

**01204312**

***Probability Theory and Random Processes***

---

Department of Computer Engineering, Faculty of Engineering,  
Kasetsart University, THAILAND

1<sup>st</sup> semester 2010  
(June – September)

# ***Lecture #9 – 1***

## ***Mixed Random Variable***

Assoc. Prof. Anan Phonphoem, Ph.D.  
*<http://www.cpe.ku.ac.th/~anan>*

Department of Computer Engineering, Faculty of Engineering,  
Kasetsart University, THAILAND

# Mixed Random Variable

- Discrete RV  $\rightarrow$  PMF & Summation
- Continuous RV  $\rightarrow$  PDF & Integral
- Combination of Discrete and Continuous RV
  - Unit impulse function
  - Can use same formulas to describe both RVs

# Unit Impulse Function

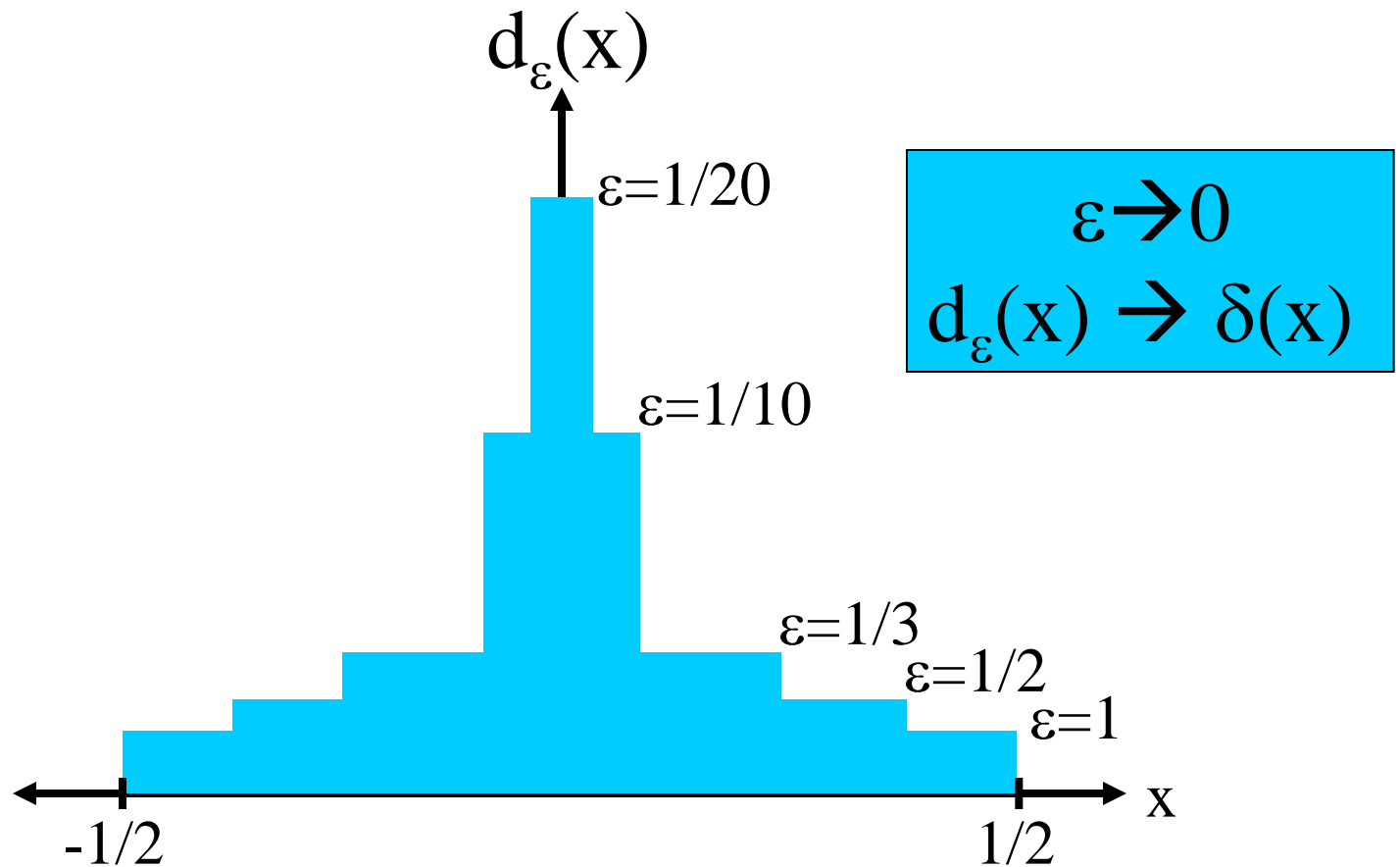
- Delta Function :  $\delta(x)$

**Definition:**

Let 
$$d_{\varepsilon}(x) = \begin{cases} 1/\varepsilon & -\varepsilon/2 \leq x \leq \varepsilon/2 \\ 0 & \text{Otherwise} \end{cases}$$

Then 
$$\delta(x) = \lim_{\varepsilon \rightarrow 0} d_{\varepsilon}(x)$$

# Delta Function



No mathematical meaning **but very useful**

# Delta Function

$$\int_{-\infty}^{\infty} d_{\varepsilon}(x) dx = \int_{-\varepsilon/2}^{\varepsilon/2} \frac{1}{\varepsilon} dx = 1$$

As  $\varepsilon \rightarrow 0$ ,  $d_{\varepsilon}(x) \rightarrow \delta(x)$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

Special case of

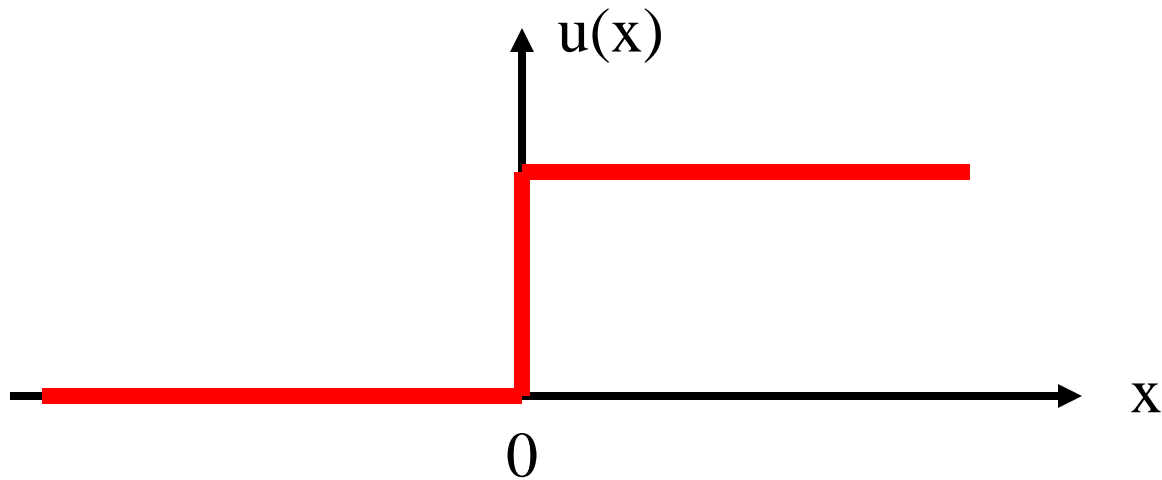
**Theorem:** (Sifting Property)

$$\int_{-\infty}^{\infty} g(x) \delta(x - x_0) dx = g(x_0)$$

# Unit Step Function

**Definition:**

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



# Unit Step Function

$$\int_{-\infty}^{-x} d_{\varepsilon}(v) dv = 0 \quad \int_{-\infty}^x d_{\varepsilon}(v) dv = 1$$

For  $x \neq 0$ ,  $\varepsilon \rightarrow 0$

$$\int_{-\infty}^x d_{\varepsilon}(v) dv = u(x)$$

**Theorem:**

$$\int_{-\infty}^x \delta(v) dv = u(x)$$

**For  $x = 0$  ??**

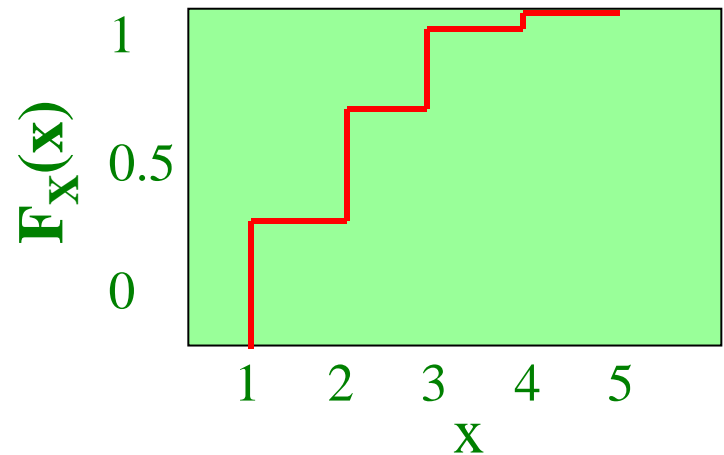
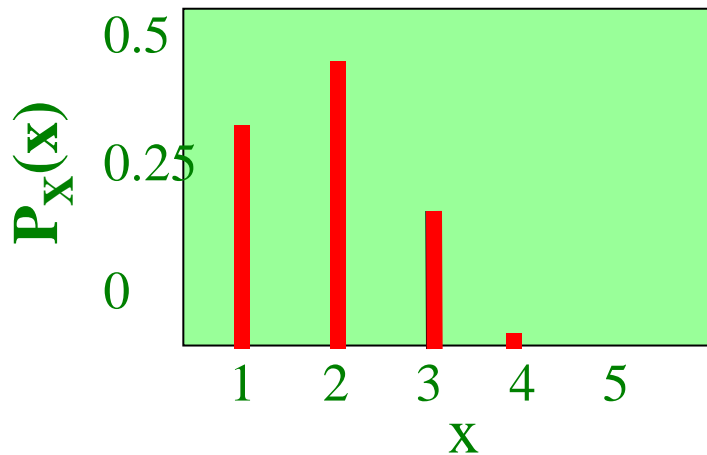
**Not exist**

$$\delta(x) = \frac{d u(x)}{dx}$$



# PMF $\rightarrow$ PDF

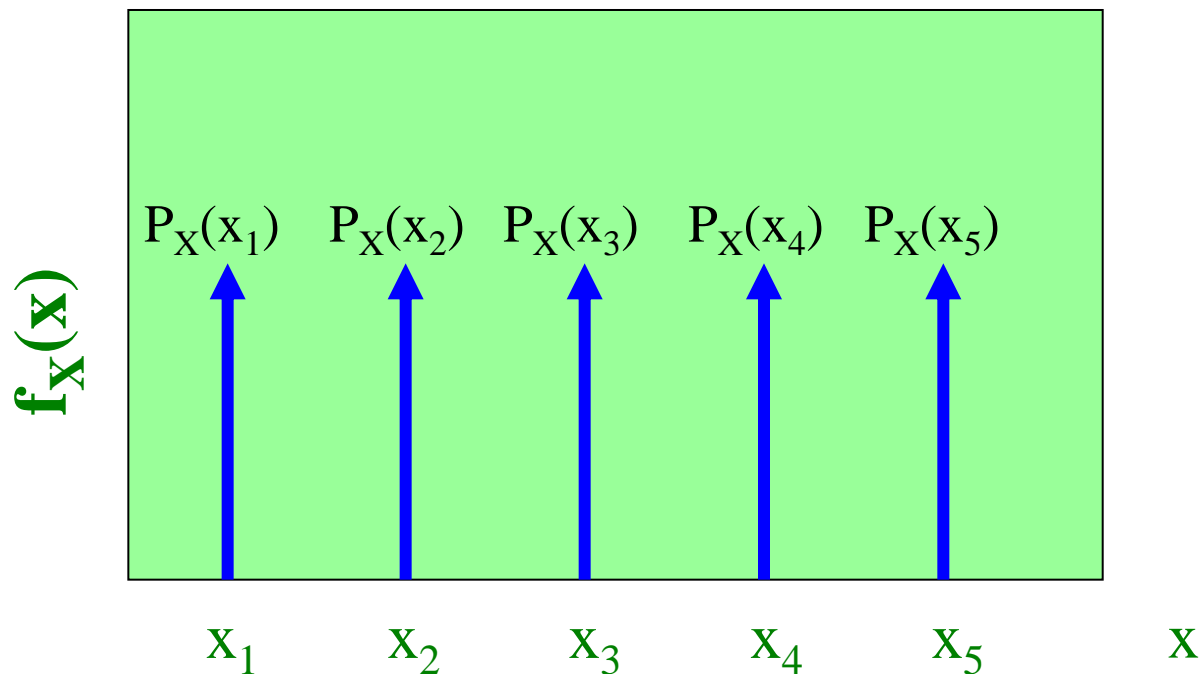
$$F_X(x) = \sum_{x_i \in S_X} P_X(x_i) u(x-x_i)$$



$u(x-x_i) \rightarrow u(x)$  shift to  $x_i$

# PMF $\rightarrow$ PDF

$$f_X(x) = \sum_{x_i \in S_X} P_X(x_i) \delta(x-x_i)$$



# PMF $\rightarrow$ PDF

$$f_X(x) = \sum_{x_i \in S_X} P_X(x_i) \delta(x-x_i)$$

$$E[X] = \int_{-\infty}^{\infty} x \sum_{x_i \in S_X} P_X(x_i) \delta(x-x_i) dx$$

$$= \sum_{x_i \in S_X} \int_{-\infty}^{\infty} x P_X(x_i) \delta(x-x_i) dx$$

$$= \sum_{x_i \in S_X} x_i P_X(x_i)$$

Sifting property

$$\int_{-\infty}^{\infty} g(x) \delta(x-x_0) dx = g(x_0)$$

# PMF $\leftrightarrow$ PDF

## Theorem :

- $P[X = x_0] = q$
- $P_X(x_0) = q$
- $F_X(x_0^+) - F_X(x_0^-) = q$     **Discontinuity at  $x_0$**
- $f_X(x_0) = q \delta(0) \quad \rightarrow q \delta(x_0) ??$

# Mixed Random Variable

**Definition:**  $X$  is a mixed RV Iff

$f_x(\mathbf{x}) =$  both impulses and nonzero, finite values

# Example

- Observe the period of telephone call
  - 1/3 of calls : never begin (no answer/busy)
  - For the success call, with probability of 2/3, call is uniformly  $[0,3]$
- Find PDF, CDF and Mean of call holding time

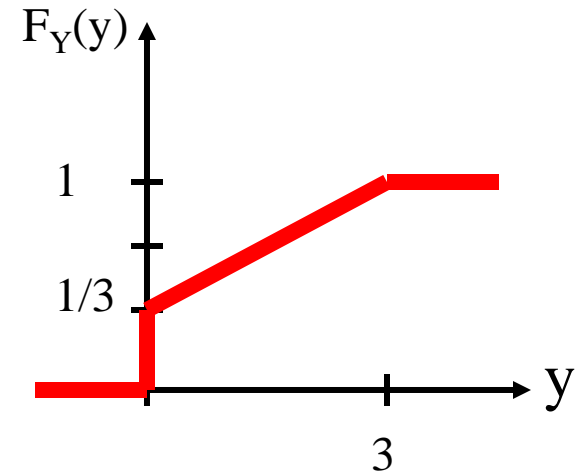
# Example

- Y: call holding time
- A: phone was answered  $\rightarrow A^c$ : not answered
- $0 \leq y \leq 3$
- $F_Y(y) = P[Y \leq y]$

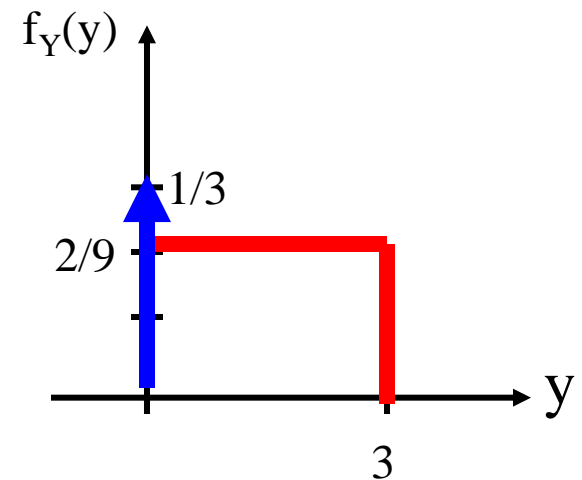
$$\begin{aligned} &= P[Y \leq y | A^c] P[A^c] + P[Y \leq y | A] P[A] \\ &= (1)(1/3) + (y/3)(2/3) \\ &= 1/3 + 2y/9 \end{aligned}$$

# Example

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 1/3 + 2y/9 & 0 \leq y \leq 3 \\ 1 & \text{Otherwise} \end{cases}$$



$$f_Y(y) = \begin{cases} \delta(y)/3 + 2/9 & 0 \leq y \leq 3 \\ 0 & \text{Otherwise} \end{cases}$$





# Example

$$\begin{aligned} E[Y] &= \int_{-\infty}^{\infty} y (1/3)\delta(y) dy + \int_0^3 y (2/9) dy \\ &= 0 + 1 = 1 \end{aligned}$$

# Derived Random Variable

$$Y = aX$$

$$F_Y(y) = P[aX \leq y] = P[X \leq y/a] = F_X(y/a)$$

$$f_Y(y) = \frac{d F_Y(y)}{dy} = (1/a) f_X(y/a)$$

## Theorem :

- $F_Y(y) = F_X(y/a)$
- $f_Y(y) = (1/a) f_X(y/a)$

# Derived Random Variable

$$\mathbf{Y = X + b}$$

$$F_Y(y) = P[X + b \leq y] = P[X \leq y - b] = F_X(y - b)$$

$$f_Y(y) = \frac{d F_Y(y)}{dy} = f_X(y - b)$$

## **Theorem :**

- $F_Y(y) = F_X(y - b)$
- $f_Y(y) = f_X(y - b)$

# Conditioning a continuous RV

$$P[A|B] = P[AB] / P[B]$$

$$P[x_1 < X \leq x_2] = \int_{x_1}^{x_2} f_X(x) dx$$

Approx:  $P[x < X \leq x+dx] = f_X(x) dx$

$$\begin{aligned} f_{X|B}(x) dx &= P[x < X \leq x+dx | B] = \frac{P[x < X \leq x+dx, B]}{P[B]} \\ &= \frac{P[x < X \leq x+dx]}{P[B]} \quad \leftarrow x \in B, x+dx \in B \\ &= \frac{f_X(x) dx}{P[B]} \end{aligned}$$

# Conditioning a continuous RV

$$f_{X|B}(x) \cancel{dx} = \frac{f_X(x) \cancel{dx}}{P[B]}$$

**Definition:**

$$f_{X|B}(x) = \begin{cases} \frac{f_X(x)}{P[B]} & x \in B \\ 0 & \text{Otherwise} \end{cases}$$

**Definition:**

$$E[g(X)|B] = \int_{-\infty}^{\infty} g(x) f_{X|B}(x) dx$$

# Example

- Observe the period of telephone call ( $T$ ) is an **exponential RV** with expected value 3 min.
- Find  $E[T|T>2]$

- **Solution:**

$$f_T(t) = \begin{cases} (1/3) e^{-t/3} & t \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

$$P[T > 2] = \int_2^{\infty} f_T(t) dt = e^{-2/3}$$

# Example

$$f_{T|T>2}(t) = \begin{cases} f_T(t) / P[T > 2] & t > 2 \\ 0 & \text{Otherwise} \end{cases}$$
$$= \begin{cases} (1/3) e^{-(t-2)/3} & t > 2 \\ 0 & \text{Otherwise} \end{cases}$$

$$E[T | T > 2] = \int_2^{\infty} t (1/3) e^{-(t-2)/3} dx$$

$$= 5 \text{ min.}$$

# ***Lecture #9 – 2***

## ***Multiple Random Variables***

Assoc. Prof. Anan Phonphoem, Ph.D.  
*<http://www.cpe.ku.ac.th/~anan>*

Department of Computer Engineering, Faculty of Engineering,  
Kasetsart University, THAILAND



# Joint CDF

- Pairs of Random Variables

- Discrete:

Joint PMF  $P_{X,Y}(x,y) = P[X=x, Y=y]$

- Continuous:

$$P_{X,Y}(x,y) = 0 \quad (P_X(x) = 0, P_Y(y) = 0)$$

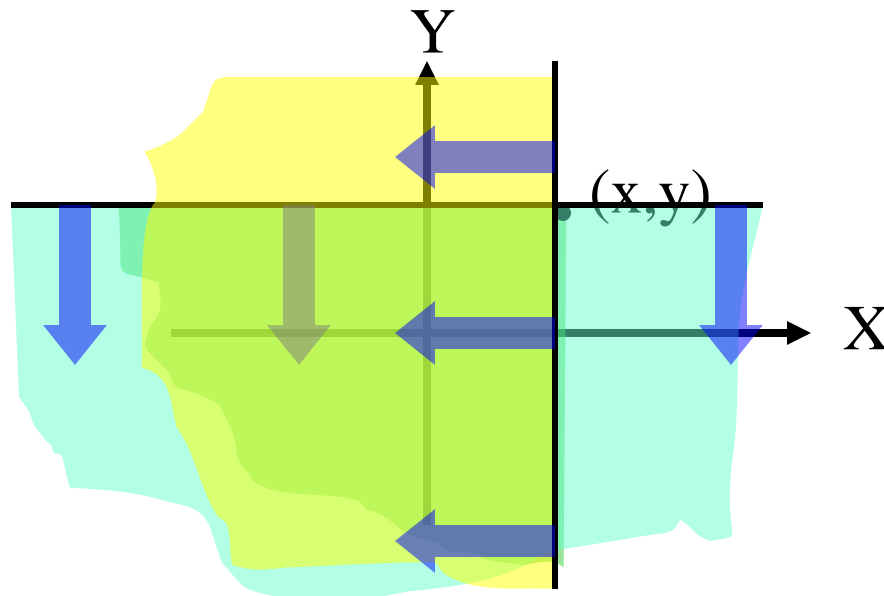
For 1 RV  $\rightarrow$  interval on real axis

For 2 RVs  $\rightarrow$  area in a plane

# Joint CDF

**Definition:** Joint CDF of X and Y

$$F_{X,Y}(x,y) = P[X \leq x, Y \leq y]$$



# Interesting Properties

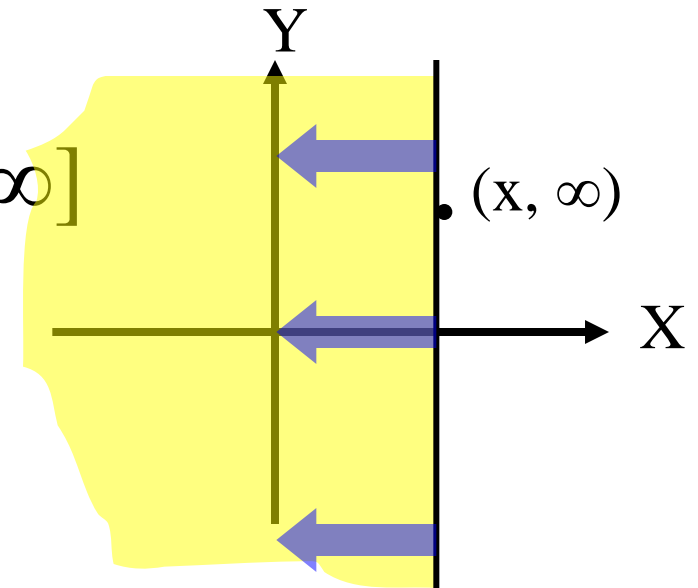
- For Event  $\{X \leq x\}$

$$F_X(x) = P[X \leq x]$$

$$= P[X \leq x, Y \leq \infty]$$

$$= \lim_{y \rightarrow \infty} F_{X,Y}(x, y)$$

$$= F_{X,Y}(x, \infty)$$



# Joint CDF

## Theorem :

(a)  $0 \leq F_{X,Y}(x,y) \leq 1$

(b)  $F_X(x) = F_{X,Y}(x, \infty)$

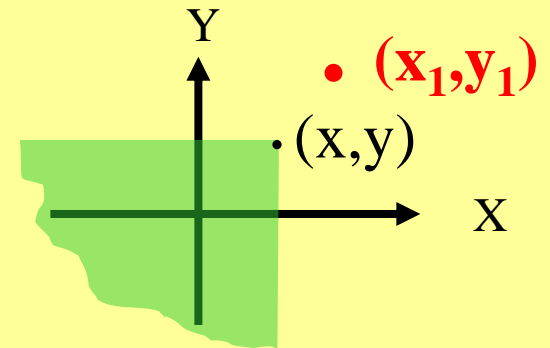
(c)  $F_Y(y) = F_{X,Y}(\infty, y)$

(d)  $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$

(e) If  $x_1 \geq x$  and  $y_1 \geq y$

then  $F_{X,Y}(x_1, y_1) > F_{X,Y}(x, y)$

(f)  $F_{X,Y}(\infty, \infty) = 1$



# Joint PDF

**Definition:** Joint PDF of X and Y is satisfied

$$F_{X,Y}(x,y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(u,v) \, dv \, du$$

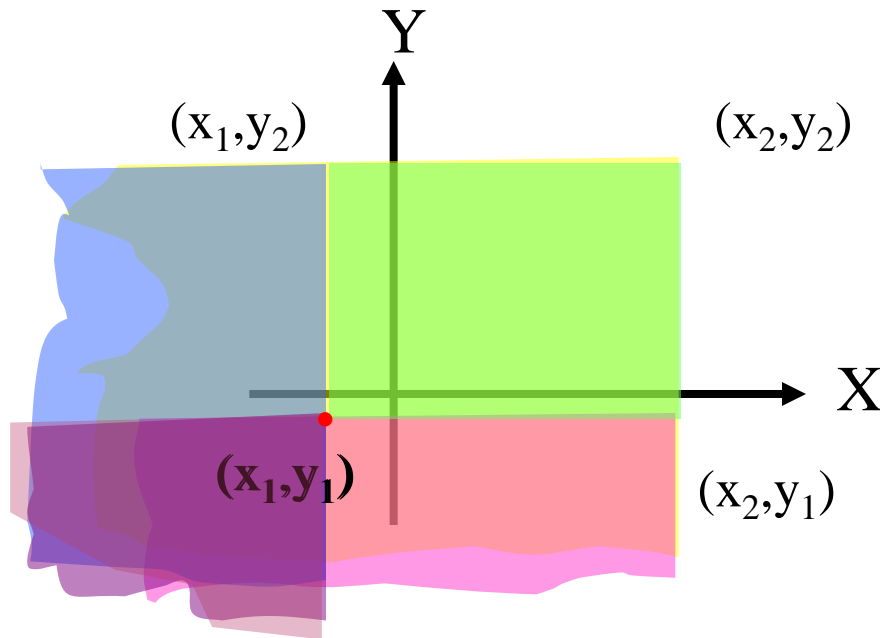
**Theorem:**

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

# Joint CDF

## Theorem:

$$\begin{aligned} &P[x_1 < X \leq x_2, y_1 < Y \leq y_2] \\ &= F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1) \end{aligned}$$



# Joint PDF

**Theorem:**

$$(a) f_{X,Y}(x,y) \geq 0 \text{ for all } (x,y)$$

$$(b) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1$$

**Theorem:**

$$P[A] = \iint_A f_{X,Y}(x,y) dx dy$$

# Example

$$f_{X,Y}(x,y) = \begin{cases} c & 0 \leq x \leq 3, 0 \leq y \leq 5 \\ 0 & \text{Otherwise} \end{cases}$$

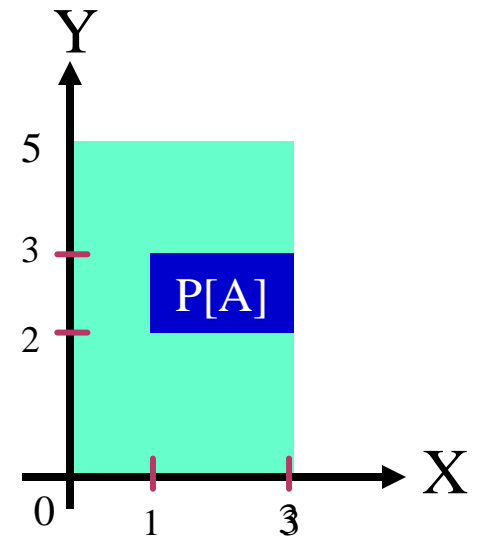
**Find constant c**

$$\int_0^3 \int_0^5 c \, dy \, dx = 15c = 1$$

$$\rightarrow c = 1/15$$

**Find  $P[A] = P[1 \leq x \leq 3, 2 \leq y \leq 3]$**

$$P[A] = \int_1^3 \int_2^3 1/15 \, dv \, du = 2/15$$





# Marginal PDF

**Theorem:**

$$f_{X,}(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$$

$$f_{Y,}(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx$$

# Example

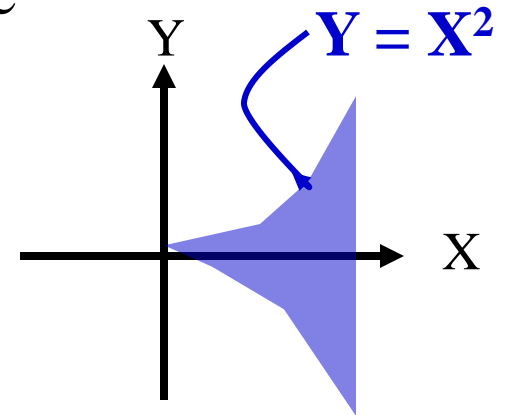
$$f_{X,Y}(x,y) = \begin{cases} cx & 0 \leq x \leq 1, |y| < x^2 \\ 0 & \text{Otherwise} \end{cases}$$

**Find constant c**

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} cx \, dx \, dy = \int_0^1 \left( \int_{-x^2}^{x^2} cx \, dy \right) dx$$

$$= \int_0^1 cx (2x^2) \, dx = \frac{cx^4}{2} \Big|_0^1$$

$$= \frac{c}{2} = 1 \quad \rightarrow \quad c = 2$$



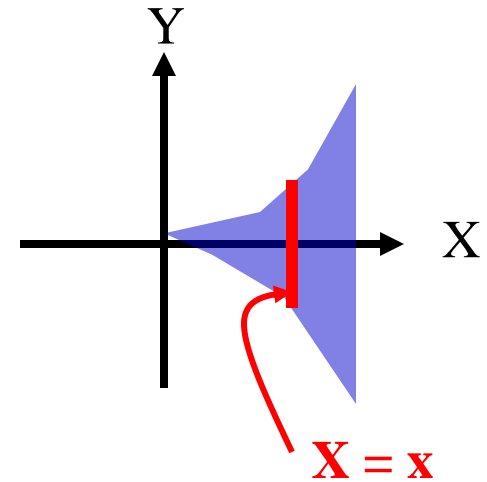
# Example

Find the marginal PDF  $f_X(x)$  and  $f_Y(y)$

Fixed  $x$  ( $X = x$ ) then integrate all  $y$

$$f_X(x) = \int_{-x^2}^{x^2} 2x \, dy = 4x^3$$

$$f_X(x) = \begin{cases} 4x^3 & 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

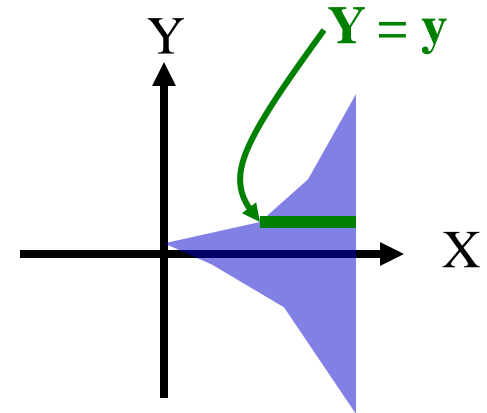


# Example

Fixed  $y$  ( $Y = y$ ) then integrate all  $x$

$$f_{Y,}(y) = \int_{\sqrt{|y|}}^1 2x \, dx = 1 - |y|$$

$$f_Y(y) = \begin{cases} 1 - |y| & -1 \leq y \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$



# Functions of 2 RVs

## Example:

Wireless based station with 2 antennas.  $X$  and  $Y$  are RVs of the signal

- Find the strongest signal

$$W = X \quad \text{if } |X| > |Y| \quad \text{or} \quad W = Y \quad \text{otherwise}$$

- Find the addition of 2 signals

$$W = X + Y$$

- Find the addition of 2 signals with weight

$$W = aX + bY$$

# Functions of 2 RVs

$$F_W(w) = P[W \leq w] = \int \int_{g(x,y) \leq w} f_{X,Y}(x,y) dx dy$$

# Example

$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \leq x \leq 3, 0 \leq y \leq 5 \\ 0 & \text{Otherwise} \end{cases}$$

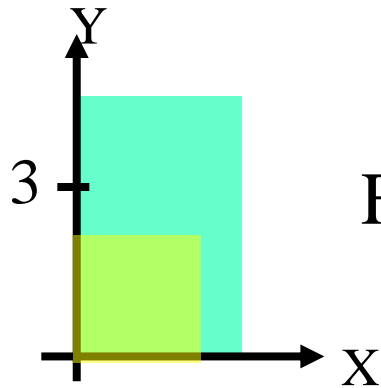
**Find PDF of  $W = \max(X,Y)$**

For  $W = \max(X,Y) \rightarrow \{W \leq w\} = \{X \leq w, Y \leq w\}$

$$\begin{aligned} F_W(w) &= P[X \leq w, Y \leq w] \\ &= \int_{-\infty}^w \int_{-\infty}^w f_{X,Y}(x,y) dx dy \end{aligned}$$

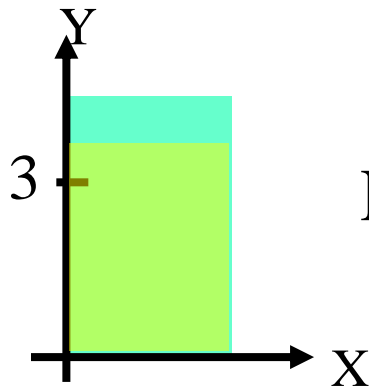
# Example

We can divide into 2 cases



$$0 \leq w \leq 3$$

$$F_W(w) = \int_0^w \int_0^w 1/15 \, dx \, dy = w^2/15$$



$$3 \leq w \leq 5$$

$$F_W(w) = \int_0^w \left( \int_0^3 1/15 \, dx \right) dy = w/5$$



# Example

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^2/15 & 0 \leq w \leq 3 \\ w/5 & 3 < w \leq 5 \\ 1 & w > 5 \end{cases}$$

$$f_W(w) = \begin{cases} 2w/15 & 0 \leq w \leq 3 \\ 1/5 & 3 < w \leq 5 \\ 0 & \text{Otherwise} \end{cases}$$

