

# LECTURE #10

## M/G/1

204528

Queueing Theory and  
Applications in Networks

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# Outline

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- M/G/1
- Imbedded Markov Chain
  - Transition Probability
  - Mean Queue Length
- Mean Residual Service time

imbed [ɪm'bed] *vb* **-beds, -bedding, -bedded** a less common spelling of embed  
<http://www.thefreedictionary.com>

# M/G/1

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- A single server
- Poisson arrivals
- Arbitrary service time
  - not necessary to be memoryless

# M/M/1

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- Let  $N(t)$  = the number of customers in the system
- If  $N(t) > 1$ 
  - A customer is in service
  - The service time distribution is memoryless
- So we can predict the future behavior of the system  
→ It's depends on the previous state
- The process  $N(t)$  is a **Markov Chain**

# M/G/1

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- Let  $N(t)$  = the number of customers in the system
- If  $N(t) > 1$ 
  - A customer is in service
  - The service time distribution is not memoryless
- So, we need to know the time spent by customer to predict the future behavior of the system
- We want to summarize the complete past history of the system
- The process  $N(t)$  is **not a Markov Chain**

# M/G/1

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- Let  $X_0(t)$  = the service time already received by the customer in service at time  $t$
- However,  $[N(t), X_0(t)]$  is a Markov Process
  - Summarize all past history
- But it is a two-dimensional state description
  - ➔ can solved by the *method of supplementary variables*

# M/G/1

- We select the method of **Imbedded Markov Chain** (*Embedded Markov Chain*)
- We wish to simplify from two-dimensional description → one-dimensional
- Must find some points in time that the number of the system will provide the future state of the system
  - Departure instance
  - For the departure point,  $X_0(t)$  can be eliminated

# Imbedded Markov Chain

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- The departure of the customers is also called **Regeneration points**
- Let  $t_n$  = the time of departure of the  $n^{\text{th}}$  customer (where  $n = 1, 2, 3, \dots$ )
- Let  $X_n$  = the number of customers in the system at time  $t_n$   
$$X_n = N(t_n)$$
- $N(t) \rightarrow$  becomes the no. of customer **left behind** by a departure customer (not the same as previous  $N(t)$ )



# For Poisson Arrival

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- $P_k(t) = P[N(t) = k]$   
= P[arrival @ t finds k in system]
- This is always true

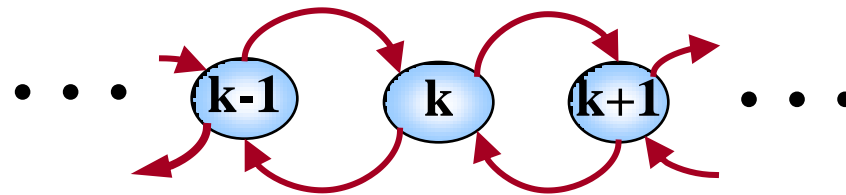
$$R_k(t) = \lim_{\Delta t \rightarrow \infty} P[N(t) = k]$$

$$P_k(t) = R_k(t)$$

# For Any System

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- With the limiting distribution



$$\lim_{t \rightarrow \infty} P[N(t) = k] = \lim_{n \rightarrow \infty} P[X_n = k]$$

- $P[\text{arrival @ } t \text{ finds } k \text{ customers in the system}]$   
 $= P[\text{departure @ } t \text{ leaves } k \text{ customers in the system}]$
- For M/G/1  
 $\rightarrow r_k = d_k = p_k$

# Imbedded Markov Chain

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- $X_n = \#$  customers in the system at time  $t_n$
- Let  $Y_n = \#$  customers arriving during the service time of the  $n^{\text{th}}$  customer

$$X_{n+1} = \begin{cases} X_n - 1 + Y_{n+1} & \text{if } X_n > 0 \\ Y_{n+1} & \text{if } X_n = 0 \end{cases}$$

# Transition Probability

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**This notation is not related to the previous notation**

- $C_n$  = the  $n^{\text{th}}$  customer to enter the system
- $\tau_n$  = arrival time of  $C_n$
- $t_n = \tau_n - \tau_{n-1}$  = interarrival time between  $C_{n-1}$  and  $C_n$
- $x_n$  = Service time of  $C_n$
- $q_n$  = number of customers left behind by departure of  $C_n$
- $v_n$  = number of customers arriving during the service of  $C_n$

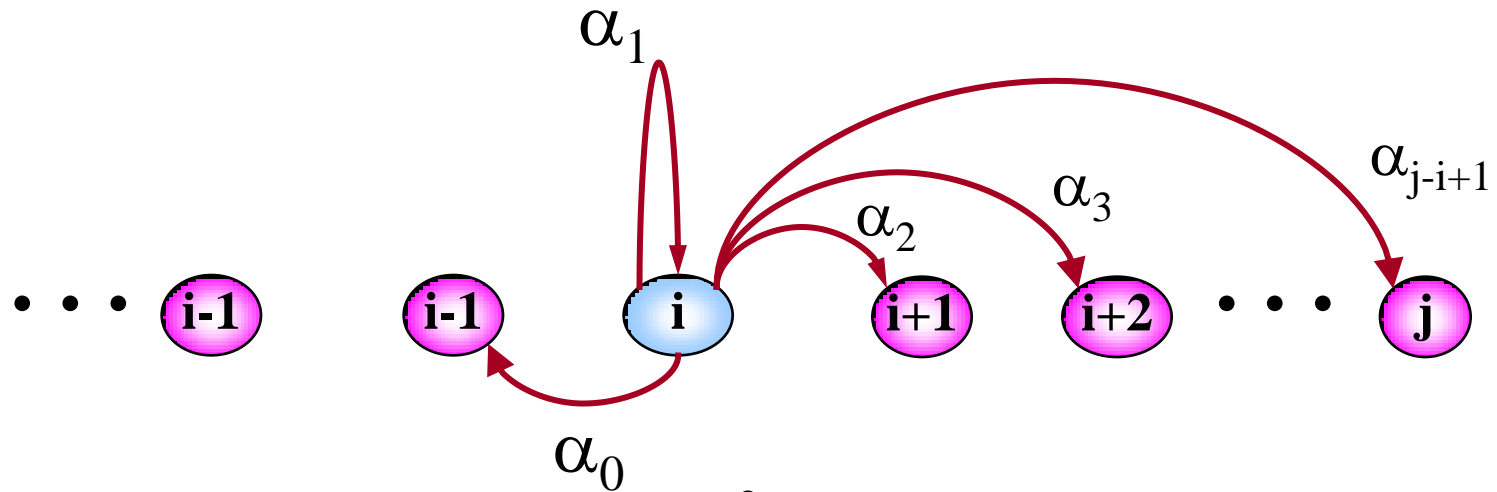
# Transition Probability

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- Want to find Distribution of  $q_n$ ,  $P[q_n = k]$ 
  - Time dependent
    - ➔ limiting distribution  $n \rightarrow \infty$
    - ➔  $P[q_n = k] = p_k = P[N(t) = k]$

# State transition diagram of M/G/1

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$$\alpha_k = P[v_{n+1} = k] = \int_0^{\infty} P[\tilde{v} = k \mid \tilde{x} = x] b(x) dx$$

$$\alpha_k = \int_0^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x} b(x) dx$$

# Transition Probability

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- Define one step transition probability
- $P = p_{ij} = P[q_{n+1} = j \mid q_n = i]$

$$P = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \dots \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \dots \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \dots \\ 0 & 0 & \alpha_0 & \alpha_1 & \dots \\ 0 & 0 & 0 & \alpha_0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

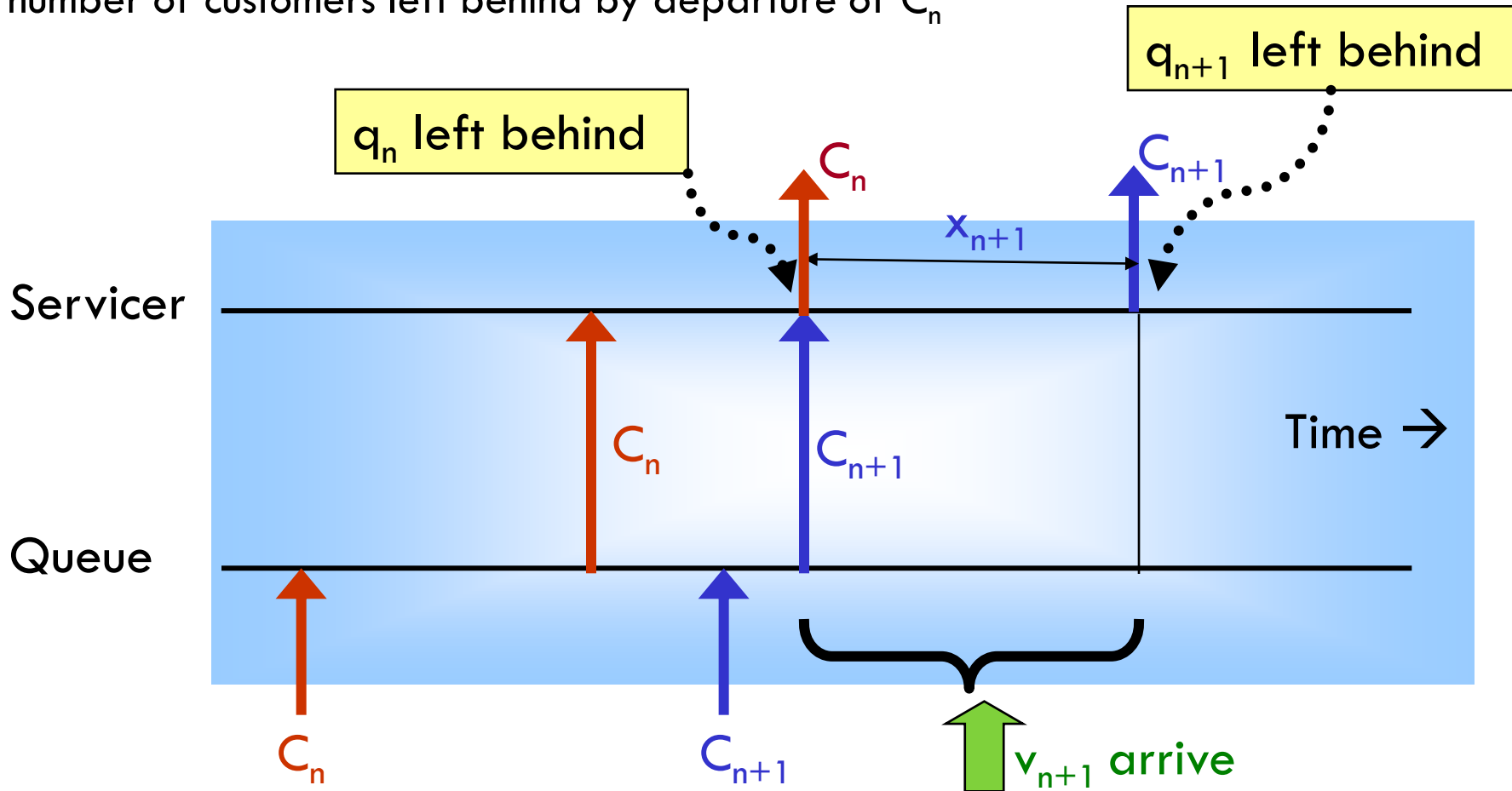
$$\alpha_k = P[v_{n+1} = k]$$

$v_n$  = number of customers arriving during the service of  $C_n$

# The Mean Queue Length ( $q_n > 0$ )

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$q_n$  = number of customers left behind by departure of  $C_n$

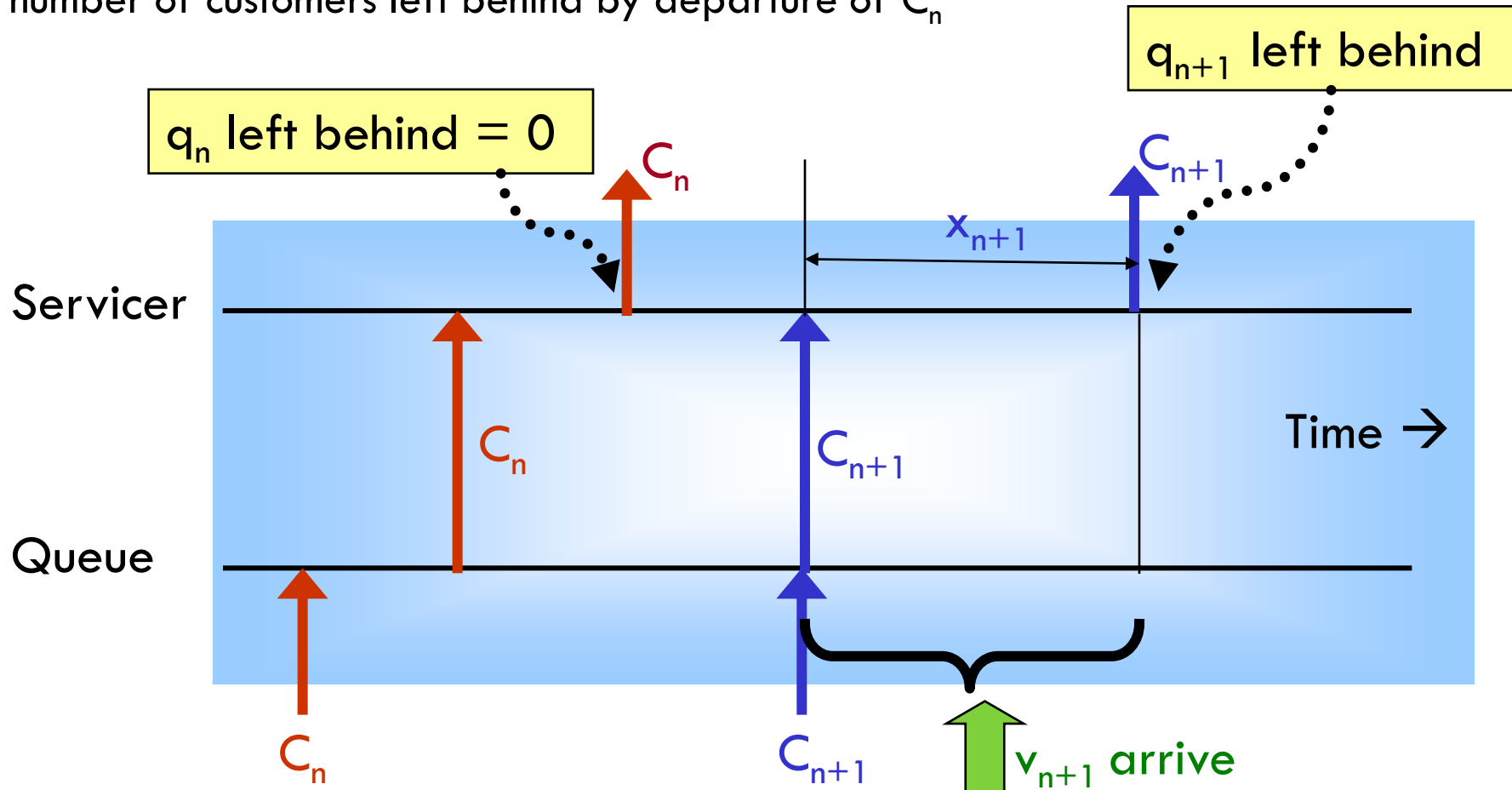




# The Mean Queue Length ( $q_n = 0$ )

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$q_n$  = number of customers left behind by departure of  $C_n$



# The Mean Queue Length, $E[q_{n+1}]$

$$E[q_{n+1}] = E[q_n] - E[\Delta_{q_n}] + E[v_{n+1}]$$

- Take the limit as  $n \rightarrow \infty$ ,  $E[\tilde{q}] = E[\tilde{q}] - E[\Delta_{\tilde{q}}] + E[\tilde{v}]$
- We get,

$$E[\Delta_{\tilde{q}}] = E[\tilde{v}] = \text{average no. of arrivals in a service time}$$

- On the other hand,

$$\begin{aligned} E[\Delta_{\tilde{q}}] &= \sum_{k=0}^{\infty} \Delta_k P[\tilde{q} = k] \\ &= \Delta_0 P[\tilde{q} = 0] + \Delta_1 P[\tilde{q} = 1] + \dots \\ &= P[\tilde{q} > 0] \end{aligned}$$

- Therefore  $E[\Delta_{\tilde{q}}] = P[\tilde{q} > 0]$ . Since we are dealing with single server, it is also equal to  $P[\text{busy system}] = \rho$ . Therefore,

$$E[\tilde{v}] = \rho$$

# The Mean Queue Length

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Since we have

$$q_{n+1} = q_n - \Delta_{q_n} + v_{n+1}$$

$$q_{n+1}^2 = q_n^2 + \Delta_{q_n}^2 + v_{n+1}^2 - 2q_n\Delta_{q_n} + 2q_nv_{n+1} - 2\Delta_{q_n}v_{n+1}$$

Note that :  $(\Delta_{q_n})^2 = \Delta_{q_n}$  and  $q_n\Delta_{q_n} = q_n$

$$\begin{aligned} \lim_{n \rightarrow \infty} E[q_{n+1}^2] &= \lim_{n \rightarrow \infty} \{E[q_n^2] + E[\Delta_{q_n}^2] + E[v_{n+1}^2] - \\ &\quad 2E[q_n] + 2E[q_nv_{n+1}] - 2E[\Delta_{q_n}v_{n+1}]\} \\ 0 &= E[\Delta_{\tilde{q}}] + E[\tilde{v}^2] - 2E[\tilde{q}] + 2E[\tilde{q}_n]E[\tilde{v}] - 2E[\Delta_{\tilde{q}}]E[\tilde{v}] \\ E[\tilde{q}] &= \rho + \frac{E[\tilde{v}^2] - E[\tilde{v}]}{2(1 - \rho)} \end{aligned}$$

Now the remaining question is how to find  $E[\tilde{v}^2]$

# The Mean Queue Length

- Let  $V(Z) = \sum_{k=0}^{\infty} P[\tilde{v} = k] Z^k$

$$\begin{aligned} V(Z) &= \sum_{k=0}^{\infty} \int_0^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x} b(x) dx Z^k \\ &= \int_0^{\infty} e^{-\lambda x} \left( \sum_{k=0}^{\infty} \frac{(\lambda x Z)^k}{k!} \right) b(x) dx \\ &= \int_0^{\infty} e^{-\lambda x} e^{\lambda x Z} b(x) dx \\ &= \int_0^{\infty} e^{-(\lambda - \lambda Z)x} b(x) dx \end{aligned}$$

- Look at  $B^*(s) = \int_0^{\infty} e^{-sx} b(x) dx$ . Therefore,

$$V(Z) = B^*(\lambda - \lambda Z)$$

# The Mean Queue Length

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- From this, we can get  $E[\tilde{v}]$ ,  $E[\tilde{v}^2]$ , ...

$$\begin{aligned}\frac{dV(Z)}{dZ} &= \frac{dB^*(\lambda - \lambda Z)}{dZ} = \frac{dB^*(\lambda - \lambda Z)}{d(\lambda - \lambda Z)} \bullet \frac{d(\lambda - \lambda Z)}{dZ} \\ &= -\lambda \frac{dB^*(y)}{dy} \\ \frac{dV(Z)}{dZ} \Big|_{Z=1} &= -\lambda \frac{dB^*(y)}{dy} \Big|_{y=0} = +\lambda \bar{x} = \rho\end{aligned}$$

# The Mean Queue Length

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$$\frac{d^2 V(Z)}{dZ^2} = \bar{v}^2 - \bar{v}, \text{ since } V(Z) = B^*(\lambda - \lambda Z)$$

$$\frac{d^2 V(Z)}{dZ^2} = \frac{d}{dZ} \left[ -\lambda \frac{dB^*(y)}{dy} \right] = -\lambda \frac{d^2 B^*(y)}{dy^2} \frac{dy}{dZ}$$

$$\frac{d^2 V(Z)}{dZ^2} \Big|_{Z=1} = \lambda^2 \frac{dB^{2*}(y)}{dy^2} \Big|_{y=0} = \lambda^2 B^{*(2)}(0)$$

$$\bar{v}^2 - \bar{v} = \lambda^2 \bar{x}^2 \quad \Rightarrow \quad \bar{v}^2 = \bar{v} + \lambda^2 \bar{x}^2$$

# The Mean Queue Length

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Go back, since

$$E[\tilde{q}] = \rho + \frac{E[\tilde{v}^2] - E[\tilde{v}]}{2(1 - \rho)}$$

$$E[\tilde{q}] = \rho + \frac{\lambda^2 \bar{x}^2}{2(1 - \rho)}$$

$$= \rho + \rho^2 \frac{(1 + C_b^2)}{2(1 - \rho)}$$

# The Mean Queue Length

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$$q_{n+1} = \begin{cases} q_n - 1 + v_{n+1} & \text{if } q_n > 0 \\ v_{n+1} & \text{if } q_n = 0 \end{cases}$$

- For  $n \rightarrow \infty$

$$N = \rho + \frac{\lambda^2 \bar{x}^2}{2(1 - \rho)}$$

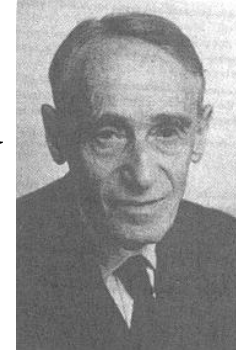
**Pollaczek-Khinchin (P-K) formula**



# Pollaczek-Khinchin (P-K) formula

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- The formula was developed by *Felix Pollaczek* (Austrian-French) and *Aleksandr Khinchin* (Russian)
- A single server situation
  - Poisson arrival distribution
  - General service time distribution
- To determine
  - Mean waiting time in queue (queuing delay)
  - Mean end-to-end time through the system



Felix Pollaczek



Aleksandr Khinchin

<http://en.wikipedia.org>

# Mean Residual Service time

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- Let  $W_i$  = Waiting time in queue of  $i^{\text{th}}$  customer
- $R_i$  = Residual service time
  - $j$  being served when  $i$  arrives,  $R_i$  = remaining time until customer  $j$ 's service time is completed
  - If no customer is in service,  $R_i = 0$
- $X_i$  = Service time of  $i^{\text{th}}$  customer
- $N_i$  = # customers found waiting in queue

# Mean Residual Service time

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$$W_i = R_i + \sum_{j=i-N_i}^{i-1} X_j$$

$$E[W_i] = E[R_i] + E\left[ \sum_{j=i-N_i}^{i-1} E[X_j | N_i] \right]$$

$$= E[R_i] + \overline{X} E[N_i]$$

$$W = R + \frac{1}{\mu} N_Q$$

# Mean Residual Service time

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$$W = R + \frac{1}{\mu} N_Q$$

- By Little's Theorem

$$N_Q = \lambda W$$

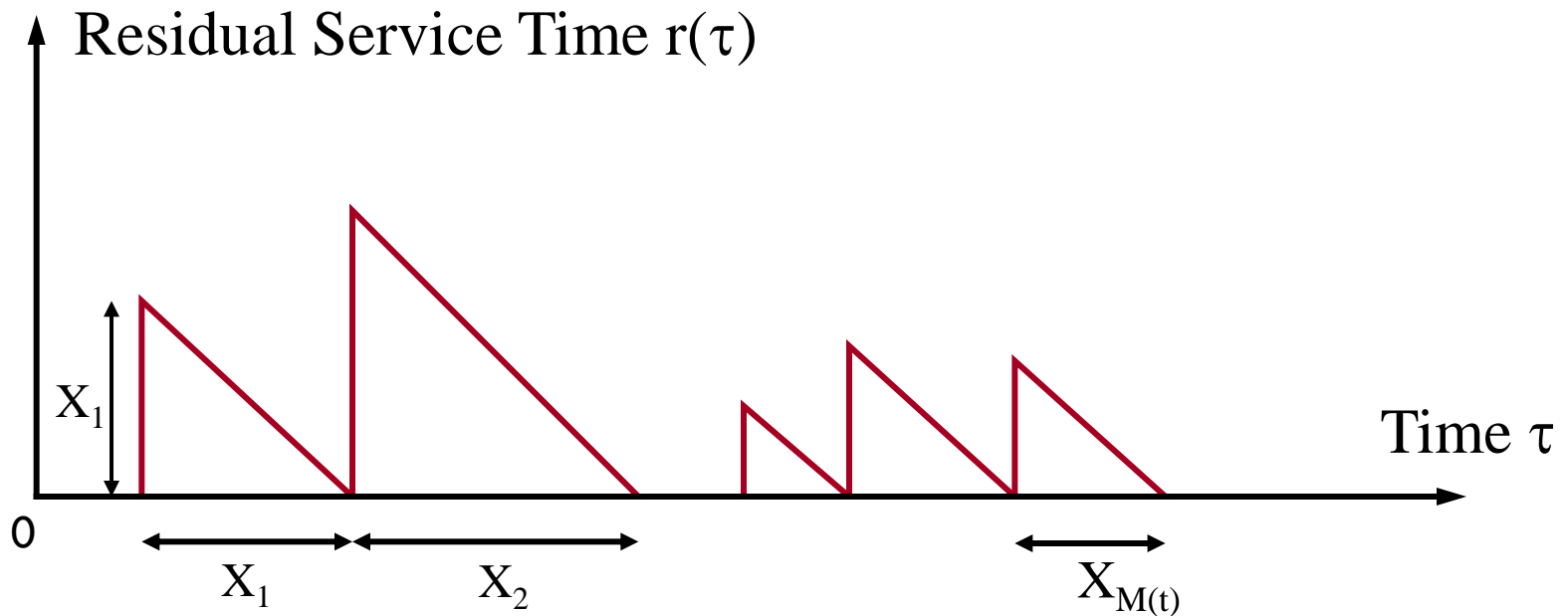
$$\rightarrow W = R + \rho W$$

$$\rightarrow W = \frac{R}{1 - \rho}$$

# Mean Residual Service time

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- Calculate R



# Mean Residual Service time

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$$\begin{aligned} R &= \frac{1}{t} \int_0^t r(\tau) d\tau = \frac{1}{t} \sum_{i=1}^{M(t)} \frac{1}{2} X_i^2 \\ &= \frac{1}{2} \frac{M(t)}{t} \frac{\sum_{i=1}^{M(t)} X_i^2}{M(t)} \end{aligned}$$

$$R = \frac{1}{2} \lambda \overline{X^2} \quad t \rightarrow \infty$$

# Mean Residual Service time

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$$W = \frac{R}{1 - \rho} \quad \text{AND} \quad R = \frac{1}{2} \lambda \bar{X}^2 \quad \Rightarrow \quad W = \frac{\lambda \bar{X}^2}{2(1 - \rho)}$$

$$T = \bar{X} + W = \bar{X} + \frac{\lambda \bar{X}^2}{2(1 - \rho)}$$

$$N = \lambda T \quad \Rightarrow \quad N = \lambda \bar{X} + \frac{\lambda^2 \bar{X}^2}{2(1 - \rho)}$$

$$\Rightarrow \quad N = \rho + \frac{\lambda^2 \bar{X}^2}{2(1 - \rho)}$$

# P-K Formula: for M/M/1

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- Service = exponential distribution

$$E[X] = \frac{1}{\mu}, \quad E[X^2] = \frac{2}{\mu^2}$$

$$E[W] = \frac{\lambda E[X^2]}{2(1-\rho)} = \frac{\rho}{\mu(1-\rho)}, \quad E[Q] = \frac{\lambda^2 E[X^2]}{2(1-\rho)} = \frac{\rho^2}{(1-\rho)}$$

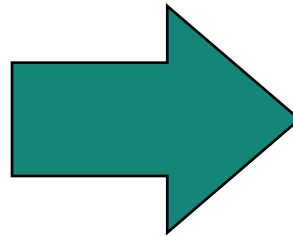
$$E[T] = \frac{1}{\mu} + \frac{\lambda E[X^2]}{2(1-\rho)} = \frac{1}{\mu} + \frac{\rho}{\mu(1-\rho)} = \frac{1}{\mu - \lambda}, \quad E[N] = \lambda E[T] = \frac{\lambda}{\mu - \lambda}$$



# P-K Formula: for M/M/1

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$$N = \rho + \frac{\lambda^2 \bar{X}^2}{2(1 - \rho)}$$



$$N = \frac{\rho}{(1 - \rho)}$$

# P-K Formula: for M/D/1

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- Service = Deterministic service times all equal to  $1/\mu$

$$E[X] = \frac{1}{\mu}, \quad E[X^2] = \frac{1}{\mu^2}$$

$$E[W] = \frac{\lambda E[X^2]}{2(1-\rho)} = \frac{\rho}{2\mu(1-\rho)}, \quad E[Q] = \frac{\lambda^2 E[X^2]}{2(1-\rho)} = \frac{\rho^2}{2(1-\rho)}$$

$$E[T] = \frac{1}{\mu} + \frac{\lambda E[X^2]}{2(1-\rho)} = \frac{1}{\mu} + \frac{\rho}{2\mu(1-\rho)} = \frac{2-\rho}{2\mu(1-\rho)}, \quad E[N] = \lambda E[T] = \frac{\rho(2-\rho)}{2(1-\rho)}$$