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Probability Theory and Random Processes

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Lecture #14
Stochastic Process – I

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Outline

- Stochastic Process
- Counting Process
- Poisson Process
- Brownian Motion Process
- Autocovariance and Autocorrelation
- Random Sequence
- Stationary Process
- Wide-sense Stationary Process

Definition

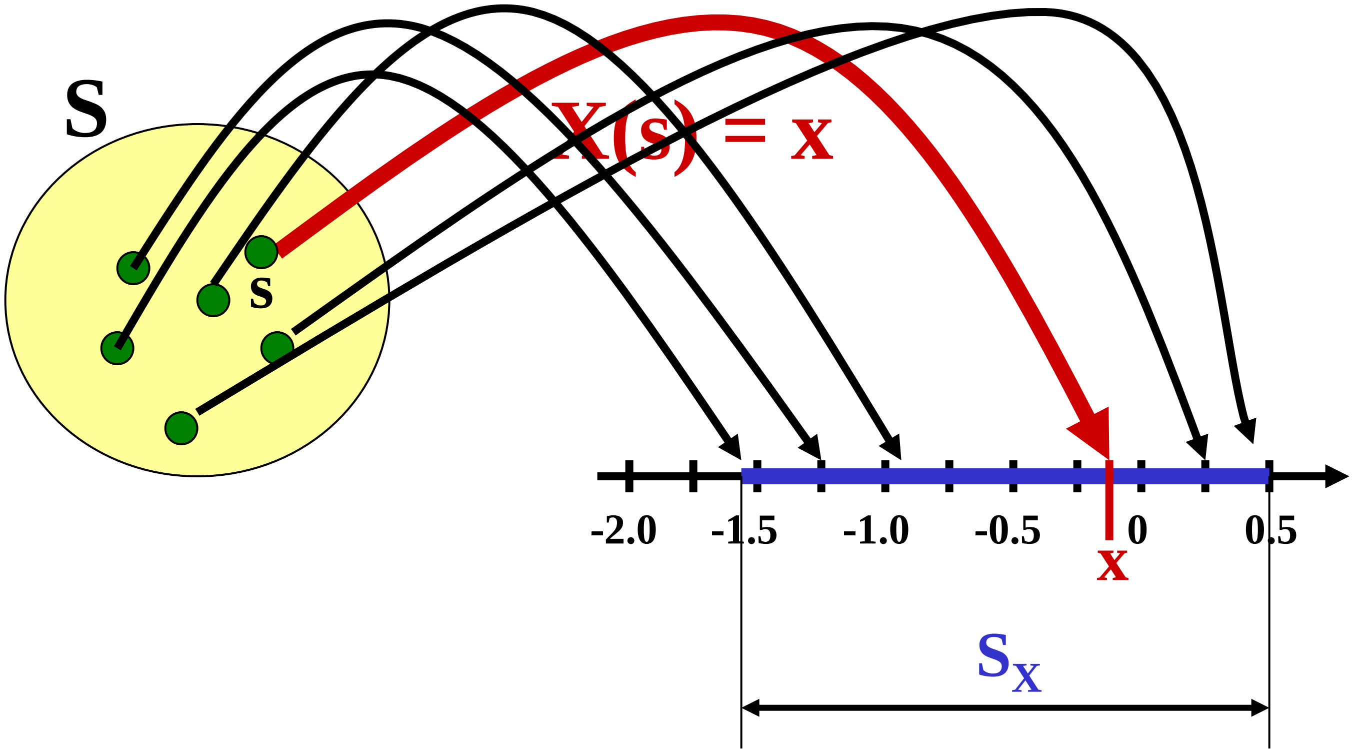
Stochastic Process

Random — **Function of Time: *f(t)***

Random Process

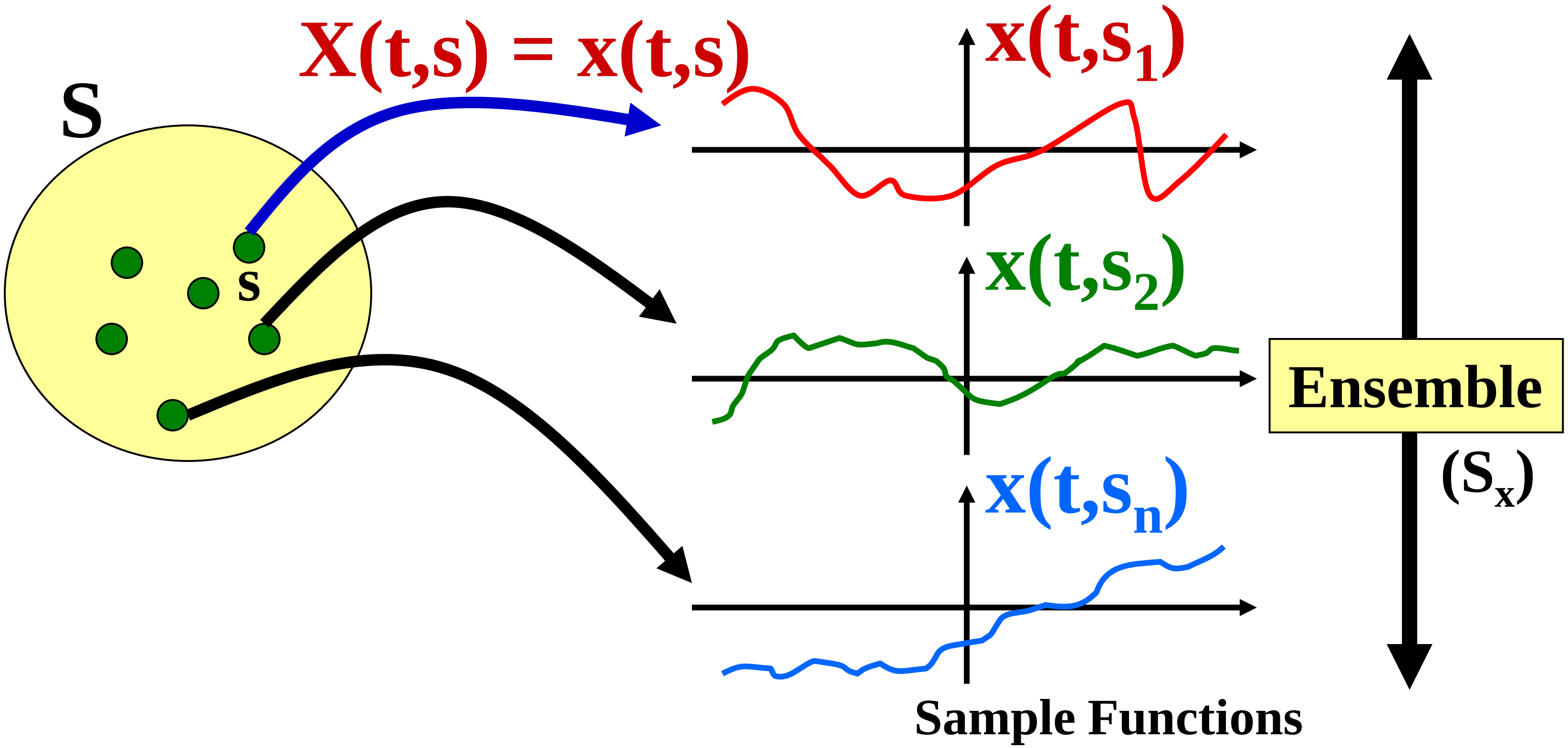
Random Variable

X is a function that maps each outcome, s, in S to a real number X(s), x



Random Process

$X(t)$ is a function that maps each outcome, s, in S to a time function $x(t,s)$

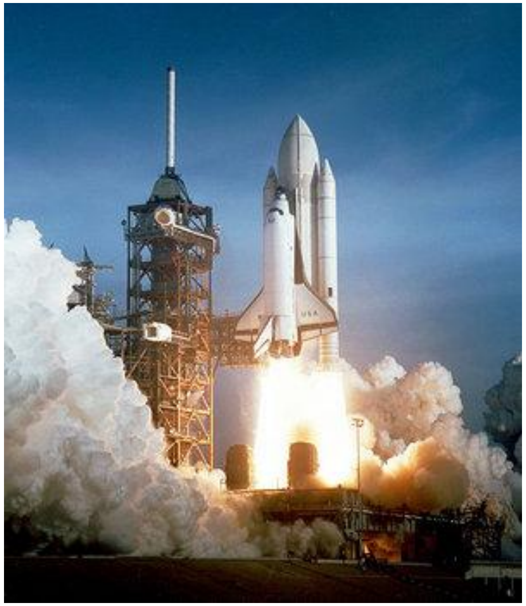


Example 1

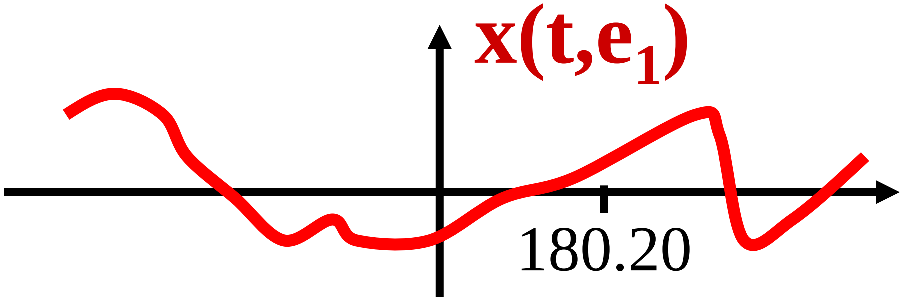
- Taking temperature at the surface of a space shuttle
- Starting at launch time t = 0
- X(t) = temp in degree Celsius on the surface
- Each launch s, record x(t,s)

http://spaceflight.nasa.gov/history/shuttle-mir/spacecraft/s-orb-sscomponents-main.htm

Example 1



http://sites.indianriverschools.org/srhs/teachers/nyberg/main.htm

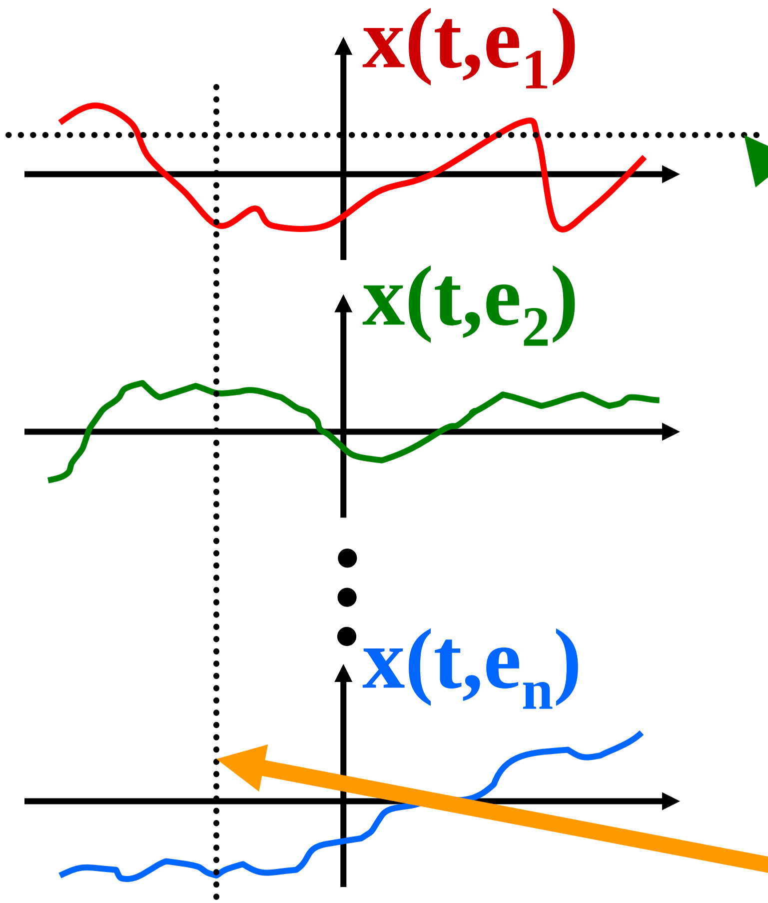


At time **t = 180.20 sec**

$x(180.20, e_1) = 450$ °C

e_1 = 1st launch measuring

Example 1



Sample functions

Average temp at $\mathbf{e_1}$
for completed mission is
4700 C ➔ **E[**x(t, e_1)**]**

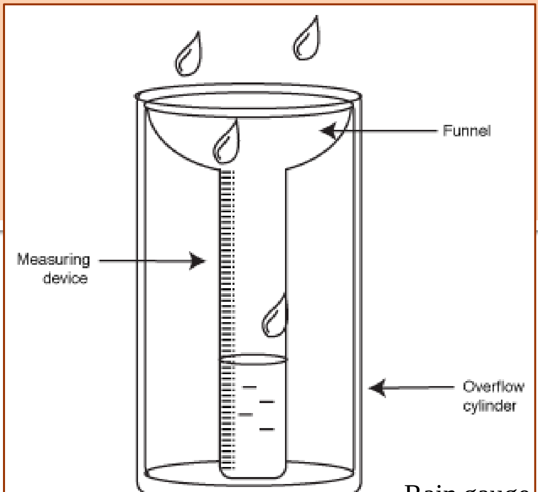
Time Average

Average temp of engine
@ t = 180.20 sec is
970 C ➔ **E[**X(180.20)**]**

Ensemble Average

Example 2

- Measure the rain fall in a day @Bangkhen every day
- Let F(t) = random process
- f(t,y) = a sample function for measuring at day "t" of the year "y"

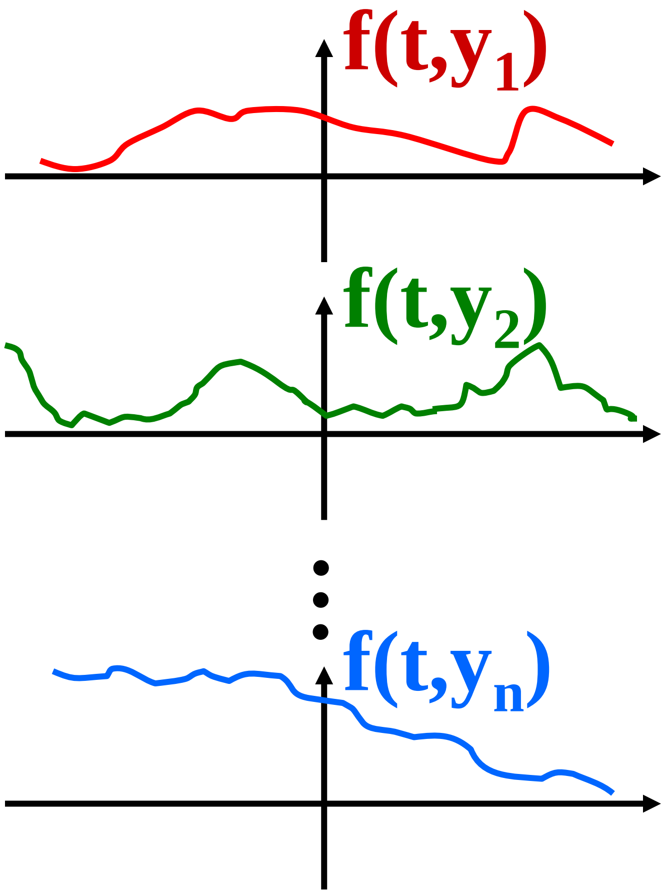


Rain gauge

http://www.infoplease.com/cig/weather/measuring-rain.html

http://snap.newsadvance.com/lna/snap/media_view/94

Example 2



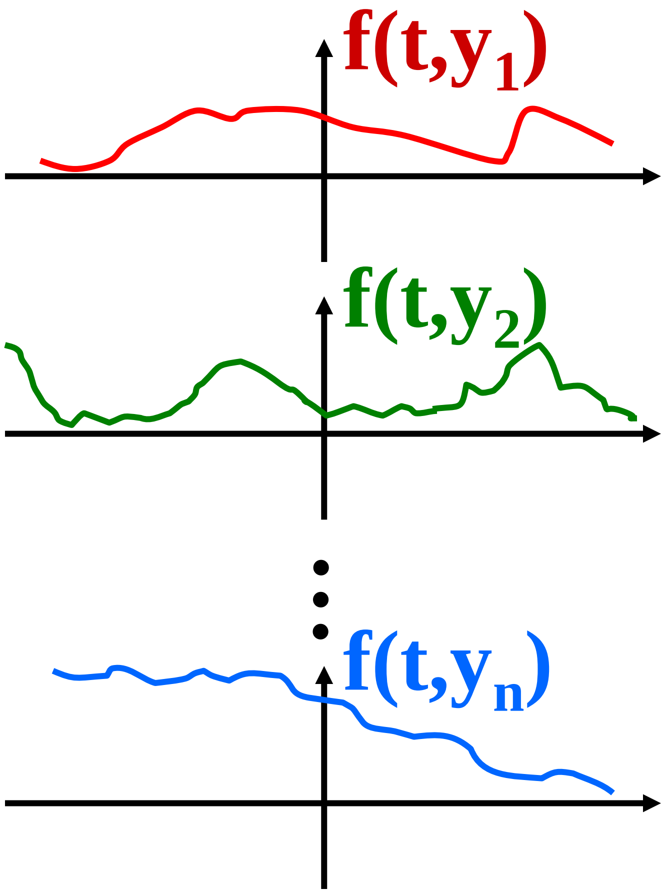
A sample function of rain fall in y_1 = year 2009 ($1 \leq t \leq 365$)

A sample function of rain fall in y_2 = year 2010 ($1 \leq t \leq 365$)

A sample function of rain fall in y_n = year 2012 ($1 \leq t \leq 365$)

Sample functions

Example 2



Sample functions

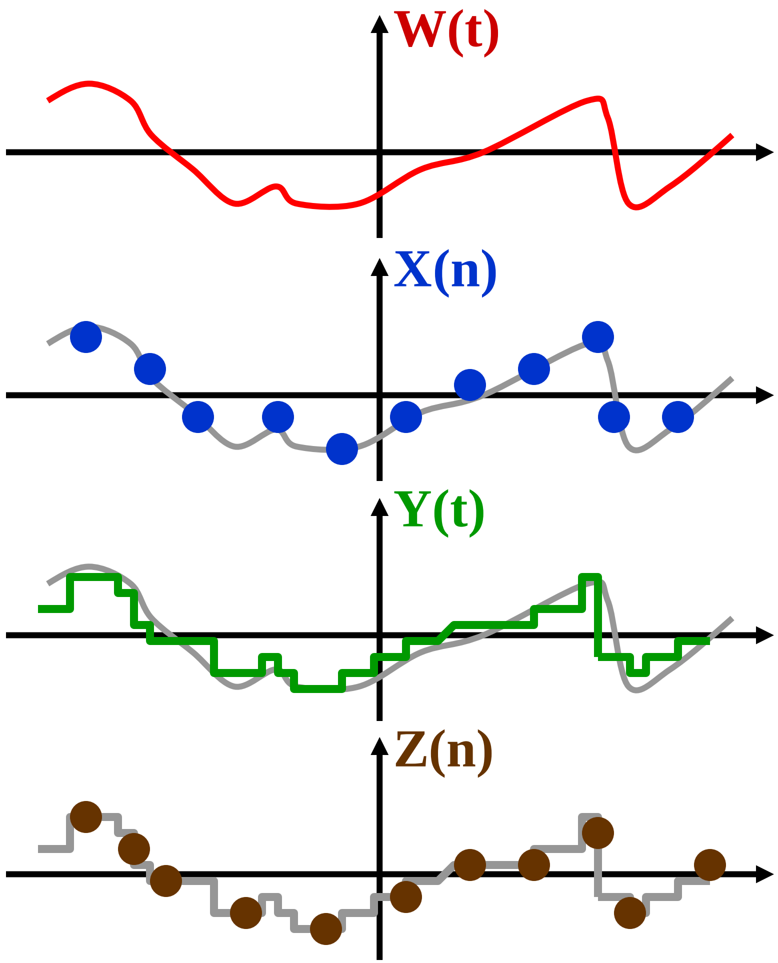
Therefore,

we might want to know

- The average rain fall in year 2010
- The average rain fall for May 22

2012 Conference
Pool side party OK ?

Types of Stochastic Process



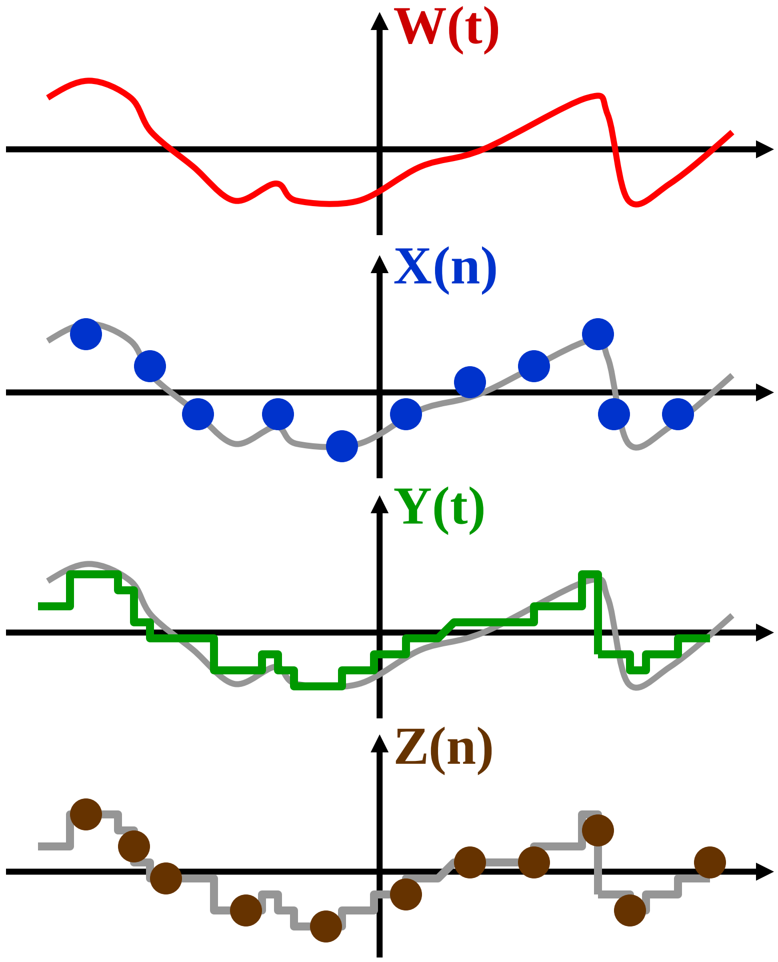
Continuous Time,
Continuous value Process

Discrete Time,
Continuous value Process

Continuous Time,
Discrete value Process

Discrete Time,
Discrete value Process

Stochastic Process Examples



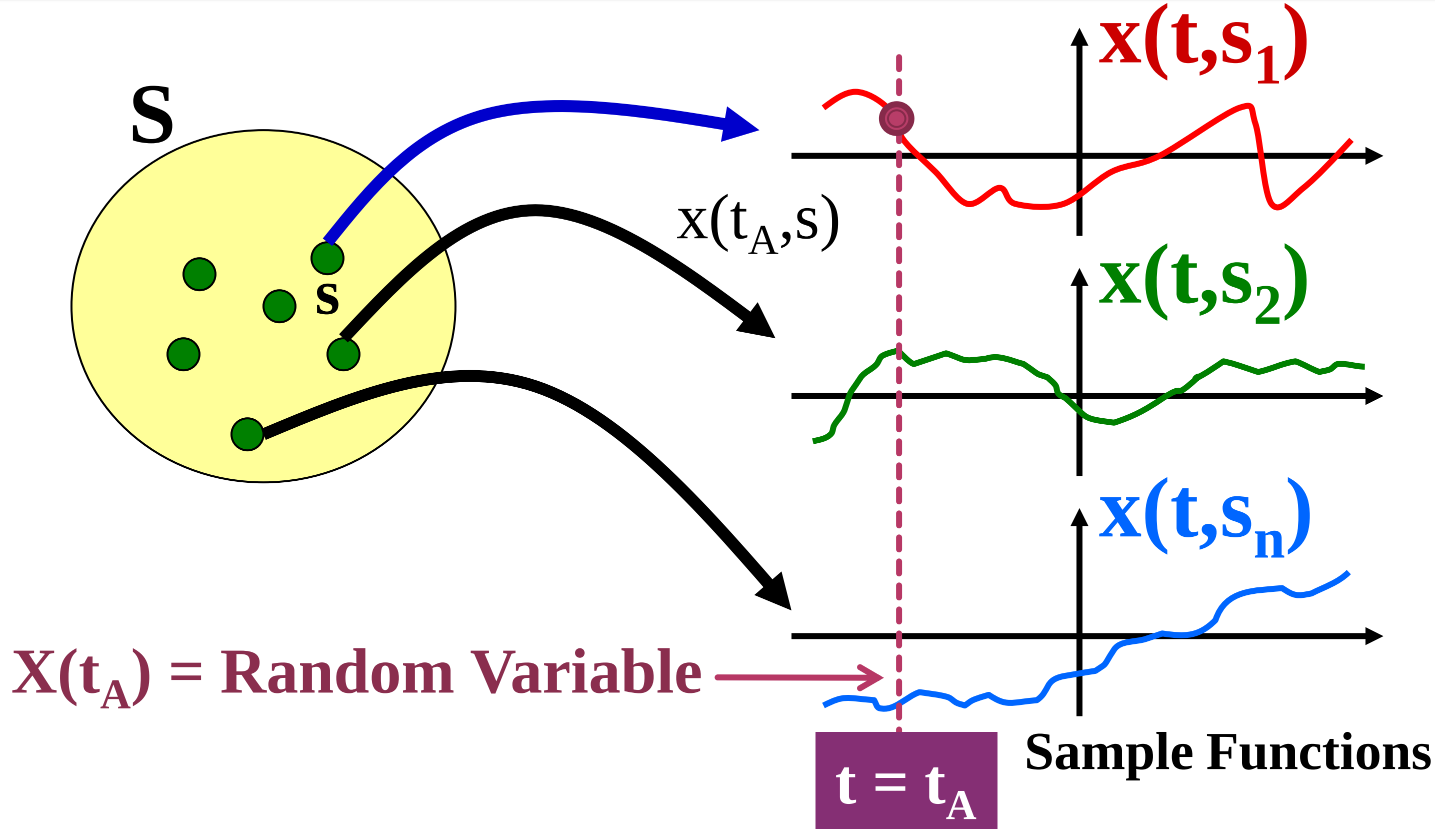
Record temperature as a continuous time

Record temperature every T seconds

Record round(temperature) as a continuous time

Record round(temperature) every T seconds

Random Variables from Random Process



So, we can find the PDF ➔ $f_{X(t_A)}(x)$

IID Random Sequence

- Independent, Identically Distributed Random Sequence
- Independent trials of an experiment at a constant rate
- Discrete / Continuous

Theorem:

$$P_{X_{n_1} \cdots X_{n_k}}(x_1, \ldots, x_k) = P_X(x_1) \cdots P_X(x_k) = \prod_{i=1}^{k} P_X(x_i)$$

Counting Process

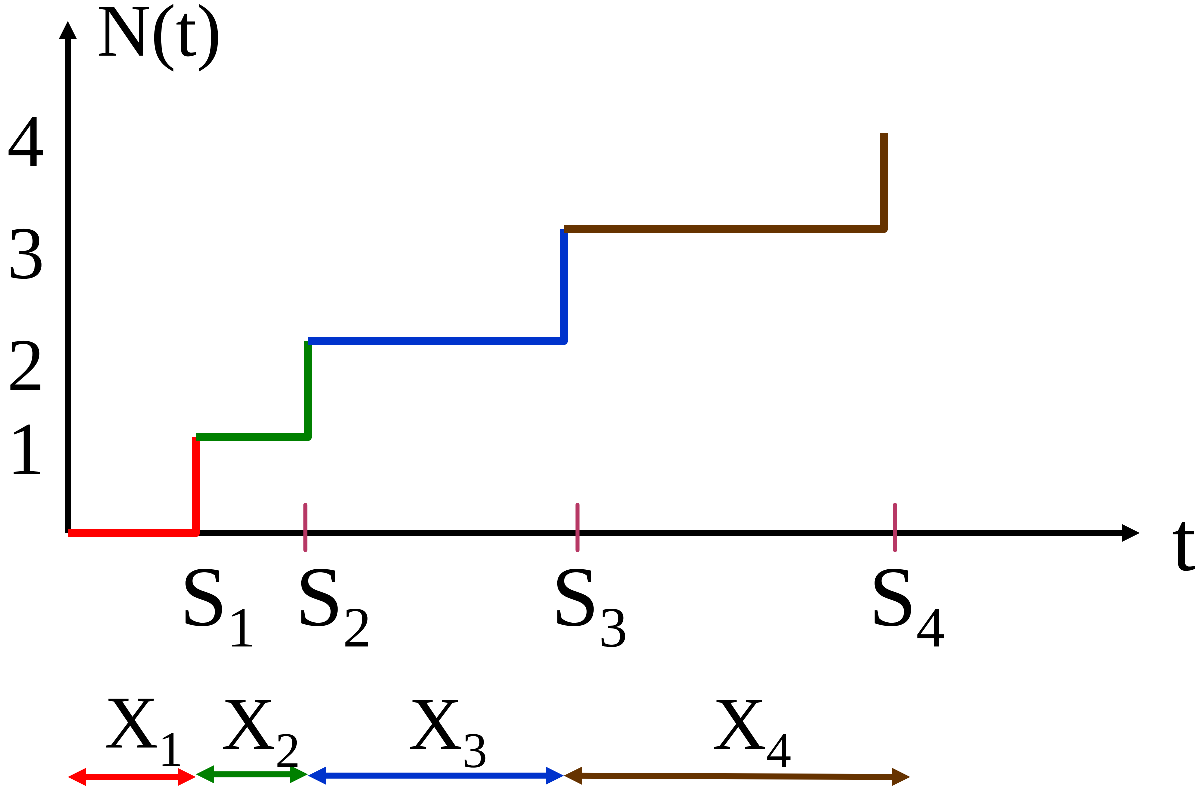
Counting Process

Definition: A Stochastic Process is a Counting Process N(t) if

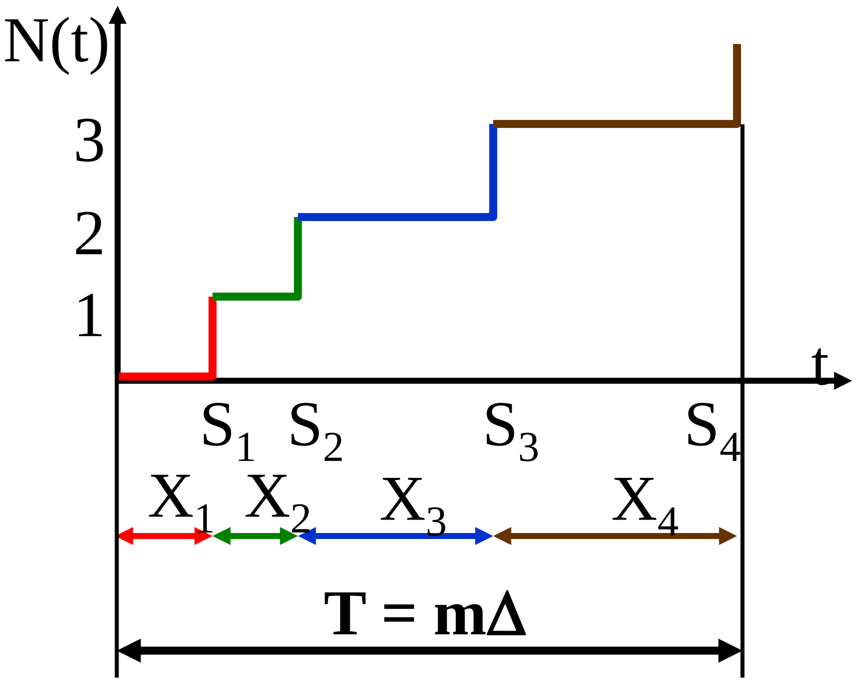
- n(t,s) = 0 for $t < 0$
- n(t,s) = integer valued and non-decreasing

Counting Process

of customers **arrive** at (0,t]



Counting Process



- For a small step **Δ**, only one arrival ($X_n = 1$)
- Success Prob. of $X_n = \lambda\, \Delta = \lambda\, T/m$

Binomial PMF

$$P_{N_m}(n) = \begin{cases} \binom{m}{n} (\lambda T/m)^n (1 - \lambda T/m)^{m-n} & n = 0,1,2,.. \\ 0 & \text{Otherwise} \end{cases}$$

Counting Process

Binomial Process

$$P_{Nm}(n) = \begin{cases} \binom{m}{n} (\lambda T/m)^n (1-\lambda T/m)^{m-n} & n = 0,1,2,.. \\ 0 & \text{Otherwise} \end{cases}$$

$m \rightarrow \infty$ or $\Delta \rightarrow 0$

Poisson Process

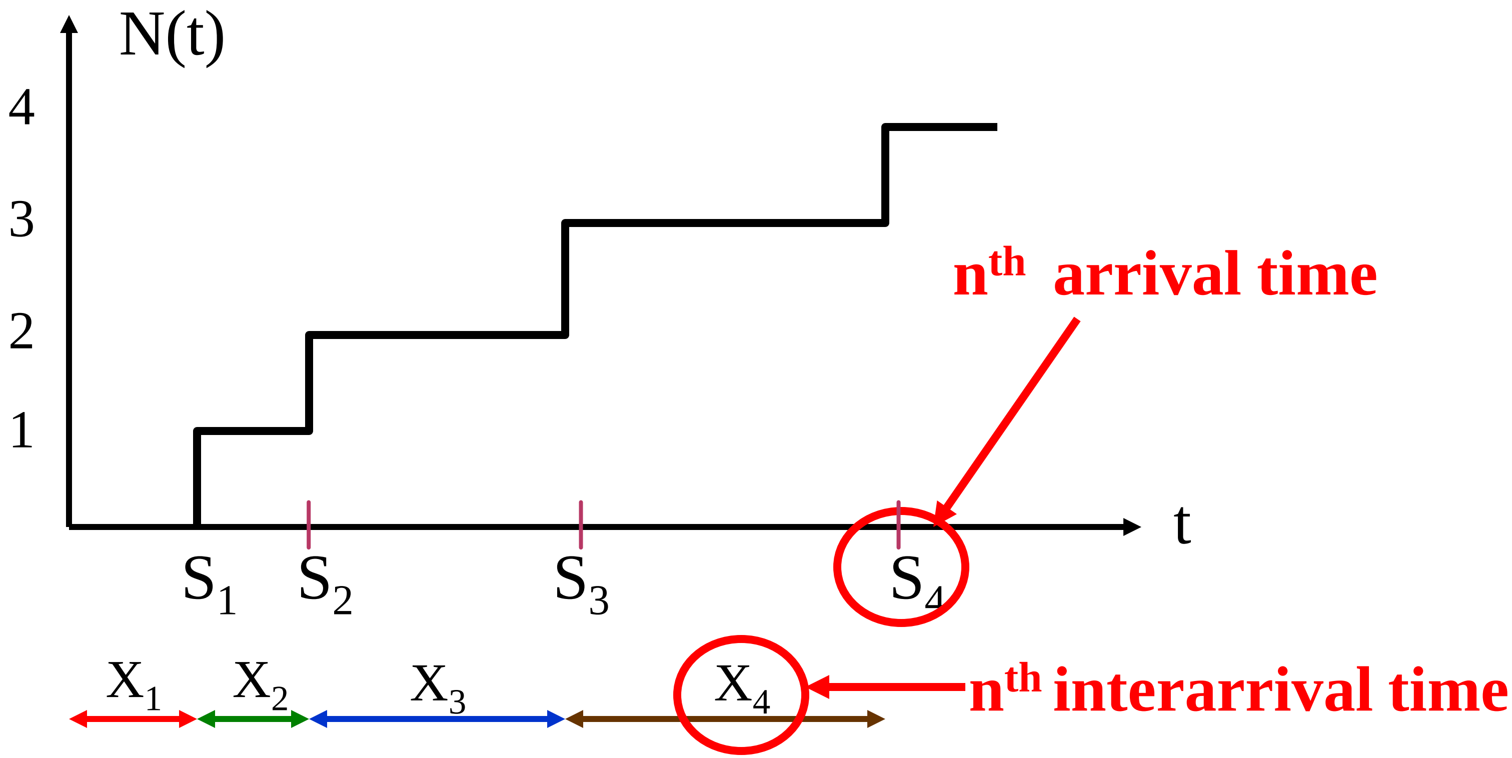
$$P_{N(t)}(n) = \begin{cases} \dfrac{(\lambda T)^n e^{-\lambda T}}{n!} & n = 0,1,2,.. \\ 0 & \text{Otherwise} \end{cases}$$

Poisson Process

Poisson Process

- Poisson Process is
  a Counting Process that the # of Arrival during any interval is Poisson RV
- An arrival during any instant is **independent** of the past history of the process → Memoryless

Poisson Process



- X_n is called **Interarrival Time**

Poisson Process

Definition:

A Counting Process N(t) is a Poisson Process N(t) if

- # of arrivals in $(t_0,t_1]$, $N(t_1) - N(t_0)$, is a Poisson RV with expected value $\lambda(t_1-t_0)$
- # of arrivals in each interval are independent random variable

Poisson Process

- Process rate (λ) = E[N(t)] / t
- M = $N(t_1) - N(t_0)$ = Poisson RV

$$P_M(m) = \begin{cases} \dfrac{[\lambda(t_1-t_0)]^m}{m!} e^{-\lambda(t_1-t_0)} & m = 0,1,2,.. \\ 0 & \text{Otherwise} \end{cases}$$

Joint PMF

Theorem: Poisson Process N(t) of rate λ,

Joint PMF of $N(t_1),\ldots,N(t_k)$, $t_1 < \ldots < t_k$

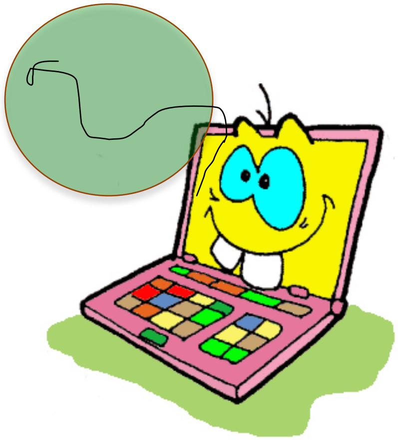
$$P_{N(t_1),\ldots,N(t_2)}(n_1,\ldots,n_k)$$

$$= \begin{cases} \dfrac{\alpha_1^{n_1} e^{-\alpha_1}}{n_1!} \dfrac{\alpha_2^{(n_2-n_1)} e^{-\alpha_2}}{(n_2 - n_1)!} \cdots \dfrac{\alpha_k^{(n_k-n_{k-1})} e^{-\alpha_k}}{(n_k - n_{k-1})!} & 0 \le n_1 \le \ldots \le n_k \\ 0 & \text{Otherwise} \end{cases}$$

$$\alpha_i = \lambda(t_i - t_{i-1})$$

Example

- A mobile station transmits data packet as **Poisson** process with rate 12 packets/sec
  - Find # of packets transmitted in the k^{th} hour
  - Find Joint PMF of # of packets transmitted in the k^{th} hour and z^{th} hour



Example

- Let N_k = # of packets transmitted in k^{th} hour
- # packets in each hour is IID

$$P_{N_i}(n) = \begin{cases} \dfrac{[12(3600-0)]^n \; e^{-12\,(3600-0)}}{n!} & n = 0,1,2,.. \\ 0 & \text{Otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{[43200]^n \; e^{-43200}}{n!} & n = 0,1,2,.. \\ 0 & \text{Otherwise} \end{cases}$$

Example

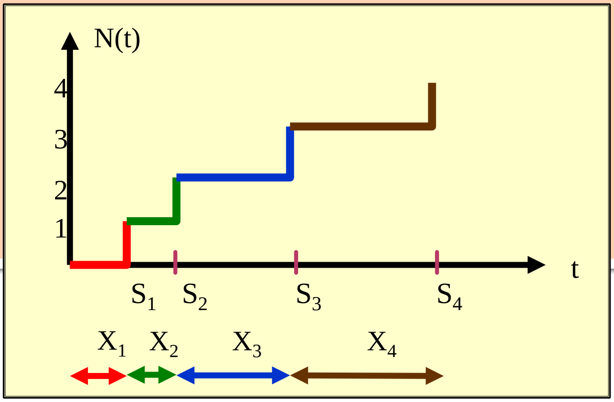
- Joint PMF of # of packets transmitted in the k^{th} hour and z^{th} hour

$$P_{N_k,N_z}(n_k,n_z) = \begin{cases} \dfrac{\alpha_k^{n_k} e^{-\alpha_k}}{n_k!} \dfrac{\alpha_z^{n_z} e^{-\alpha_z}}{n_z!} & n_k = 0,1,\ldots \\ & n_z = 0,1,\ldots \\ 0 & \text{Otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{\alpha^{(n_k+n_z)}}{n_k!\,n_z!} e^{-2\alpha} & n_k = 0,1,\ldots \\ & n_z = 0,1,\ldots \\ 0 & \text{Otherwise} \end{cases}$$

$$\alpha = \alpha_k = \alpha_Z = \lambda T = [12(3600-0)] = 43200$$

Interarrival Time

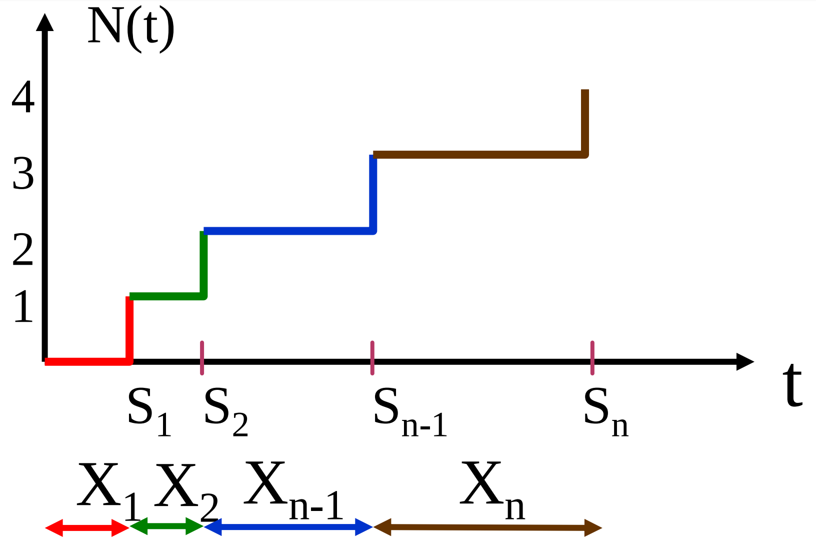


Theorem:

Poisson Process of rate λ, the interarrival times X_1, X_2, … are an iid random sequence with **Exponential PDF**

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

Proof of Interarrival Time



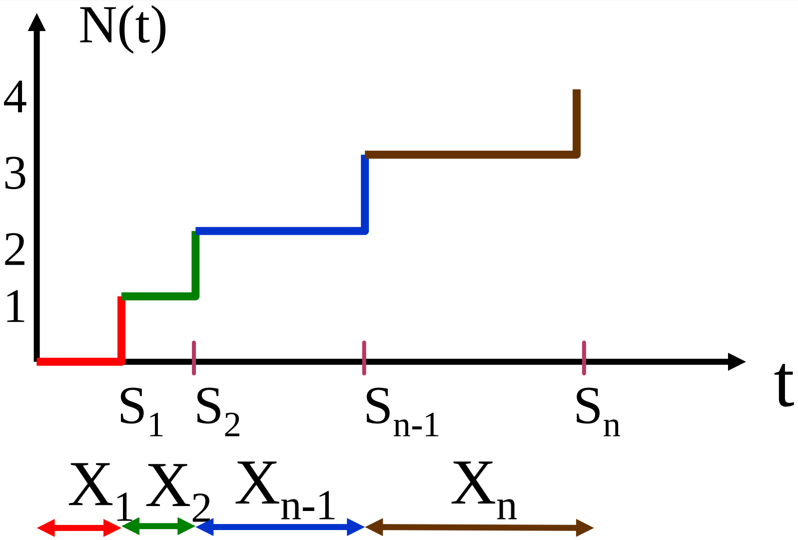
- Given
  - $X_1 = x_1, X_2 = x_2, \ldots, X_{n-1} = x_{n-1}$
- For $x > 0$, **$X_n > x$**
  - iff **No Arrival** in $(t_{n-1}, t_{n-1}+x]$
  - # arrival in $(t_{n-1}, t_{n-1}+x]$ is independent past history of $X_1, X_2, \ldots, X_{n-1}$

$$P[X_n > x \mid X_1 = x_1, X_2 = x_2, \ldots, X_{n-1} = x_{n-1}] = P[N(t_{n-1}+x) - N(t_{n-1}) = 0]$$

$$= \frac{[\lambda x]^0 \; e^{-\lambda x}}{0!} = e^{-\lambda x}$$

X_n is independent past history of $X_1, X_2, \ldots, X_{n-1}$

Proof of Interarrival Time

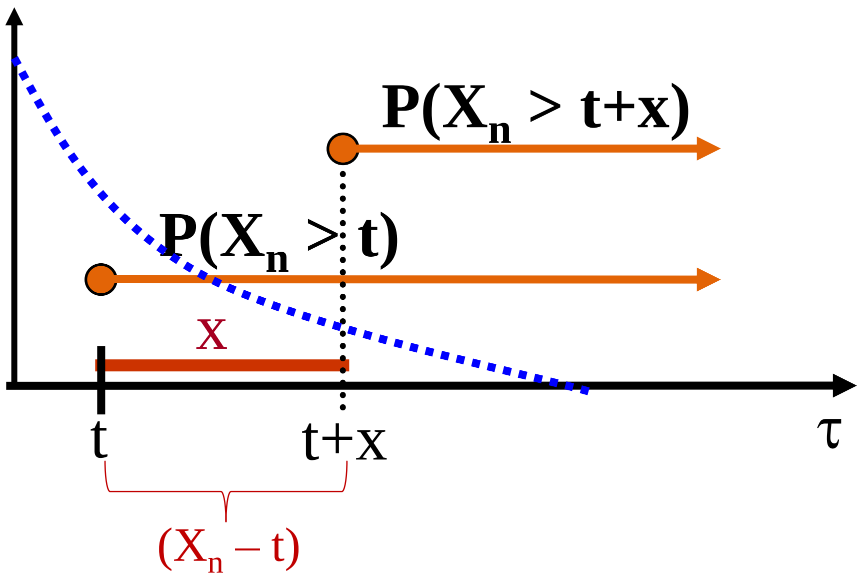


$$F_{X_n}(x) = 1 - P[X_n > x]$$

$$= \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

$$f_{X_n}(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

Properties of Poisson Process



- Memoryless Property
- Exponential Interarrival time
  - $P[X_n > x] = e^{-\lambda x}$

$$P([\, X_n > t + x \mid X_n > t \,])$$

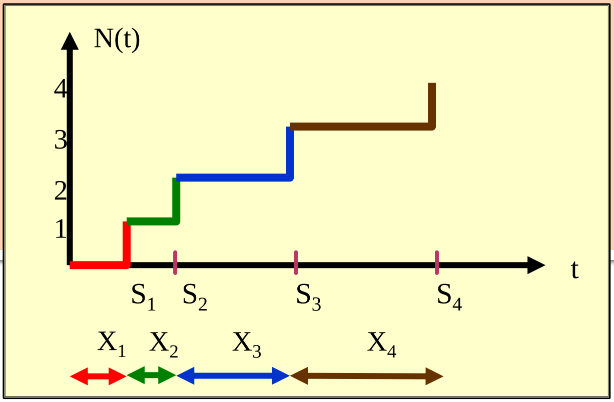
$$= \frac{P([\, X_n > t+x,\, X_n > t \,])}{P[X_n > t\,]}$$

$$= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x}$$

If the arrival has not occur at time t,
the additional time until the arrival $(X_n - t)$
has the same exponential distribution as X_n

No matter how long to wait for the arrival
→ the remaining time until arrival is still an exponential with rate λ

Interarrival Time



Theorem:

A Counting Process with ***independent exponential interarrival time*** $X_1, X_2, \ldots$ with $E[X_i] = 1/\lambda$ is a ***Poisson Process*** of rate λ

Two independent Poisson processes

Theorem:

- Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes of rate λ_1 and λ_2
- The counting process

$$N(t) = N_1(t) + N_2(t)$$

is a Poisson process of rate $(\lambda_1 + \lambda_2)$