

LECTURE #3

PROBABILITY REVIEW (II)

204528

Queueing Theory and
Applications in Networks

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Outline

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- Probability meaning
- Random Variable
- Discrete RV
- Continuous RV
- Multiple RVs

Cumulative Distribution Function (CDF)

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- Definition:

$$F_X(x) = P[X \leq x]$$

- Contain complete information about the probability model of the random variable
- PMF  CDF

CDF Theorem

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Theorem: For a discrete random variable X

with $S_X = \{x_1, x_2, \dots\}$ & $x_1 \leq x_2 \leq \dots$

1) $F_X(-\infty) = 0$ and $F_X(\infty) = 1$ → **From 0 to 1**

2) $\forall x' \geq x, F_X(x') \geq F_X(x)$ → **Monotonic Increasing**

3) For $x_i \in S_X$ and $\varepsilon = +\text{small number}$

$F_X(x_i) - F_X(x_i - \varepsilon) = P_X(x_i)$ → **Discontinuity = $P_X(x)$**

4) $F_X(x) = F_X(x_i) \quad \forall x, x_i \leq x < x_{i+1}$ → **Horizon line**

CDF Example

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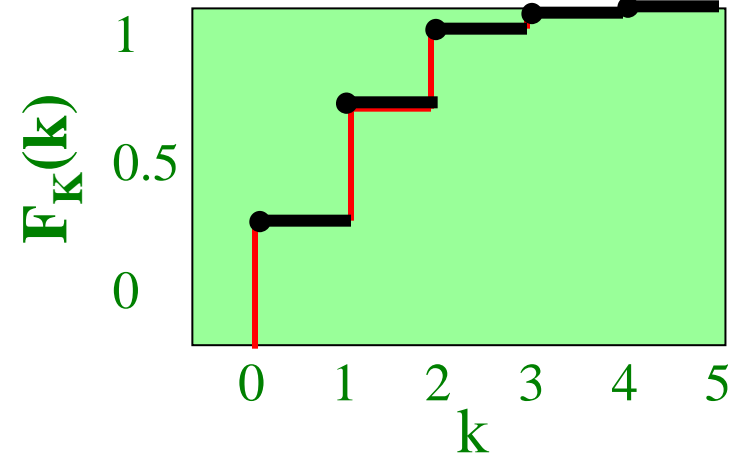
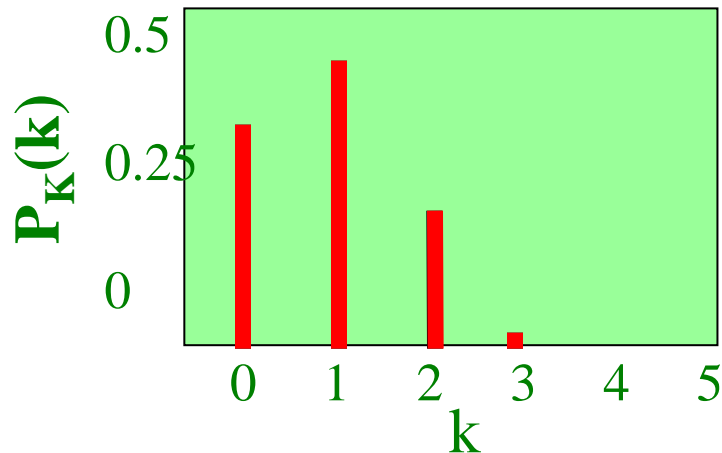
- For a binomial RV, # of fail CDs in 5 tests with $p = 0.2$

$$P_K(k) = \begin{cases} \binom{5}{k} (0.2)^k (0.8)^{5-k} & k = 0, 1, 2, \dots, 5 \\ 0 & \text{Otherwise} \end{cases}$$

CDF Example

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k	$P_K(k)$	k	$P_K(k)$
0	0.33	3	0.05
1	0.41	4	0.01
2	0.20	5	0



Average

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- Study RV \rightarrow average
- What is the average of an RV?
 - A single number that describes the RV
 - An example of statistic
- What is Statistic?
 - Numbers that collect all information of things under our interesting
 - Averages: mean, mode, and median

Average

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- Mean:
 - Sum / #terms
- Mode:
 - Most common value
 - $P_X(x_{\text{mod}}) \geq P_X(x) \quad \forall x$
- Median:
 - The middle of the data set
 - $P[X < x_{\text{med}}] = P[X > x_{\text{med}}]$

Mean \rightarrow Expected Value

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- Adding all measurements / #terms

$$E[X] = \mu_X = \sum_{x \in S_X} xP_X(x)$$

- **Example:**

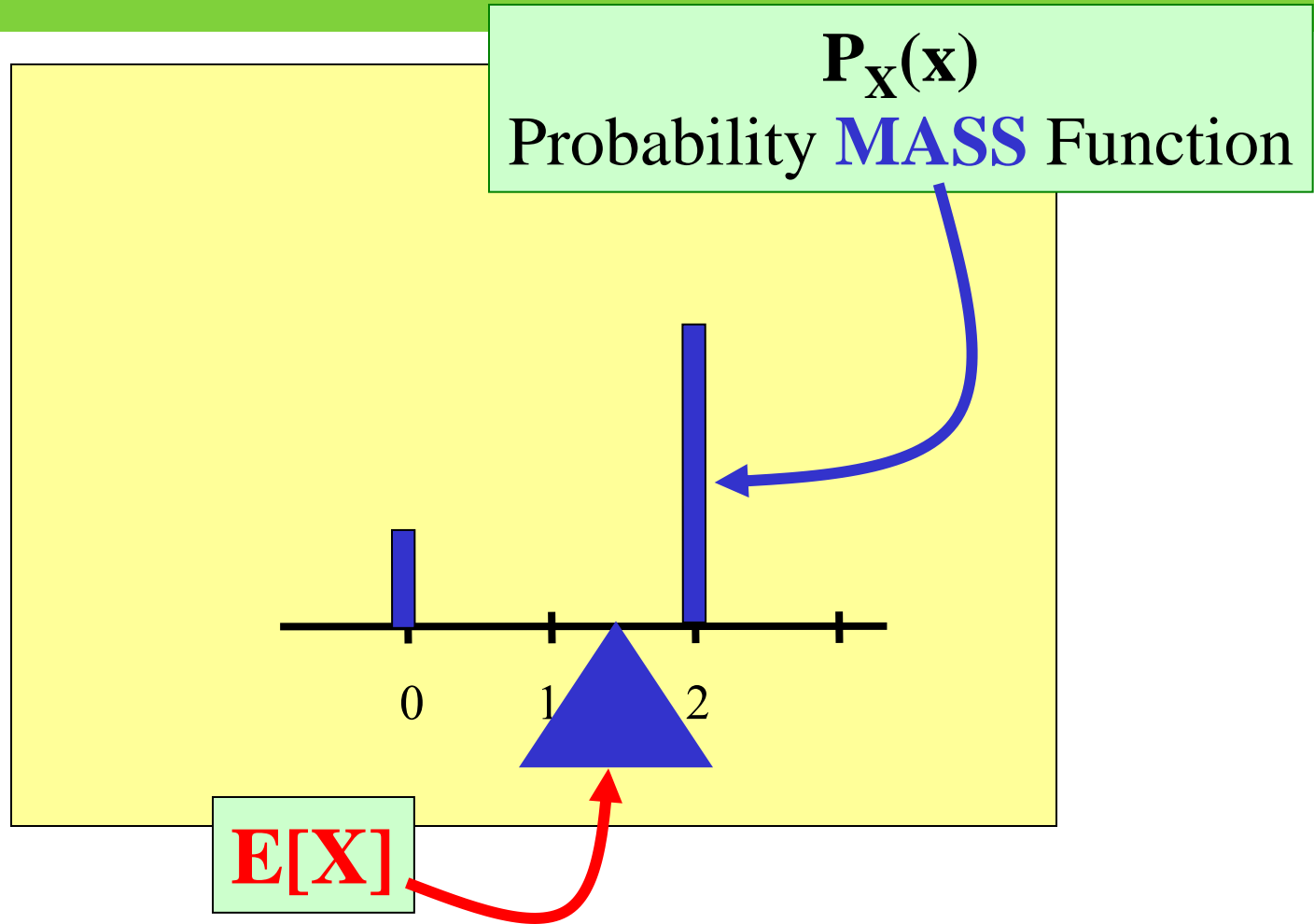
$$P_T(t) = \begin{cases} 1/4 & t = 0 \\ 3/4 & t = 2 \\ 0 & \text{Otherwise} \end{cases}$$

- $E[T] = ?$

$$= 0(1/4) + 2(3/4) = 3/2$$

Expected Value

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Useful Discrete RV

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<u>Uniform</u> Equiprobable outcomes	$\begin{cases} 1/(j-k+1) & x = k, k+1, k+2, \dots, j \\ 0 & \textit{Otherwise} \end{cases}$	$E[X] = \frac{(j+k)}{2}$
<u>Bernoulli</u> Pass/Fail	$\begin{cases} 1-p & x = 0 \\ p & x = 1 \\ 0 & \textit{Otherwise} \end{cases}$	$E[X] = p$
<u>Geometric</u> # tests until fail	$\begin{cases} p(1-p)^{x-1} & x = 1, 2, 3, \dots \\ 0 & \textit{Otherwise} \end{cases}$	$E[X] = 1/p$

Useful Discrete RV

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<p><u>Binomial</u></p> <p># fails in n tests</p>	$\begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x=1,2,\dots,n \\ 0 & \textit{Otherwise} \end{cases}$	<p>$E[X] = np$</p>
<p><u>Pascal</u></p> <p># tests until k fails</p>	$\begin{cases} \binom{x-1}{k-1} p^k (1-p)^{x-k} & x = k, k+1, \dots \\ 0 & \textit{Otherwise} \end{cases}$	<p>$E[X] = k/p$</p>
<p><u>Poisson</u></p> <p>occurrence in a period</p>	$\begin{cases} \frac{(\lambda T)^x e^{-\lambda T}}{x!} & x = 0, 1, 2, \dots \\ 0 & \textit{Otherwise} \end{cases}$	<p>$E[X] = \alpha$ $\alpha = \lambda T$</p>

Variance & Standard Deviation

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- We knew average, $E[X]$,
why do we need these Variance &
Standard Deviation?
- How far from the average?
- $T = X - \mu_x$
 $E[T] = E[X - \mu_x]$
 $= 0$

Variance

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- The useful measurement is $\mathbf{E}[|T|]$
- $\mathbf{E}[T^2] = \mathbf{E}[(X - \mu_x)^2] \rightarrow \mathbf{Variance}$

Definition:

$$\text{Var}[X] = \mathbf{E} [(X - \mu_x)^2]$$

Standard Deviation

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Definition:

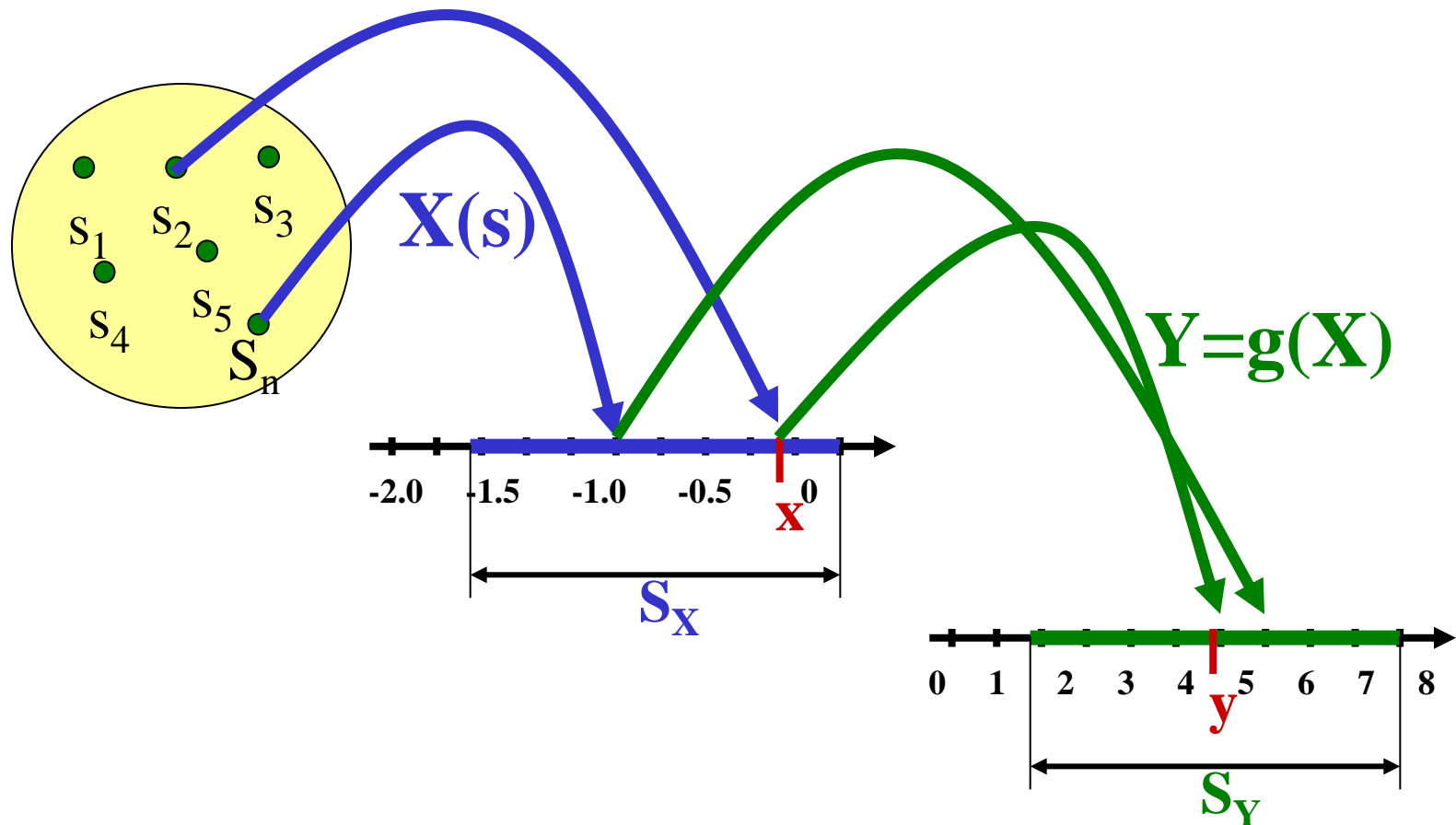
Sigma X

$$\sigma_X = \sqrt{\text{Var}[X]}$$

- σ_X can compare to μ_x
- Ex. $\sigma_X = 15$, Score +6 from mean
→ OK. Middle of class
- Ex. $\sigma_X = 3$, Score +6 from mean
→ V.Good In Top class group

Derived Random Variable

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Why do we need a Derived Random Variable?

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- From sample values of the random variable, use these values to compute other quantities.
- Example:
 - Find a decibel value form signal-to-noise ratio
- $Y = g(X)$

Example-1

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- Random Variable $X = \#$ pages in one fax
- $P_X(x) =$ number of pages in each fax
- Charging plan
 - 1st page = 10 Baht
 - 2nd page = 9 Baht
 - ...
 - 5th page = 6 Baht
 - 6 – 10 pages = 50 Baht
- Find the charge in Baht for sending one fax

Example-1

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- Random Variable Y = the charge in Baht for sending one fax

$$Y = g(X) = \begin{cases} 10.5X - 0.5X^2 & 1 \leq X \leq 5 \\ 50 & 6 \leq X \leq 10 \end{cases}$$

PMF of Y

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Theorem:

$$P_Y(y) = \sum_{x:g(x)=y} P_X(x)$$

$P[Y=y]$ = Σ of all outcomes $X = x$ for which $Y = y$

Conditional PMF

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$$P[A|B] = P[X = x|B]$$

Definition: Given event B , $P[B] > 0$

$$P_{X|B}(x) = P[X=x|B]$$

Theorem:
$$P[A] = \sum_{i=1}^n P[A|B_i]P[B_i]$$

Theorem:
$$P_X(x) = \sum_{i=1}^n P_{X|B_i}(x)P[B_i]$$

Outline

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- Probability meaning
- Random Variable
- Discrete RV
- Continuous RV
- Multiple RVs

MULTIPLE DISCRETE RVS



What is Multiple Discrete RV?

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- Each observation \rightarrow Random Variable
- 2 observations \rightarrow 2 Random Variables
- ≥ 2 observations \rightarrow Multiple RVs

Joint PMF

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- For an experiment, Observe one thing
 - Model with one Random Variable
 - Describe the prob. model by using PMF
- For the same experiment, Observe 2 things
 - 2 Random Variables $\rightarrow X$ and Y
 - Joint PMF
- $P_{X,Y}(x,y)$

Joint PMF

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Definition:

$$P_{X,Y}(x,y) = P[X=x, Y=y]$$

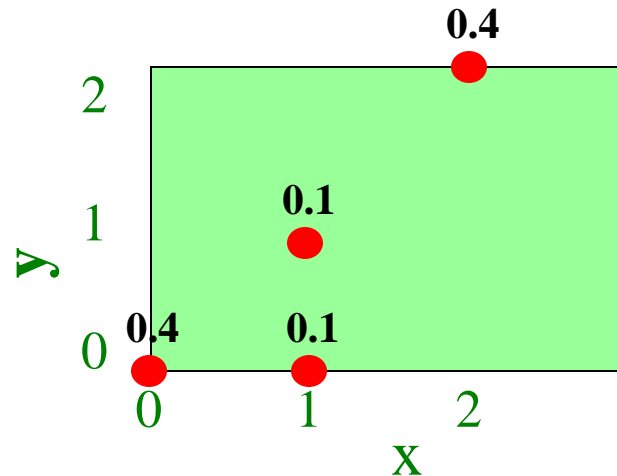
$$S_{X,Y} = \{ (x,y) \mid P_{X,Y}(x,y) > 0 \}$$

3 forms of Joint PMF

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$$P_{X,Y}(x,y) = \begin{cases} 0.4 & x=2, y=2 \\ 0.1 & x=1, y=1 \\ 0.1 & x=1, y=0 \\ 0.4 & x=0, y=0 \\ 0 & \text{Otherwise} \end{cases}$$

$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$
$x=0$	0.4	0	0
$x=1$	0.1	0.1	0
$x=2$	0	0	0.4



Marginal PMF

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- In an experiment with 2 RVs, X and Y
 - Possible to consider only one (X) and ignore Y
 - $P_X(x)$

Theorem: For random variables X and Y with

joint PMF $P_{X,Y}(x,y)$:

$$P_X(x) = \sum_y P_{X,Y}(x,y)$$
$$P_Y(y) = \sum_x P_{X,Y}(x,y)$$

Marginal PMF

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Margin

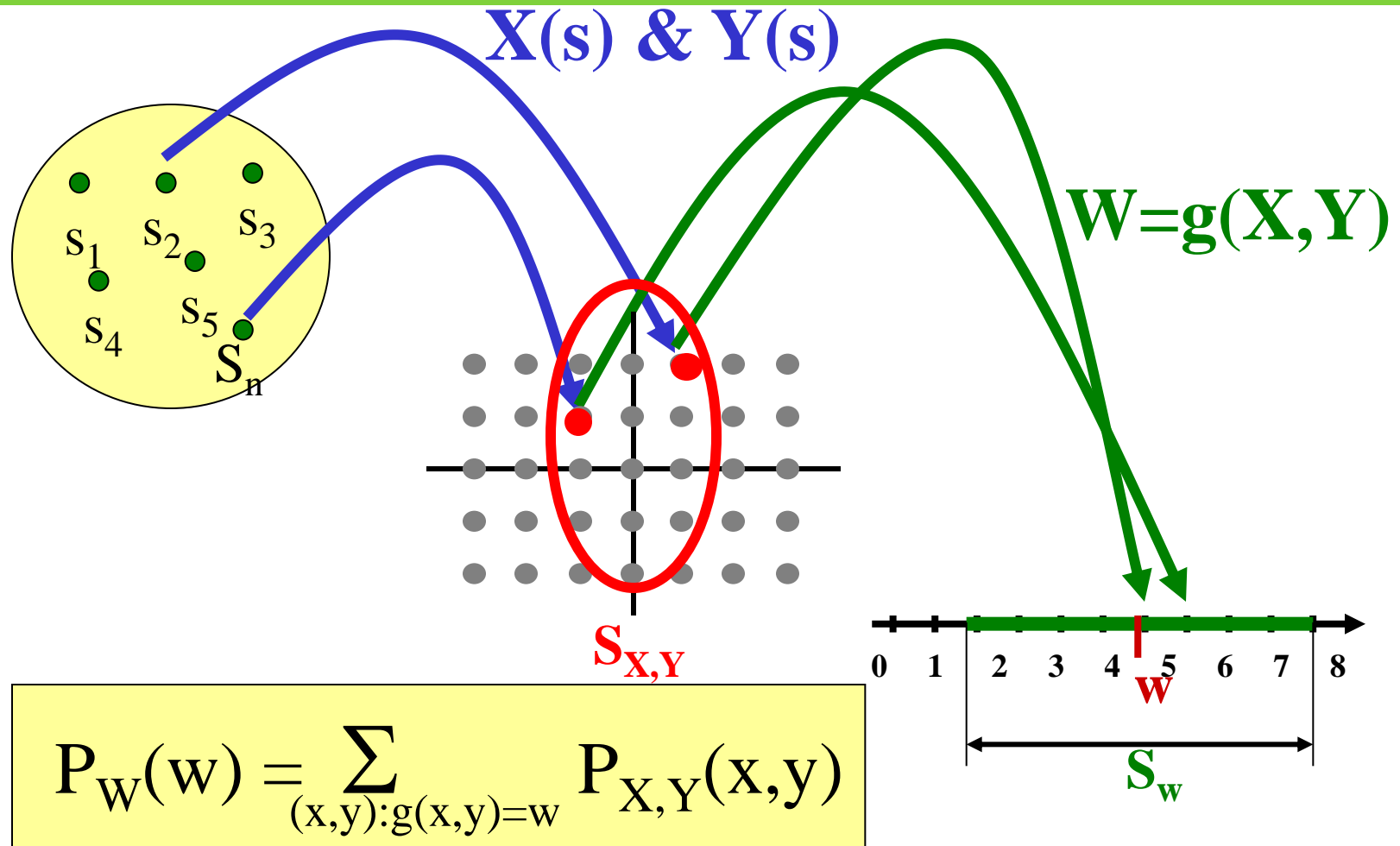
$P_{X,Y}(x,y)$	$y=0$	$y=1$	$y=2$	$P_X(x)$
$x=0$	0.4	0	0	0.4
$x=1$	0.1	0.1	0	0.2
$x=2$	0	0	0.4	0.4

$P_Y(y)$	0.5	0.1	0.4	1
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Margin

Derived Random Variable Functions of 2 RVs

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Expected Value of $g(X, Y)$

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Theorem: for $W = g(X, Y)$

$$E[W] = \sum_{x \in S_X} \sum_{y \in S_Y} g(X, Y) P_{X, Y}(x, y)$$

For any 2 RVs

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Theorem:

$$E[X + Y] = E[X] + E[Y]$$

- Find $E[X]$ and $E[Y]$
→ Marginal PMF

Var[X+Y]

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Definition: $\text{Var}[X] = E[(X - \mu_x)^2]$

$$\begin{aligned}\text{Var}[X+Y] &= E[((X+Y) - \mu_{x+Y})^2] \\ &= E[((X+Y) - (\mu_x + \mu_y))^2] \\ &= E[((X-\mu_x) + (Y-\mu_y))^2] \\ &= E[(X-\mu_x)^2 + 2(X-\mu_x)(Y-\mu_y) + (Y-\mu_y)^2] \\ &= E[(X-\mu_x)^2] + 2E[(X-\mu_x)(Y-\mu_y)] + E[(Y-\mu_y)^2]\end{aligned}$$

Theorem:

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2 E[(X-\mu_x)(Y-\mu_y)]$$

Covariance

Covariance of X and Y

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Definition: $\text{Cov}[X, Y] = E[(X - \mu_x)(Y - \mu_y)]$

$$\begin{aligned}\text{Cov}[X, Y] &= E[XY - \mu_x Y - \mu_y X + \mu_x \mu_y] \\ &= E[XY] - \mu_x E[Y] - \mu_y E[X] + \mu_x \mu_y \\ &= E[XY] - \mu_x \mu_y - \mu_x \mu_y + \mu_x \mu_y\end{aligned}$$

Theorem: $\text{Cov}[X, Y] = E[XY] - \mu_x \mu_y$

Covariance of X and Y

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Theorem: $\text{Cov}[X, Y] = E[XY] - \mu_x \mu_y$

Correlation

$$\begin{aligned} \text{If } X = Y \rightarrow \text{Cov}[X, X] &= E[XX] - \mu_x \mu_x \\ &= E[X^2] - \mu_x^2 \\ &= E[X^2 - 2\mu_x^2 + \mu_x^2] \\ &= E[X^2 - 2\mu_x X + \mu_x^2] \\ &= E[(X - \mu_x)^2] \\ &= \text{Var}[X] \end{aligned}$$

$$\text{If } \mu_x \text{ or } \mu_y = 0 \rightarrow \text{Cov}[X, Y] = E[XY]$$

Correlation

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Definition: The correlation of X and Y is $r_{X,Y}$

$$r_{X,Y} = E[XY]$$

Theorem: $\text{Cov}[X, Y] = r_{X,Y} - \mu_x \mu_y$

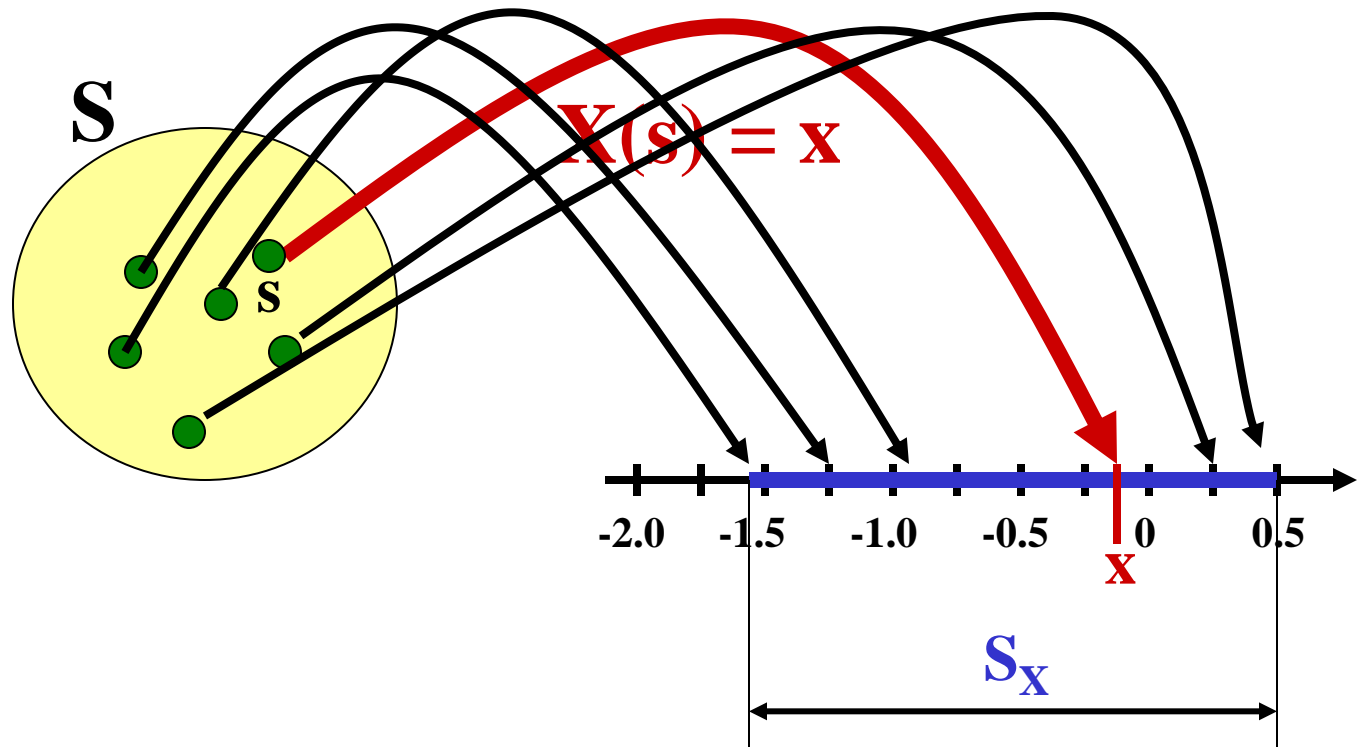
CONTINUOUS RANDOM VARIABLE



Random Variable

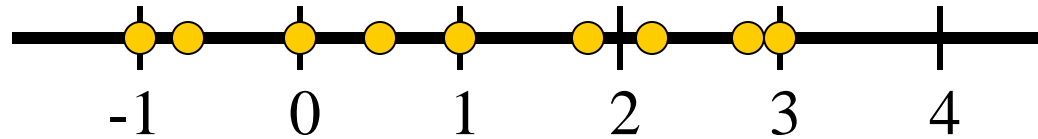
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X is a function that maps each outcome, s , in S to a real number $X(s)$, x



Continuous Sample Space

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In **Discrete**: countable set of numbers

$$S_X = \{-1, 0, 1, 3, 4\}$$

$$S_Y = \{-1, -0.9, 0, 0.5, 1, 1.8, 2.25, 2.9, 3\}$$

In **Continuous**: uncountable set of numbers

$S_X =$ **Interval** between 2 limits

$$S_X = (x_1, x_2) = (-1, 3)$$

Probability of a continuous RV

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- Measuring T , the download time

$$S_T = \{t \mid 0 < t < 12\}$$

- Guess the download time is $(0, 10]$ minutes
- Guess the download time is $[5, 8]$ minutes
- Guess the download time is $[5, 5.5]$ minutes

Chance that our guess is correct is decreasing

- Guess the download time is exactly 5.25 min.

**Probability of each individual outcome is zero.
The interesting probability is an interval.**

CDF

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- In discrete:
 - Probability Mass Function (PMF), $P_X(x)$
- In continuous:
 - Impossible to define PMF
 - Cumulative Distribution Function (CDF)

Definition: $F_X(x) = P[X \leq x]$

- PMF  CDF

CDF Theorem

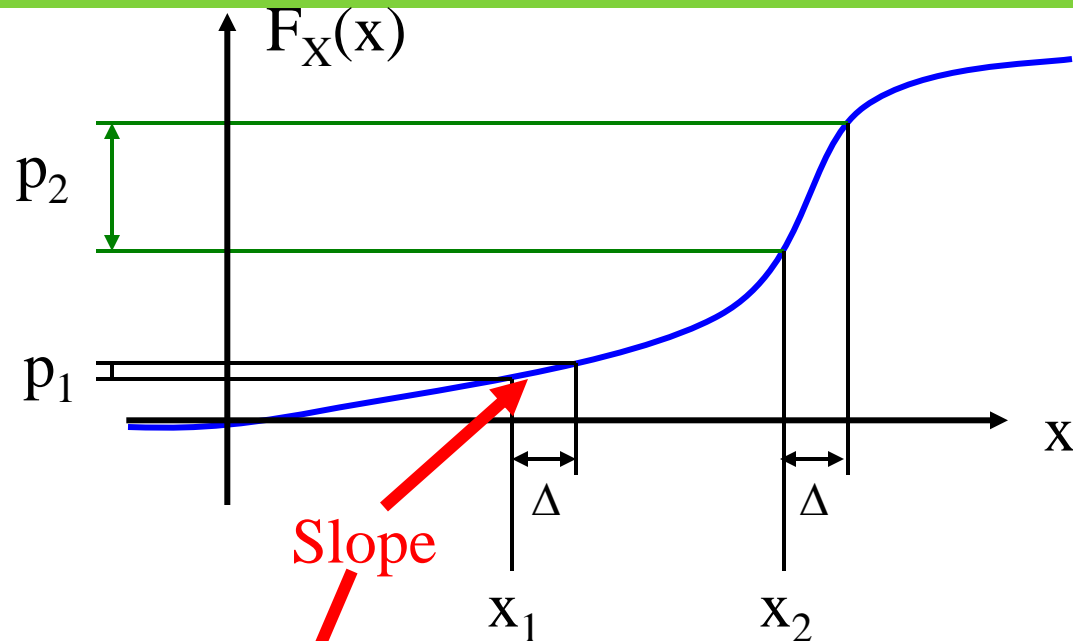
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Theorem:

- $F_X(-\infty) = 0$
- $F_X(\infty) = 1$
- $P[x_1 < X \leq x_2] = F_X(x_2) - F_X(x_1)$

Probability Density Function

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$$\begin{aligned} p_1 &= P[x_1 < X \leq x_1 + \Delta] \\ &= F_X(x_1 + \Delta) - F_X(x_1) \\ &= \frac{F_X(x_1 + \Delta) - F_X(x_1)}{\Delta} \Delta \end{aligned}$$

$$\begin{aligned} p_2 &= P[x_2 < X \leq x_2 + \Delta] \\ &= F_X(x_2 + \Delta) - F_X(x_2) \end{aligned}$$

For $\Delta \rightarrow 0$,
Slope $\rightarrow dF_X(x)/dx$ at x_1

Probability Density Function

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- The slope of CDF in a region near x
 - Probability of random variable X near x
 - The prob. in a small region(Δ) = slope * Δ
- Slope of CDF → PDF

Definition:

Probability Density Function (PDF) is

$$f_X(x) = \frac{dF_X(x)}{dx}$$

PDF Theorem

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Theorem:

- $f_X(x) \geq 0$ for all x
- $F_X(x) = \int_{-\infty}^x f_X(u) du$
- $\int_{-\infty}^{\infty} f_X(x) dx = 1$

Expected Values

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For Discrete Random Variable:

$$E[X] = \sum_{x \in S_X} x P_X(x)$$

For Continuous Random Variable:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

Expected Value & Variance

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- Find $E[X]$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Find $E[X^2]$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

- Find $\text{Var}[X]$

$$\text{Var}[X] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx$$

Some Useful Continuous RVs

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- Uniform
- Exponential
- Gaussian

Uniform Continuous RV

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Definition:

$$f_X(x) = \begin{cases} 1/(b - a) & a \leq x < b \\ 0 & \text{Otherwise} \end{cases}$$

where $b > a$

Uniform Continuous RV

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Theorem:

- $$F_X(x) = \begin{cases} 0 & x \leq a \\ (x - a)/(b - a) & a < x \leq b \\ 1 & x > b \end{cases}$$

- $$E[X] = (b + a)/2$$

- $$\text{Var}[X] = (b - a)^2/12$$

Exponential Continuous RV

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Definition:

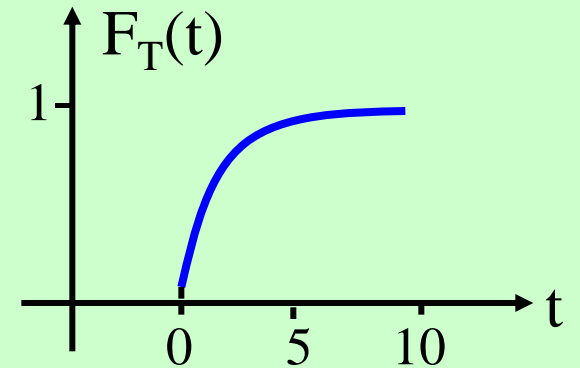
$$f_X(x) = \begin{cases} a e^{-ax} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

where $a > 0$

Exponential Example

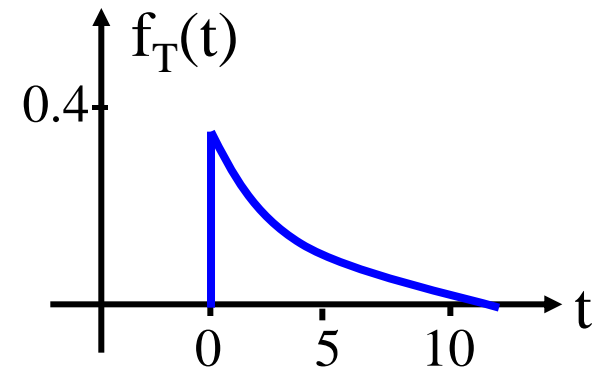
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$$F_T(t) = \begin{cases} 1 - e^{-t/3} & t \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$



Find PDF

$$\begin{aligned} f_T(t) &= \frac{dF_T(t)}{dt} \\ &= \begin{cases} (1/3) e^{-t/3} & t \geq 0 \\ 0 & \text{Otherwise} \end{cases} \end{aligned}$$



Exponential Continuous RV

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Theorem:

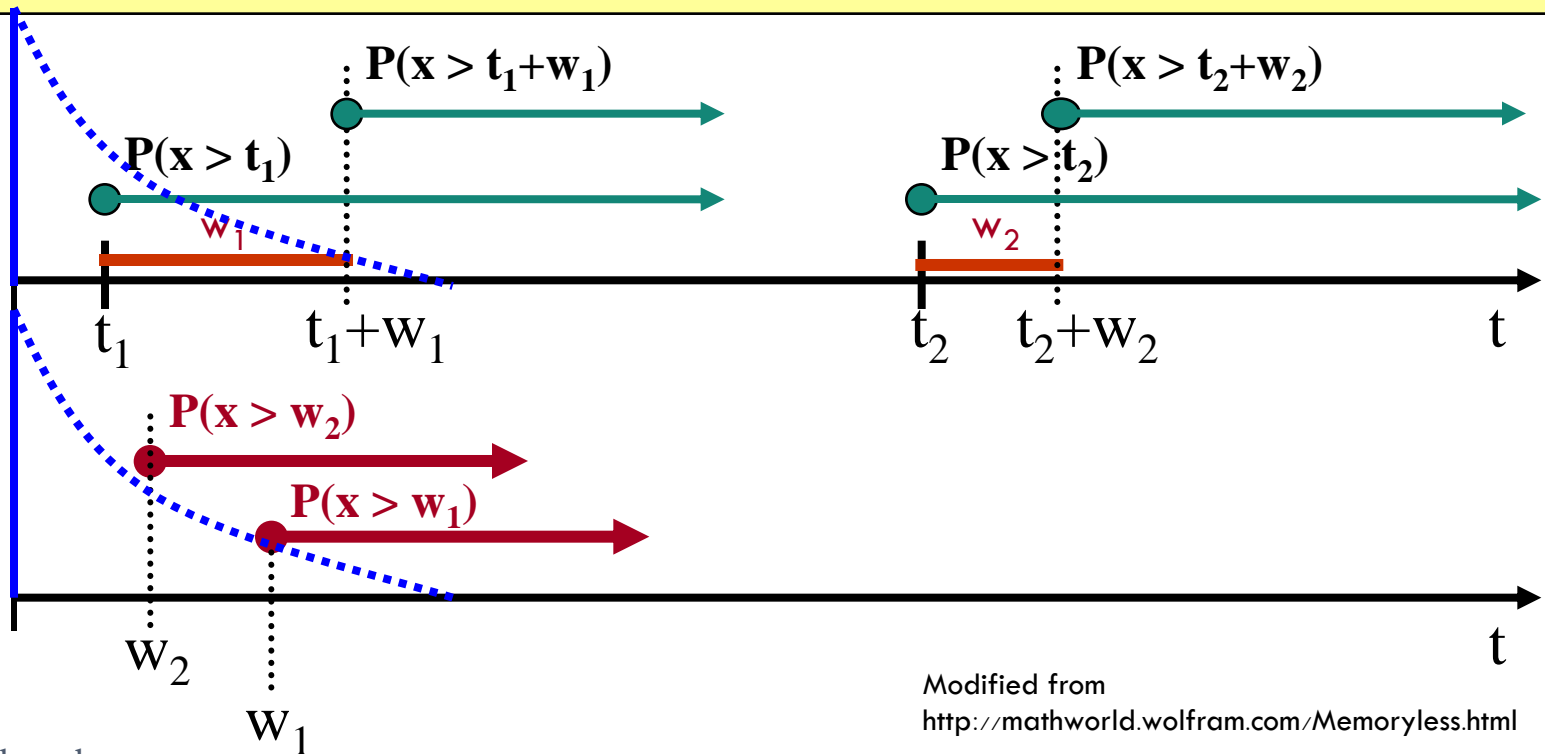
- $F_X(x) = \begin{cases} 1 - e^{-ax} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$
- $E[X] = 1/a$
- $\text{Var}[X] = 1/a^2$

Memoryless Property

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A variable x is memoryless with respect to t if,

$$P(x > t+w \mid x > t) = P(x > w) \quad \forall w \text{ with } t \neq 0$$



Modified from
<http://mathworld.wolfram.com/Memoryless.html>

Memoryless Property

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$$P(x > t+w \mid x > t) = P(x > w)$$

$$\frac{P(x > t+w, x > t)}{P(x > t)} = P(x > w)$$

$$P(x > t+w, x > t) = P(x > w) P(x > t)$$

$$P(x > t+w) = P(x > w) P(x > t)$$

Memoryless Property (Exponential Distribution)

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- For Exponential Distribution

$$P(x > t) = e^{-\lambda t}$$

$$P(x > t+w) = e^{-\lambda(t+w)}$$

Exponential distribution
is the only **Memoryless**
random distribution

- Therefore,

$$P(x > t+w) = P(x > w) P(x > t)$$

$$= e^{-\lambda w} e^{-\lambda t}$$

$$= e^{-\lambda(t+w)}$$

If t and w are integers, then the **Geometric distribution** is **Memoryless**

Gaussian Random Variables

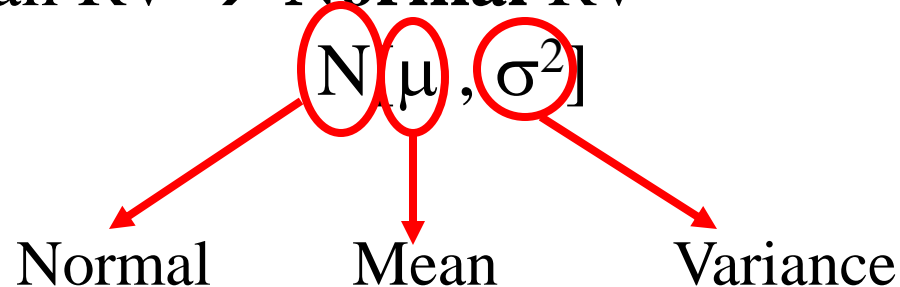
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Definition:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x - \mu)^2}{2\sigma^2}}$$

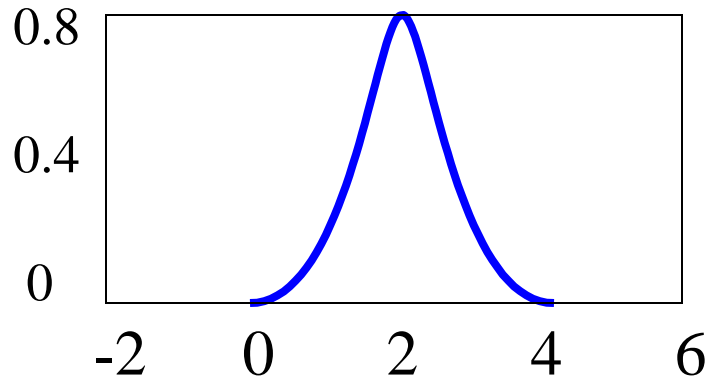
where $\mu \in \text{Real}$, and $\sigma > 0$

- Gaussian RV \rightarrow Normal RV

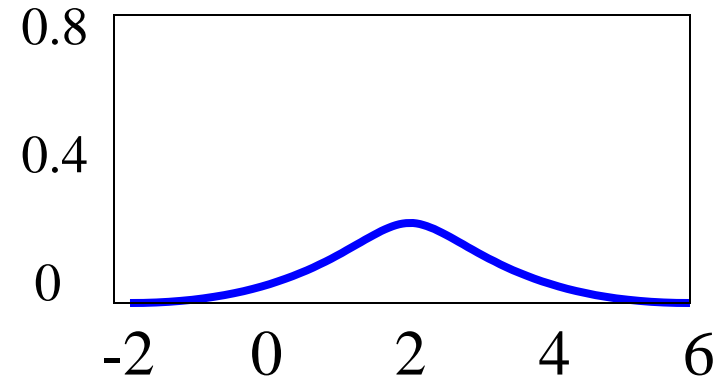


Gaussian Random Variables

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$$\mu = 2, \sigma = 1/2$$



$$\mu = 2, \sigma = 2$$

$f_X(x) \rightarrow$ Bell Shape with 2 parameters: μ and σ

$$\int_{-\infty}^{\infty} f_X(x) dx = 1$$

$$E[X] = \mu \quad \text{Var}[X] = \sigma^2$$

Mixed Random Variable

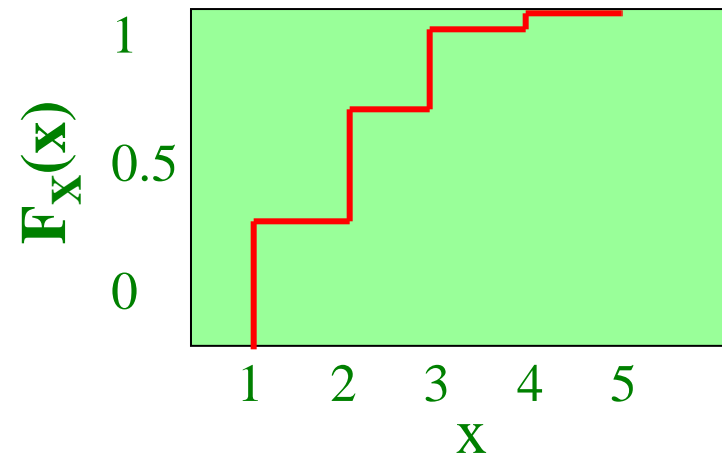
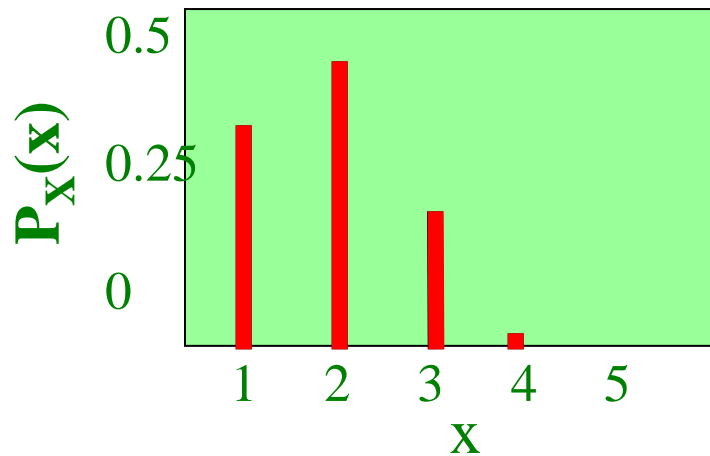
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- Discrete RV \rightarrow PMF & Summation
- Continuous RV \rightarrow PDF & Integral
- Combination of Discrete and Continuous RV
 - \rightarrow **Unit impulse function**
 - \rightarrow **Can use same formulas to describe both RVs**

PMF \rightarrow PDF

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$$F_X(x) = \sum_{x_i \in S_X} P_X(x_i) u(x-x_i)$$

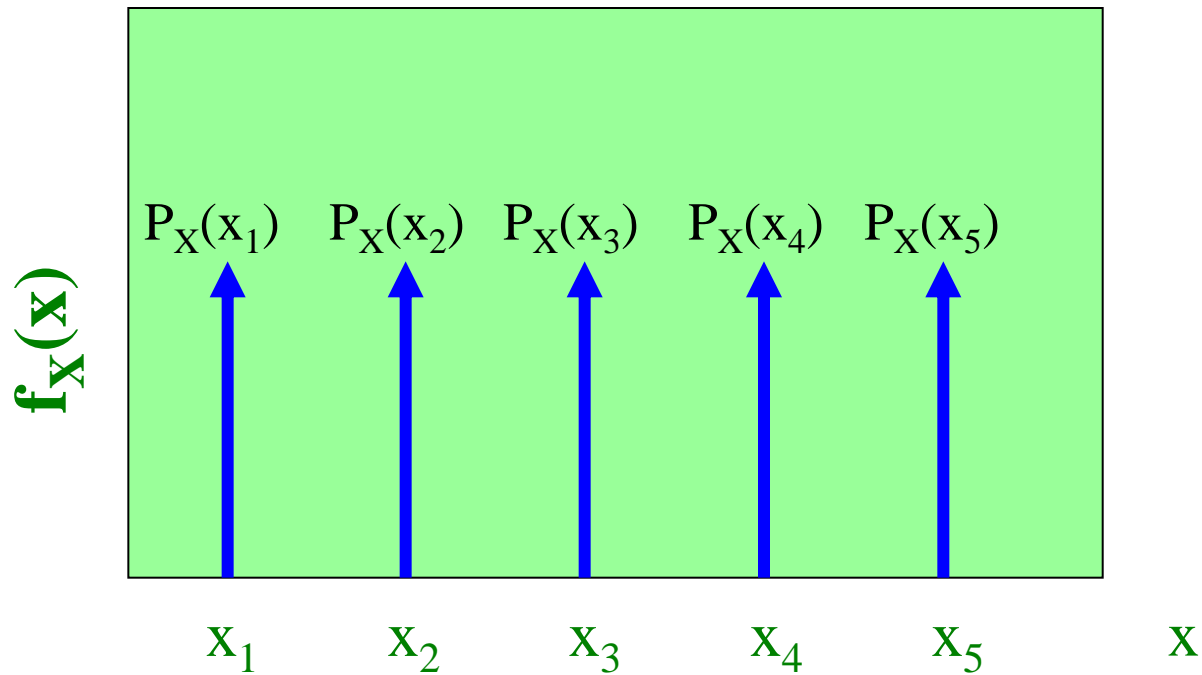


$u(x-x_i) \rightarrow u(x)$ shift to x_i

PMF \rightarrow PDF

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$$f_X(x) = \sum_{x_i \in S_X} P_X(x_i) \delta(x-x_i)$$

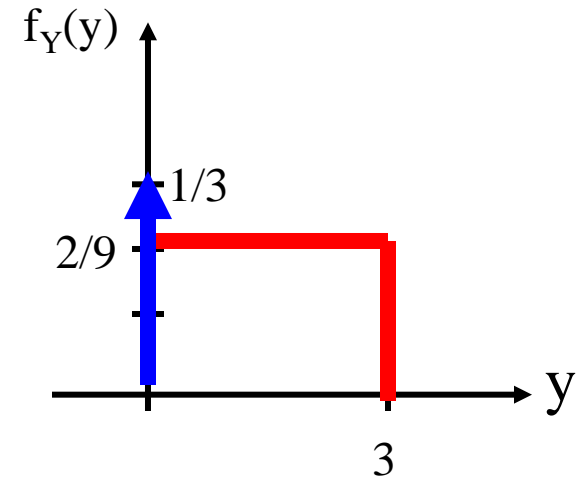
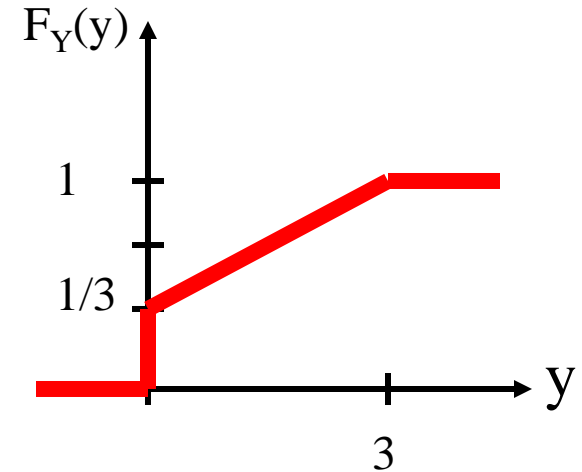


Example

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$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 1/3 + 2y/9 & 0 \leq y \leq 3 \\ 0 & \text{Otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \delta(y)/3 + 2/9 & 0 \leq y \leq 3 \\ 0 & \text{Otherwise} \end{cases}$$



Summary

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- Probability and Random Variable
 - Discrete Random Variable
 - Uniform/Bernoulli/Geometric/...
 - PMF & CDF
 - Expected Value
 - Variance & Standard Deviation
 - Continuous Random Variable
 - PDF
 - Uniform/Exponential/Gaussian
- Multiple Random Variables
- Stochastic Process