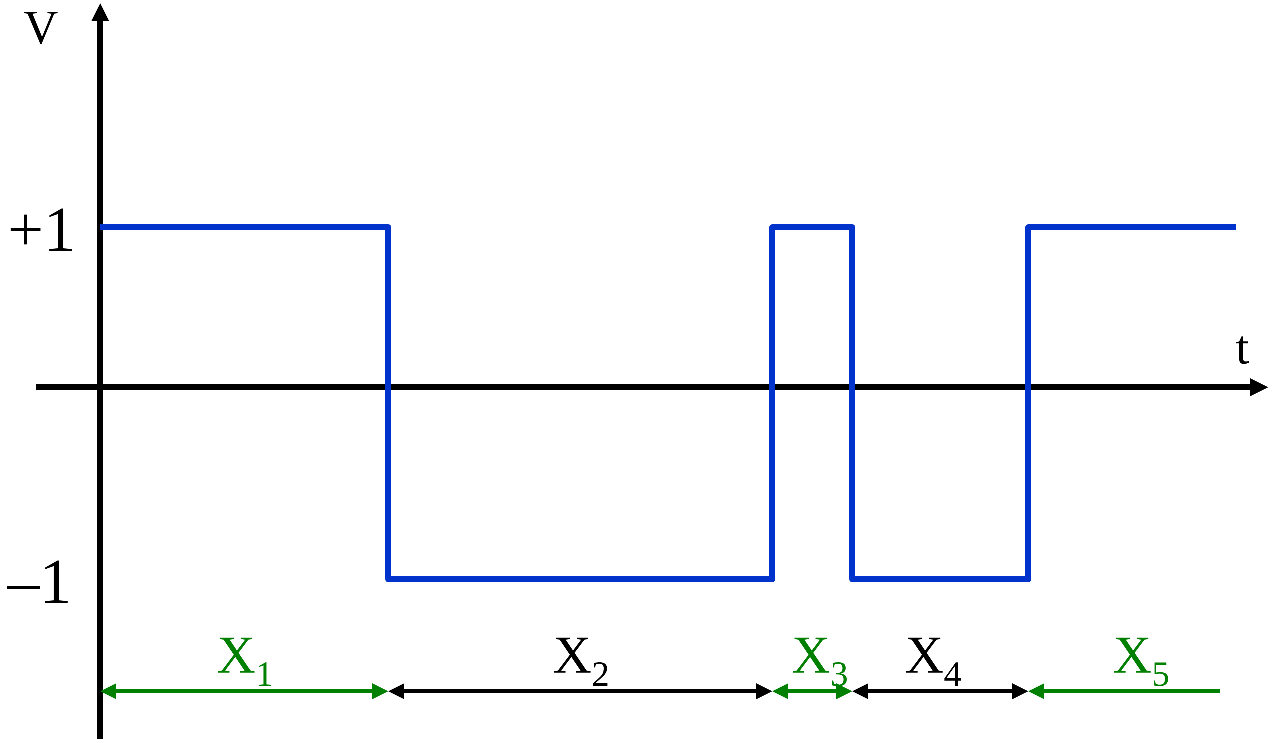
$$R_X[m.k] = R_X[0,k] = R_X[k]$$

$$C_X[m,k] = R_X[k] - \mu^2_X = C_X[k]$$

Example

- Telegraph Signal, X(t) take value ± 1
- X(0) = ± 1 with probability = 0.5
- Let X(t) toggles the polarity with each occurrence of an event in a Poisson process rate α

Example



Example

- Find PMF of X(t), $f_{X(t)}(x)$
- $P[X(t)] = P[X(t) \mid X(0) = 1]\, P[X(0) = 1]$

  $+ P[X(t) \mid X(0) = -1]\, P[X(0) = -1]$

- $P[X(t) \mid X(0) = 1] = P[N(t) = \text{even}]$

$$= \sum_{j=0}^{\infty} \frac{(\alpha t)^{2j}}{(2j)!} e^{-\alpha t}$$

$$= e^{-\alpha t} (1/2) (e^{\alpha t} + e^{-\alpha t})$$

$$= (1/2)(1 + e^{-2\alpha t})$$

Example

- $P[X(t) \mid X(0) = -1] = P[N(t) = \text{odd}]$

$$= \sum_{j=0}^{\infty} \frac{(\alpha t)^{2j+1}}{(2j+1)!} e^{-\alpha t}$$

$$= e^{-\alpha t} (1/2) (e^{\alpha t} - e^{-\alpha t})$$

$$= (1/2) (1 - e^{-2\alpha t})$$

Example

- $P[X(t) = 1]$

  $= \mathbf{P[X(t) \mid X(0) = 1]}\ P[X(0) = 1] + \mathbf{P[X(t) \mid X(0) = -1]}\ P[X(0) = -1]$

  $= (1/2)(1 + e^{-2\alpha t}) \quad (1/2) \quad + \quad (1/2)(1 - e^{-2\alpha t}) \quad (1/2)$

  $= 1/2$
- $P[X(t) = -1] = 1 - P[X(t) = 1] = 1/2$

$$f_{X(t)}(x) = \begin{cases} 1/2 & X(t) = -1, 1 \\ 0 & \text{Otherwise} \end{cases}$$

Example

- $\mu_X(t) = 1\ (1/2) + (-1)(1/2) = 0$
- $Var\ [X(t)] = E[X^2(t)] = 1^2(1/2) + (-1)^2(1/2) = 1$

Autocovariance, $C_X(t,\tau) = e^{-2\alpha\tau}$

$$f_{X(t_1),\ldots,X(t_m)}(x_1,\ldots,x_m) = f_{X(t_1+\tau),\ldots,X(t_m+\tau)}(x_1,\ldots,x_m)$$

Wide Sense Stationary

Definition: $X(t)$ is a wide sense stationary random process iff for all t,

$$E[X(t)] = \mu_X$$

$$R_X(t,\tau) = R_X(0,\tau) = R_X(\tau)$$

Definition: X_n is a wide sense stationary random sequence iff for all n,

$$E[X_n] = \mu_X$$

$$R_X[n,k] = R_X[0,k] = R_X[k]$$

Wide Sense Stationary

- For every **stationary** process or sequence, it is also **wide sense stationary**.
- However, if it is a **wide sense stationary** it may or may not be **stationary**.

Example

- Let $X_n = \pm 1$ with prob = ½ (n = even)
- For n = odd

  $X_n = -1/3$ with prob = 9/10

  $X_n = 3$ with prob = 1/10
- Stationary ?
  - No
- Wide sense stationary ?
  - Mean = 0 for all n
  - $C_X(t,\tau) = 0$ for $\tau > 0$
  - $C_X(t,\tau) = 1$ for $\tau = 0$
  - Yes , it's wide sense stationary

Wide Sense Stationary

Theorem: For a wide sense stationary process $X(t)$,

$$R_X(0) \geq 0$$

$$R_X(\tau) = R_X(-\tau)$$

$$|R_X(\tau)| \leq R_X(0)$$

Wide Sense Stationary

Theorem: For a wide sense stationary sequence X_n,

$$R_X[0] \geq 0$$

$$R_X[k] = R_X[-k]$$

$$|R_X[k]| \leq R_X[0]$$