

**01204312**

***Probability Theory and Random Processes***

---

Department of Computer Engineering, Faculty of Engineering,  
Kasetsart University, THAILAND

1<sup>st</sup> semester 2010  
(June – September)

# ***Lecture #11***

## ***Mixed Random Variable***

Assoc. Prof. Anan Phonphoem, Ph.D.  
*<http://www.cpe.ku.ac.th/~anan>*

Department of Computer Engineering, Faculty of Engineering,  
Kasetsart University, THAILAND

# Mixed Random Variable

- Discrete RV  $\rightarrow$  PMF & Summation
- Continuous RV  $\rightarrow$  PDF & Integral
- Combination of Discrete and Continuous RV
  - Unit impulse function
  - Can use same formulas to describe both RVs

# Unit Impulse Function

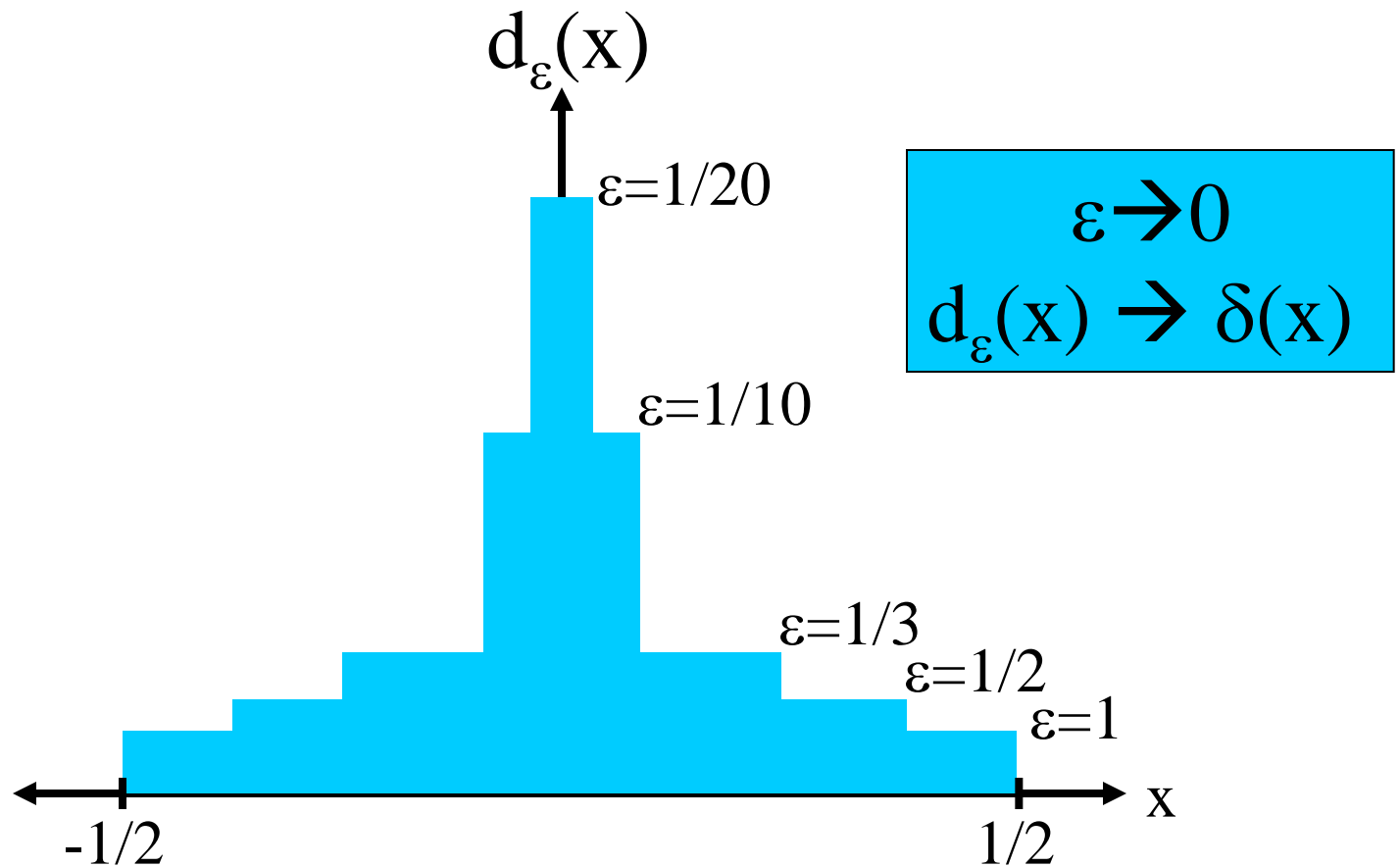
- Delta Function :  $\delta(x)$

**Definition:**

Let 
$$d_{\varepsilon}(x) = \begin{cases} 1/\varepsilon & -\varepsilon/2 \leq x \leq \varepsilon/2 \\ 0 & \text{Otherwise} \end{cases}$$

Then 
$$\delta(x) = \lim_{\varepsilon \rightarrow 0} d_{\varepsilon}(x)$$

# Delta Function



No mathematical meaning **but very useful**

# Delta Function

$$\int_{-\infty}^{\infty} d_{\varepsilon}(x) dx = \int_{-\varepsilon/2}^{\varepsilon/2} \frac{1}{\varepsilon} dx = 1$$

As  $\varepsilon \rightarrow 0$ ,  $d_{\varepsilon}(x) \rightarrow \delta(x)$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

Special case of

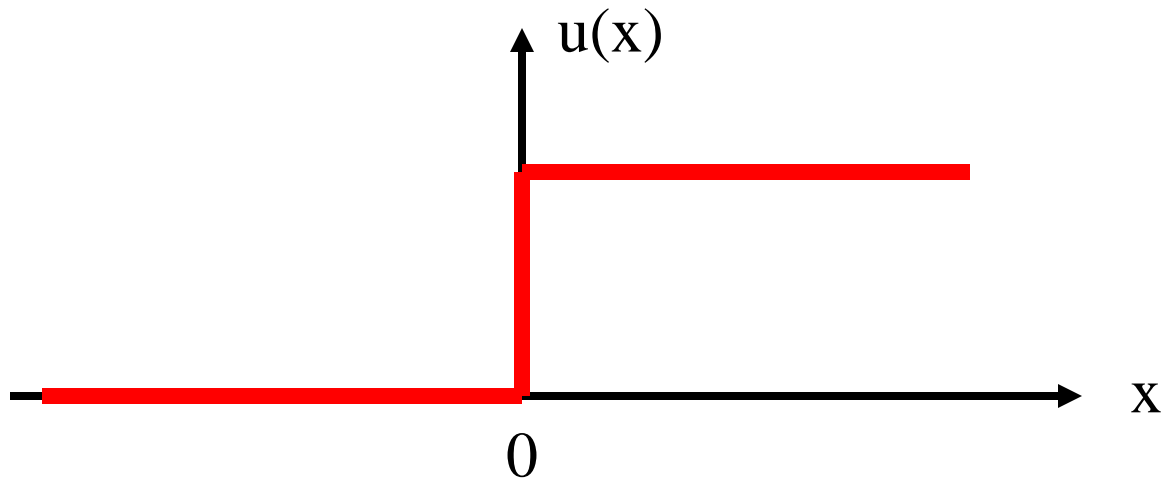
**Theorem:** (Sifting Property)

$$\int_{-\infty}^{\infty} g(x) \delta(x - x_0) dx = g(x_0)$$

# Unit Step Function

**Definition:**

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



# Unit Step Function

$$\int_{-\infty}^{-x} d_{\varepsilon}(v) dv = 0 \quad \int_{-\infty}^x d_{\varepsilon}(v) dv = 1$$

For  $x \neq 0$ ,  $\varepsilon \rightarrow 0$

$$\int_{-\infty}^x d_{\varepsilon}(v) dv = u(x)$$

**Theorem:**

$$\int_{-\infty}^x \delta(v) dv = u(x)$$

**For  $x = 0$  ??**

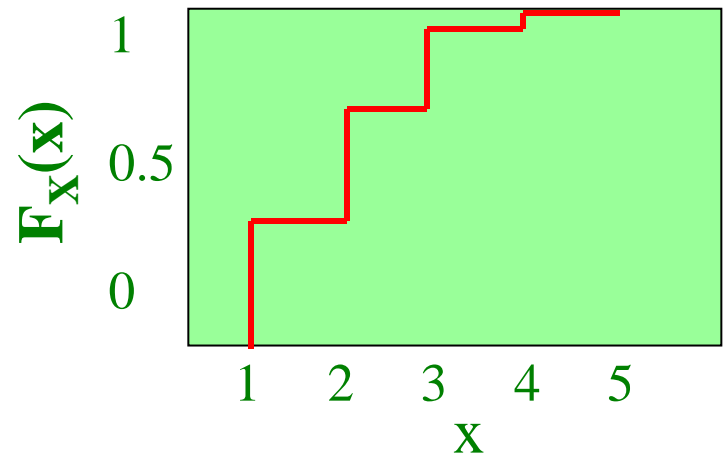
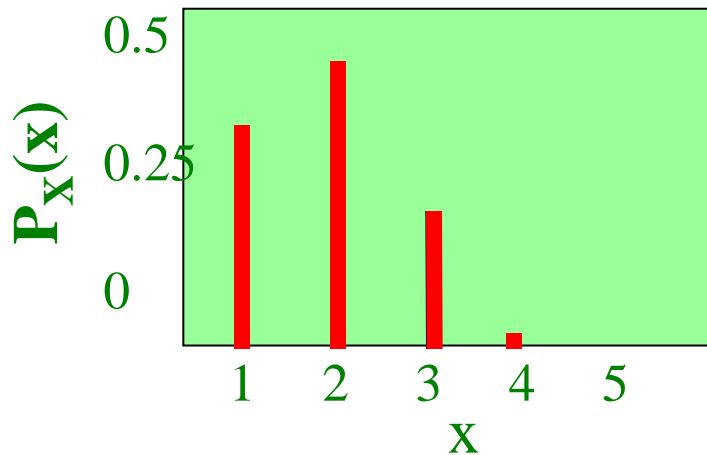
**Not exist**

$$\delta(x) = \frac{d u(x)}{dx}$$



# PMF $\rightarrow$ PDF

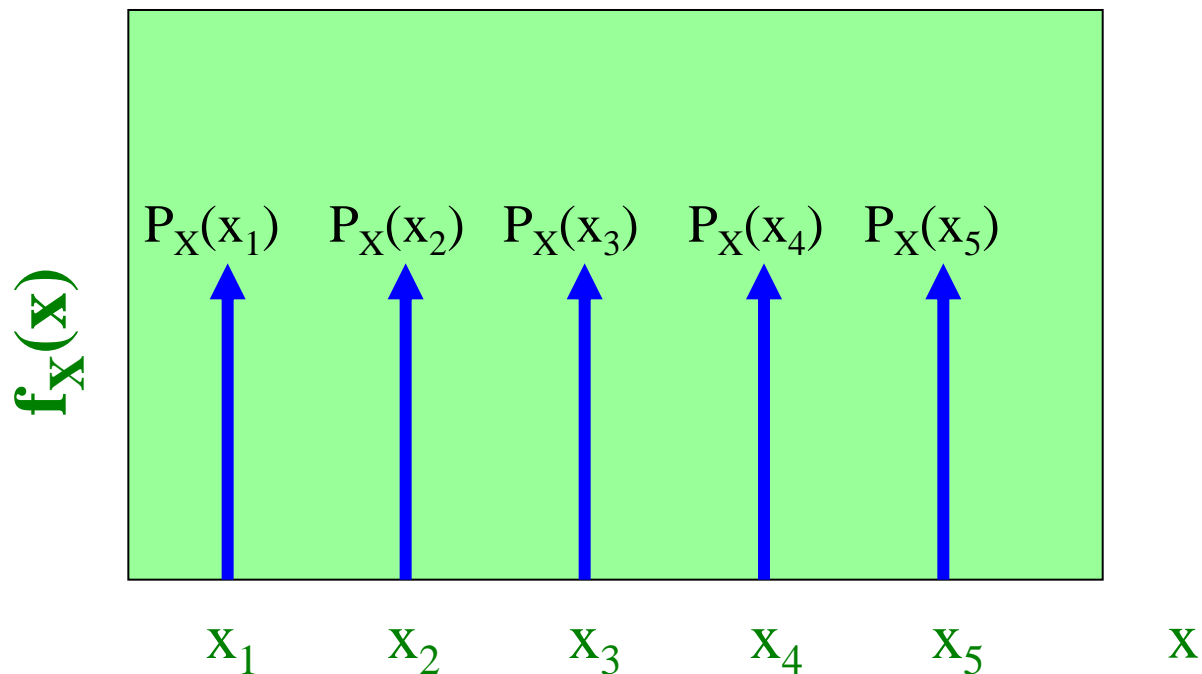
$$F_X(x) = \sum_{x_i \in S_X} P_X(x_i) u(x-x_i)$$



$u(x-x_i) \rightarrow u(x)$  shift to  $x_i$

# PMF $\rightarrow$ PDF

$$f_X(x) = \sum_{x_i \in S_X} P_X(x_i) \delta(x-x_i)$$



# PMF $\rightarrow$ PDF

$$f_X(x) = \sum_{x_i \in S_X} P_X(x_i) \delta(x-x_i)$$

$$E[X] = \int_{-\infty}^{\infty} x \sum_{x_i \in S_X} P_X(x_i) \delta(x-x_i) dx$$

$$= \sum_{x_i \in S_X} \int_{-\infty}^{\infty} x P_X(x_i) \delta(x-x_i) dx$$

$$= \sum_{x_i \in S_X} x_i P_X(x_i)$$

Sifting property

$$\int_{-\infty}^{\infty} g(x) \delta(x-x_0) dx = g(x_0)$$

# PMF $\leftrightarrow$ PDF

## Theorem :

- $P[X = x_0] = q$
- $P_X(x_0) = q$
- $F_X(x_0^+) - F_X(x_0^-) = q$     **Discontinuity at  $x_0$**
- $f_X(x_0) = q \delta(0) \quad \rightarrow q \delta(x_0) ??$

# Mixed Random Variable

**Definition:**  $X$  is a mixed RV Iff

$f_x(\mathbf{x}) =$  both impulses and nonzero, finite values

# Example

- Observe the period of telephone call
  - 1/3 of calls : never begin (no answer/busy)
  - For the success call, with probability of 2/3, call is uniformly  $[0,3]$
- Find PDF, CDF and Mean of call holding time

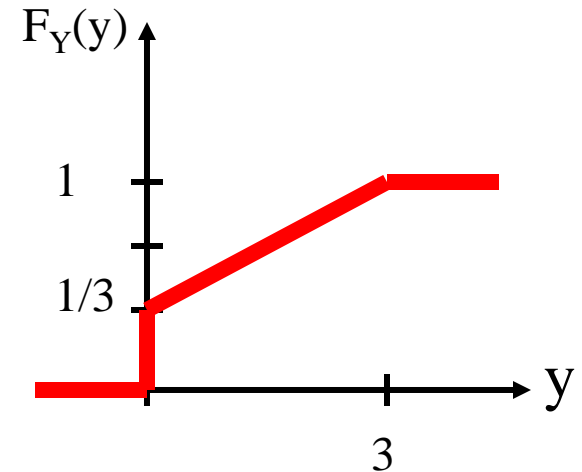
# Example

- Y: call holding time
- A: phone was answered  $\rightarrow A^c$ : not answered
- $0 \leq y \leq 3$
- $F_Y(y) = P[Y \leq y]$

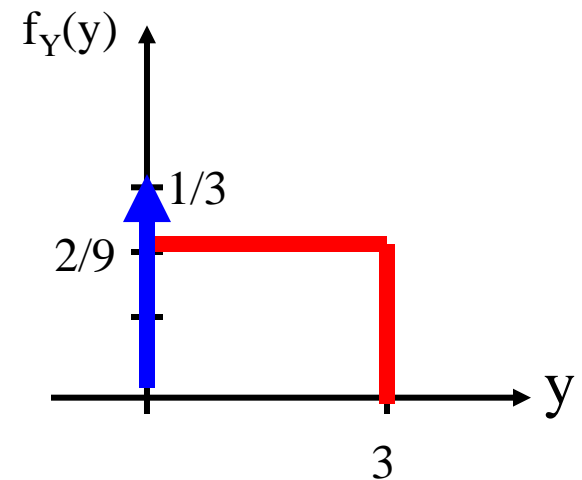
$$\begin{aligned} &= P[Y \leq y | A^c] P[A^c] + P[Y \leq y | A] P[A] \\ &= (1)(1/3) + (y/3)(2/3) \\ &= 1/3 + 2y/9 \end{aligned}$$

# Example

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 1/3 + 2y/9 & 0 \leq y \leq 3 \\ 1 & \text{Otherwise} \end{cases}$$



$$f_Y(y) = \begin{cases} \delta(y)/3 + 2/9 & 0 \leq y \leq 3 \\ 0 & \text{Otherwise} \end{cases}$$





# Example

$$\begin{aligned} E[Y] &= \int_{-\infty}^{\infty} y (1/3)\delta(y) dy + \int_0^3 y (2/9) dy \\ &= 0 + 1 = 1 \end{aligned}$$

# Derived Random Variable

$$Y = aX$$

$$F_Y(y) = P[aX \leq y] = P[X \leq y/a] = F_X(y/a)$$

$$f_Y(y) = \frac{d F_Y(y)}{dy} = (1/a) f_X(y/a)$$

## Theorem :

- $F_Y(y) = F_X(y/a)$
- $f_Y(y) = (1/a) f_X(y/a)$

# Derived Random Variable

$$\mathbf{Y = X + b}$$

$$F_Y(y) = P[X + b \leq y] = P[X \leq y - b] = F_X(y - b)$$

$$f_Y(y) = \frac{d F_Y(y)}{dy} = f_X(y - b)$$

## **Theorem :**

- $F_Y(y) = F_X(y - b)$
- $f_Y(y) = f_X(y - b)$

# Conditioning a continuous RV

$$P[A|B] = P[AB] / P[B]$$

$$P[x_1 < X \leq x_2] = \int_{x_1}^{x_2} f_X(x) dx$$

Approx:  $P[x < X \leq x+dx] = f_X(x) dx$

$$\begin{aligned} f_{X|B}(x) dx &= P[x < X \leq x+dx | B] = \frac{P[x < X \leq x+dx, B]}{P[B]} \\ &= \frac{P[x < X \leq x+dx]}{P[B]} \quad \leftarrow x \in B, x+dx \in B \\ &= \frac{f_X(x) dx}{P[B]} \end{aligned}$$

# Conditioning a continuous RV

$$f_{X|B}(x) \cancel{dx} = \frac{f_X(x) \cancel{dx}}{P[B]}$$

**Definition:**

$$f_{X|B}(x) = \begin{cases} \frac{f_X(x)}{P[B]} & x \in B \\ 0 & \text{Otherwise} \end{cases}$$

**Definition:**

$$E[g(X)|B] = \int_{-\infty}^{\infty} g(x) f_{X|B}(x) dx$$

# Example

- Observe the period of telephone call ( $T$ ) is an **exponential RV** with expected value 3 min.
- Find  $E[T|T>2]$

- **Solution:**

$$f_T(t) = \begin{cases} (1/3) e^{-t/3} & t \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

$$P[T > 2] = \int_2^{\infty} f_T(t) dt = e^{-2/3}$$

# Example

$$f_{T|T>2}(t) = \begin{cases} f_T(t) / P[T > 2] & t > 2 \\ 0 & \text{Otherwise} \end{cases}$$
$$= \begin{cases} (1/3) e^{-(t-2)/3} & t > 2 \\ 0 & \text{Otherwise} \end{cases}$$

$$E[T | T > 2] = \int_2^{\infty} t (1/3) e^{-(t-2)/3} dx$$

$$= 5 \text{ min.}$$