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Probability Theory and Random Processes

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Lecture #13

Law of large number & Central limit theorem

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Outline

- Law of Large Numbers
 - Weak law
 - Strong law
- Central Limit Theorem

Law of Large Numbers (LLN)

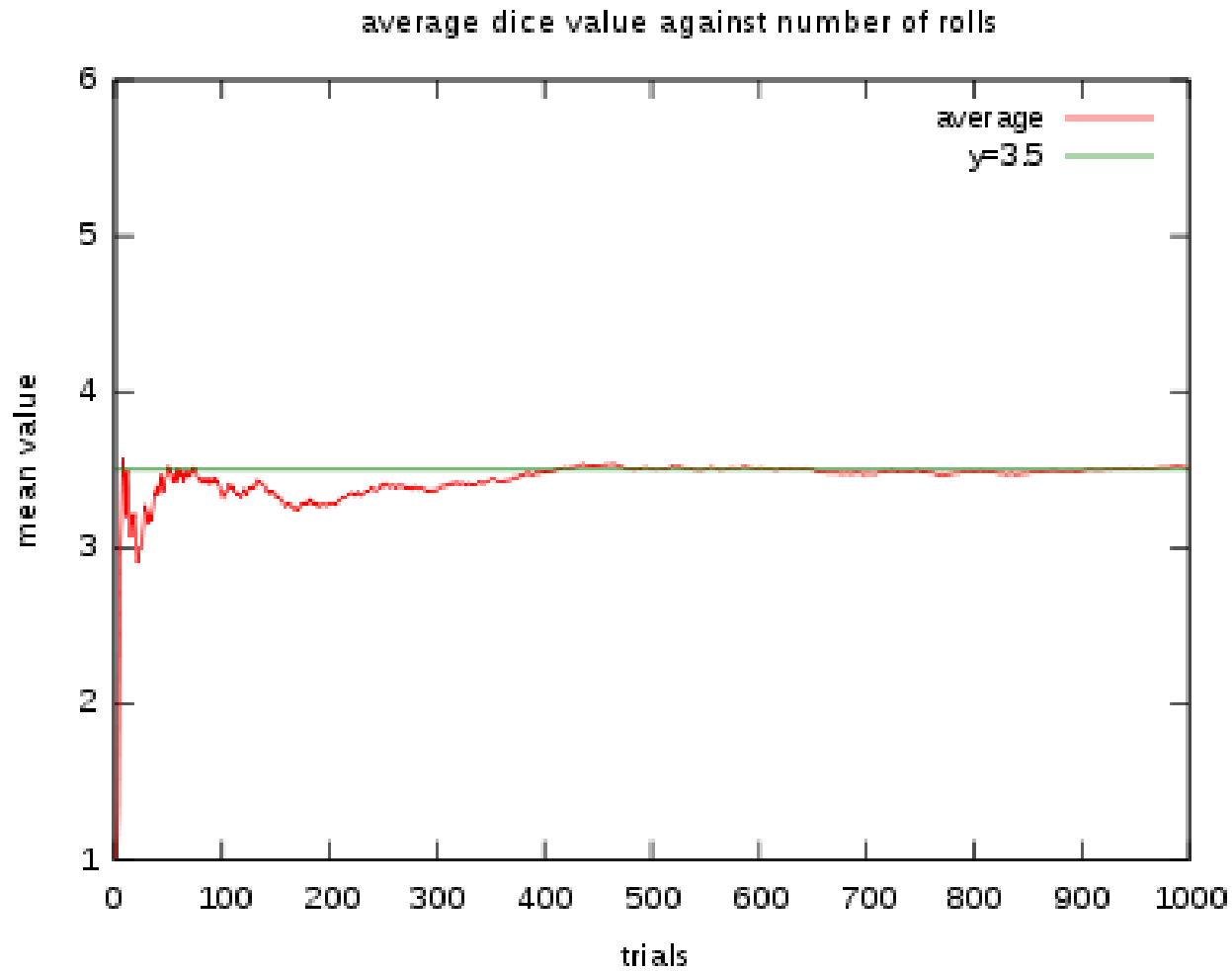
- Performing the same experiment a large number of times
- The average of the results obtained from a large number of trials should be close to the expected value
- Tend to become closer as more trials are performed

Example 1

- Roll a fair die
- 5 times, 10 times, 50 times
- The expected value is
 - $(1+2+3+4+5+6)/6 = 3.5$
- From the law of large numbers
 - If a large number of dice are rolled
 - The average is likely to be close to 3.5
 - More trials \rightarrow More accuracy

Example 1

http://en.wikipedia.org/wiki/Law_of_large_numbers



Law of Large Numbers

- Let X be a random variable
 - Unknown mean, $E(X) = \mu$
- For n independent trials
 - $X_1, X_2, X_3, \dots, X_n$
 - X_j are IID

Sample mean,
$$M_n = \frac{1}{n} \sum_{j=1}^n X_j$$

Sample Mean

Two properties of a good estimator

- 1) On average, should provide the correctness

$$E[M_n] = \mu$$

- 2) Should not vary too much

$$E[(M_n - \mu)^2] \rightarrow \varepsilon$$

Variance of Sample Mean

Let $S_n = X_1 + X_2 + \dots + X_n$

$$\begin{aligned} \text{Var}[S_n] &= \text{Var}[X_1] + \text{Var}[X_2] + \dots + \text{Var}[X_n] \\ &= n\text{Var}[X_j] = n\sigma^2 \end{aligned}$$

$$M_n = \frac{S_n}{n} \quad \Rightarrow \quad \text{Var}[M_n] = \frac{1}{n^2} \text{Var}[S_n]$$
$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

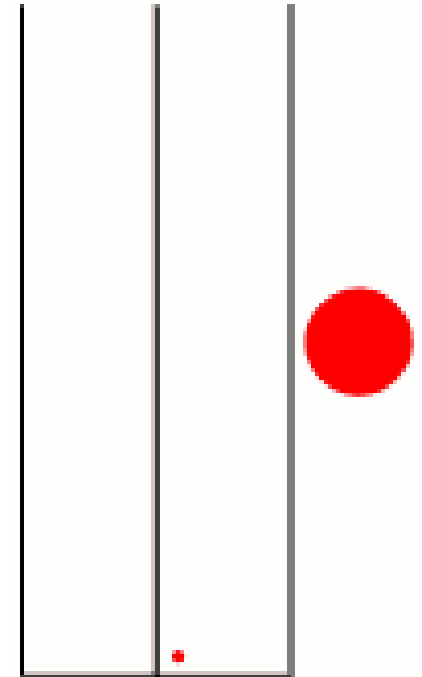
For $n \rightarrow \infty$, $\text{Var}[M_n] \rightarrow 0$

Example 2

http://en.wikipedia.org/wiki/Law_of_large_numbers

Flip a coin

- Red for head
- Blue for tail
- A pie chart shows the proportion of red and blue so far
- The proportion varies a lot at first, but gradually approaches 50%



Example 3

http://en.wikipedia.org/wiki/Law_of_large_numbers

Diffusion in applied chemistry

1. With a **single** molecule

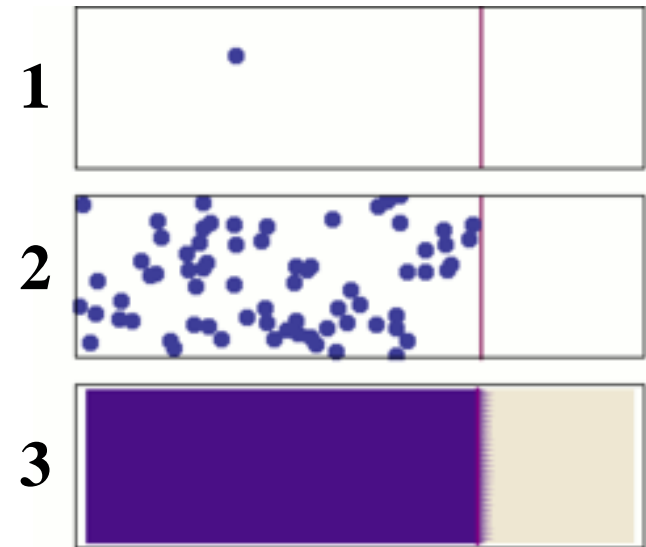
- The motion appears to be quite random

2. With **more** molecules

- Solute trends to fill the container more uniformly (random fluctuations)

3. With an **enormous** number of solute molecules

- Randomness is essentially gone
- Solute appears to move smoothly and systematically from high-concentration areas to low-concentration areas



Law of Large Numbers

- **Weak law**

$$E[M_n] \xrightarrow{p} \mu \quad \text{for } n \rightarrow \infty \quad \text{Convergence in probability}$$

$$\lim_{n \rightarrow \infty} P[|M_n - \mu| < \varepsilon] = 1$$

- **Strong law**

$$E[M_n] \xrightarrow{a.s.} \mu \quad \text{for } n \rightarrow \infty \quad \text{Convergence almost surely}$$

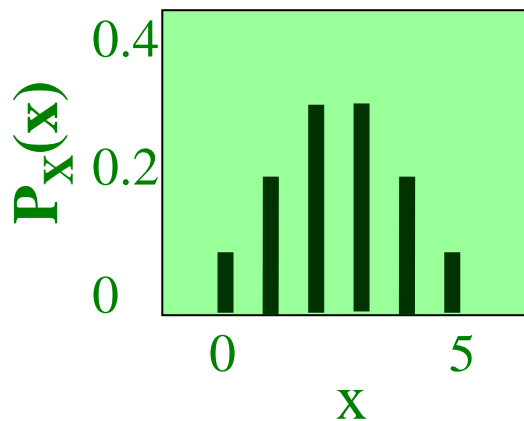
$$P[\lim_{n \rightarrow \infty} M_n = \mu] = 1$$

Central Limit Theorem (CLT)

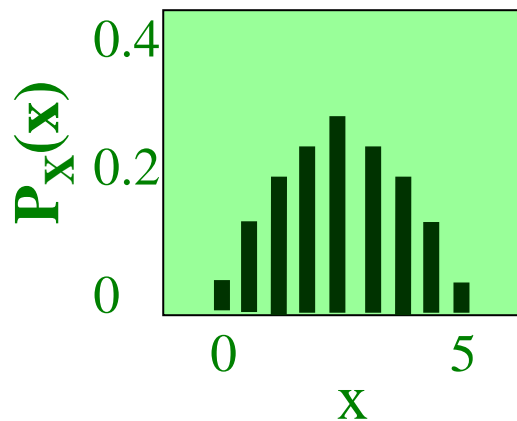
- The central limit theorem
 - **Mean of a sufficiently large number** of independent random variables
 - With finite mean and variance
 - Will be **approximately Normally Distributed (Bell-shaped curve)**
- Requires
 - Random variables to be **identically** distributed (unless certain conditions are met)

Example

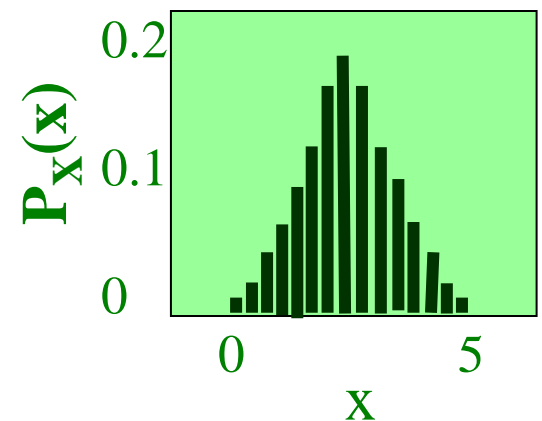
- Flip a coin
- Binomial ($n, 1/2$)
- $n \rightarrow$ large, PMF \rightarrow bell-shaped curve



$n = 5$



$n = 10$



$n = 20$

Central Limit Theorem

Let
$$W_n = X_1 + X_2 + \dots + X_n$$

For $n \rightarrow \infty$

$$\begin{aligned} E[W_n] &= E[X_1] + E[X_2] + \dots + E[X_n] \\ &= n\mu_x \end{aligned}$$

$$\begin{aligned} \text{Var}[W_n] &= \text{Var}[X_1] + \text{Var}[X_2] + \dots + \text{Var}[X_n] \\ &= n\text{Var}[X] \end{aligned}$$

Finding the convergence of CDF, $F_{W_n}(w)$, is not easy

Central Limit Theorem

Definition: in term of standardized RV. for all n

$$Z_n = \frac{\sum_{i=1}^n X_i - n\mu_X}{\sqrt{n\sigma_X^2}}$$

$$E[Z_n] = 0 \quad \text{Var}[Z_n] = 1$$

Central Limit Theorem

Theorem: Given X_1, X_2, \dots is a sequence of IID random variable with expected value μ_X and variance σ^2_X , the **CDF of Z_n** has the property

$$\lim_{n \rightarrow \infty} F_{Z_n}(z) = \Phi(z)$$

To use the CLT

- For IID sum $W_n = X_1 + X_2 + \dots + X_n$

$$Z_n = \frac{\sum_{i=1}^n X_i - n\mu_X}{\sqrt{n\sigma_X^2}} \quad \Rightarrow \quad W_n = \sqrt{n\sigma_X^2} Z_n + n\mu_X$$

The CDF of W_n can be expressed in terms of the CDF of Z_n as

$$F_{W_n}(w) = P[\sqrt{n\sigma_X^2} Z_n + n\mu_X \leq w]$$

$$= F_{Z_n}\left(\frac{w - n\mu_X}{\sqrt{n\sigma_X^2}}\right) \quad \Rightarrow \quad \lim_{n \rightarrow \infty} F_{Z_n}(z) \approx \Phi(z)$$

Central Limit Theorem Approximation (Gaussian approximation)

Definition: Let $W_n = X_1 + X_2 + \dots + X_n$ be the sum of n IID random variables, each with expected value μ_X and variance σ_X^2 , the **CLT approximation to the CDF of W_n** is

$$F_{W_n}(w) \approx \Phi \left(\frac{w - n\mu_X}{\sqrt{n\sigma_X^2}} \right)$$

Example

- A modem transmits one millions bits
- Each bit is “1” or “0” independently with equal probability
- Estimate the probability of at least 502,000 ones

Example

Solution

- Let X_i = be value of “1” or “0”

$$W = \sum_{i=1}^{10^6} X_i$$

- X_i is Bernoulli (0.5) $\rightarrow E[X_i] = 0.5$ and $\text{Var}[X_i] = 2.5$
- $E[W] = 10^6 E[X_i] = 500,000$
- $\text{Var}[W] = 10^6 \text{Var}[X_i] = 250,000$
- $\sigma_W = 500$

Example

- By CLT approximation,

$$\begin{aligned} P[W \geq 502,000] &= 1 - P[W \leq 502,000] \\ &\approx 1 - \Phi\left(\frac{502,000 - 500,000}{500}\right) = 1 - \Phi(4) \end{aligned}$$

- Using the table \rightarrow ans = $Q(4) = 3.17 \times 10^{-5}$