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Probability Theory and Random Processes

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Lecture #12

Multiple Random Variables

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Joint CDF

- Pairs of Random Variables

- Discrete:

Joint PMF $P_{X,Y}(x,y) = P[X=x, Y=y]$

- Continuous:

$$P_{X,Y}(x,y) = 0 \quad (P_X(x) = 0, P_Y(y) = 0)$$

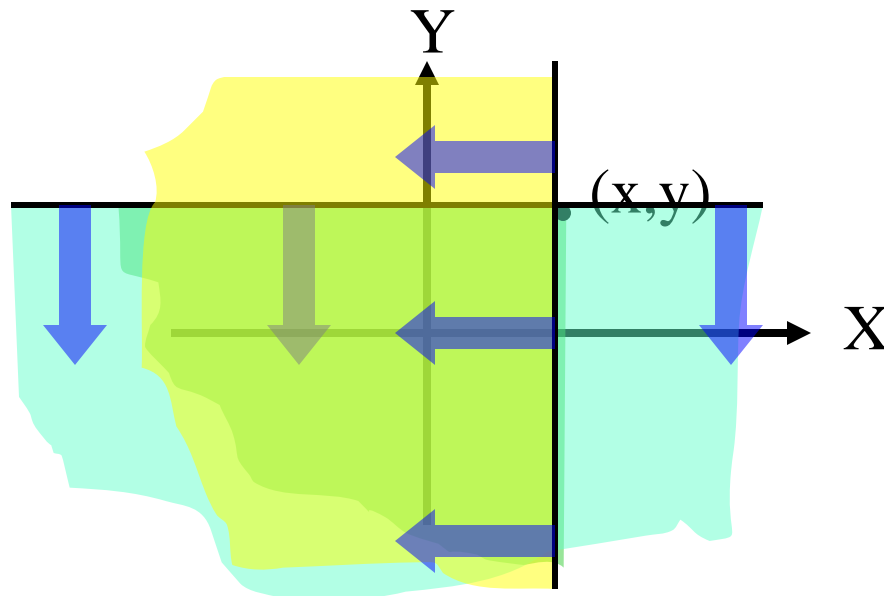
For 1 RV \rightarrow interval on real axis

For 2 RVs \rightarrow area in a plane

Joint CDF

Definition: Joint CDF of X and Y

$$F_{X,Y}(x,y) = P[X \leq x, Y \leq y]$$



Interesting Properties

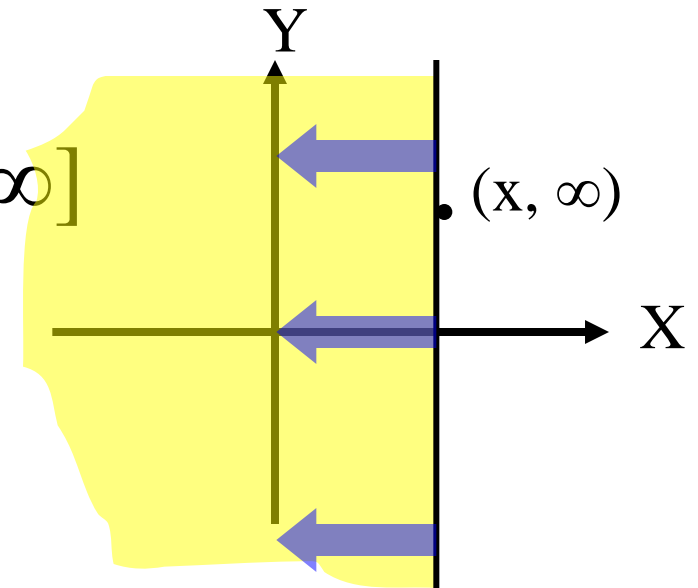
- For Event $\{X \leq x\}$

$$F_X(x) = P[X \leq x]$$

$$= P[X \leq x, Y \leq \infty]$$

$$= \lim_{y \rightarrow \infty} F_{X,Y}(x, y)$$

$$= F_{X,Y}(x, \infty)$$



Joint CDF

Theorem :

(a) $0 \leq F_{X,Y}(x,y) \leq 1$

(b) $F_X(x) = F_{X,Y}(x, \infty)$

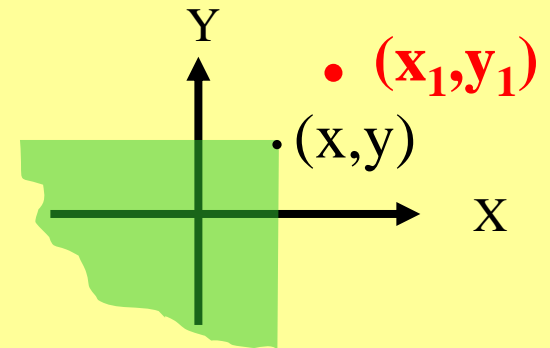
(c) $F_Y(y) = F_{X,Y}(\infty, y)$

(d) $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$

(e) If $x_1 \geq x$ and $y_1 \geq y$

then $F_{X,Y}(x_1, y_1) > F_{X,Y}(x, y)$

(f) $F_{X,Y}(\infty, \infty) = 1$



Joint PDF

Definition: Joint PDF of X and Y is satisfied

$$F_{X,Y}(x,y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(u,v) \, dv \, du$$

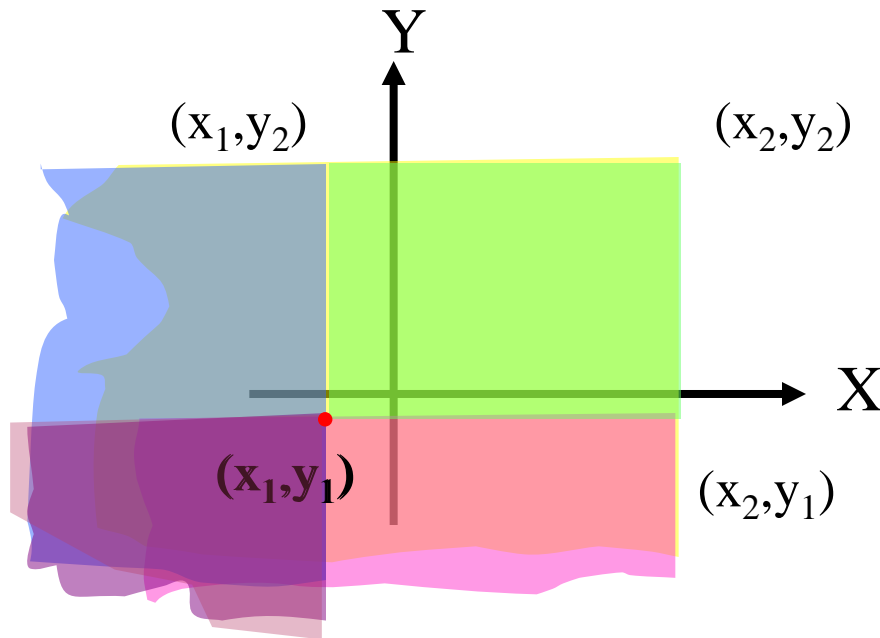
Theorem:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

Joint CDF

Theorem:

$$\begin{aligned} &P[x_1 < X \leq x_2, y_1 < Y \leq y_2] \\ &= F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1) \end{aligned}$$



Joint PDF

Theorem:

$$(a) f_{X,Y}(x,y) \geq 0 \text{ for all } (x,y)$$

$$(b) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1$$

Theorem:

$$P[A] = \iint_A f_{X,Y}(x,y) dx dy$$

Example

$$f_{X,Y}(x,y) = \begin{cases} c & 0 \leq x \leq 3, 0 \leq y \leq 5 \\ 0 & \text{Otherwise} \end{cases}$$

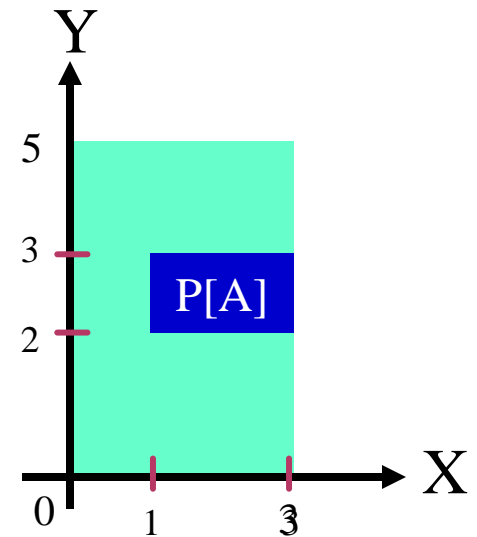
Find constant c

$$\int_0^3 \int_0^5 c \, dy \, dx = 15c = 1$$

$$\rightarrow c = 1/15$$

Find $P[A] = P[1 \leq x \leq 3, 2 \leq y \leq 3]$

$$P[A] = \int_1^3 \int_2^3 1/15 \, dv \, du = 2/15$$



Marginal PDF

Theorem:

$$f_{X,}(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$$

$$f_{Y,}(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx$$

Example

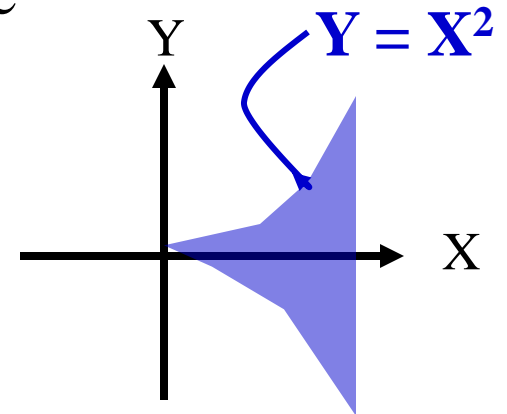
$$f_{X,Y}(x,y) = \begin{cases} cx & 0 \leq x \leq 1, |y| < x^2 \\ 0 & \text{Otherwise} \end{cases}$$

Find constant c

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} cx \, dx \, dy = \int_0^1 \left(\int_{-x^2}^{x^2} cx \, dy \right) dx$$

$$= \int_0^1 cx (2x^2) \, dx = \frac{cx^4}{2} \Big|_0^1$$

$$= \frac{c}{2} = 1 \quad \rightarrow \quad c = 2$$



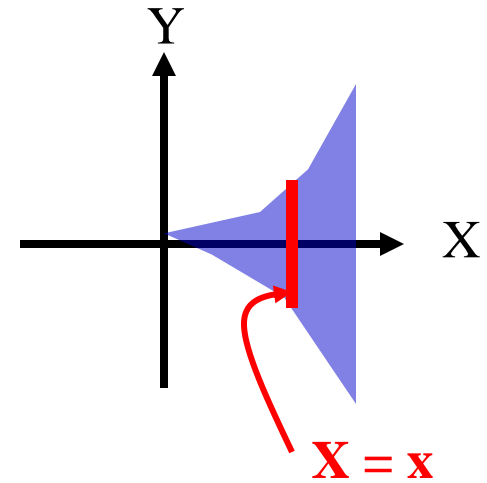
Example

Find the marginal PDF $f_X(x)$ and $f_Y(y)$

Fixed x ($X = x$) then integrate all y

$$f_X(x) = \int_{-x^2}^{x^2} 2x \, dy = 4x^3$$

$$f_X(x) = \begin{cases} 4x^3 & 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

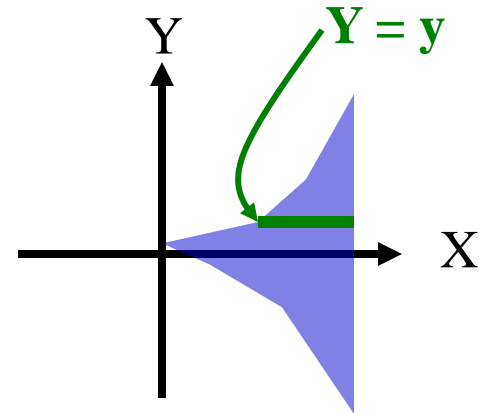


Example

Fixed y ($Y = y$) then integrate all x

$$f_{Y,}(y) = \int_{\sqrt{|y|}}^1 2x \, dx = 1 - |y|$$

$$f_Y(y) = \begin{cases} 1 - |y| & -1 \leq y \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$



Functions of 2 RVs

Example:

Wireless based station with 2 antennas. X and Y are RVs of the signal

- Find the strongest signal

$$W = X \quad \text{if } |X| > |Y| \quad \text{or} \quad W = Y \quad \text{otherwise}$$

- Find the addition of 2 signals

$$W = X + Y$$

- Find the addition of 2 signals with weight

$$W = aX + bY$$

Functions of 2 RVs

$$F_W(w) = P[W \leq w] = \int \int_{g(x,y) \leq w} f_{X,Y}(x,y) dx dy$$

Example

$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \leq x \leq 3, 0 \leq y \leq 5 \\ 0 & \text{Otherwise} \end{cases}$$

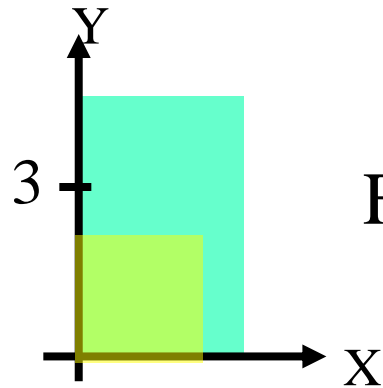
Find PDF of $W = \max(X,Y)$

For $W = \max(X,Y) \rightarrow \{W \leq w\} = \{X \leq w, Y \leq w\}$

$$\begin{aligned} F_W(w) &= P[X \leq w, Y \leq w] \\ &= \int_{-\infty}^w \int_{-\infty}^w f_{X,Y}(x,y) dx dy \end{aligned}$$

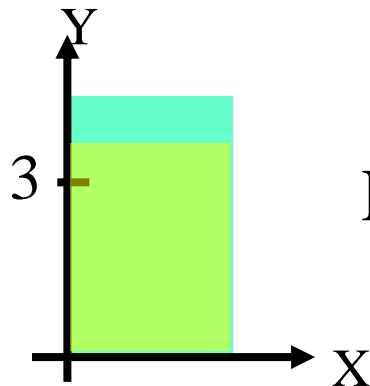
Example

We can divide into 2 cases



$$0 \leq w \leq 3$$

$$F_W(w) = \int_0^w \int_0^w \frac{1}{15} dx dy = w^2/15$$



$$3 \leq w \leq 5$$

$$F_W(w) = \int_0^w \left(\int_0^3 \frac{1}{15} dx \right) dy = w/5$$

Example

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^2/15 & 0 \leq w \leq 3 \\ w/5 & 3 < w \leq 5 \\ 1 & w > 5 \end{cases}$$

$$f_W(w) = \begin{cases} 2w/15 & 0 \leq w \leq 3 \\ 1/5 & 3 < w \leq 5 \\ 0 & \text{Otherwise} \end{cases}$$

