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Probability Theory and Random Processes

Department of Computer Engineering, Faculty of Engineering,
Kasetsart University, THAILAND

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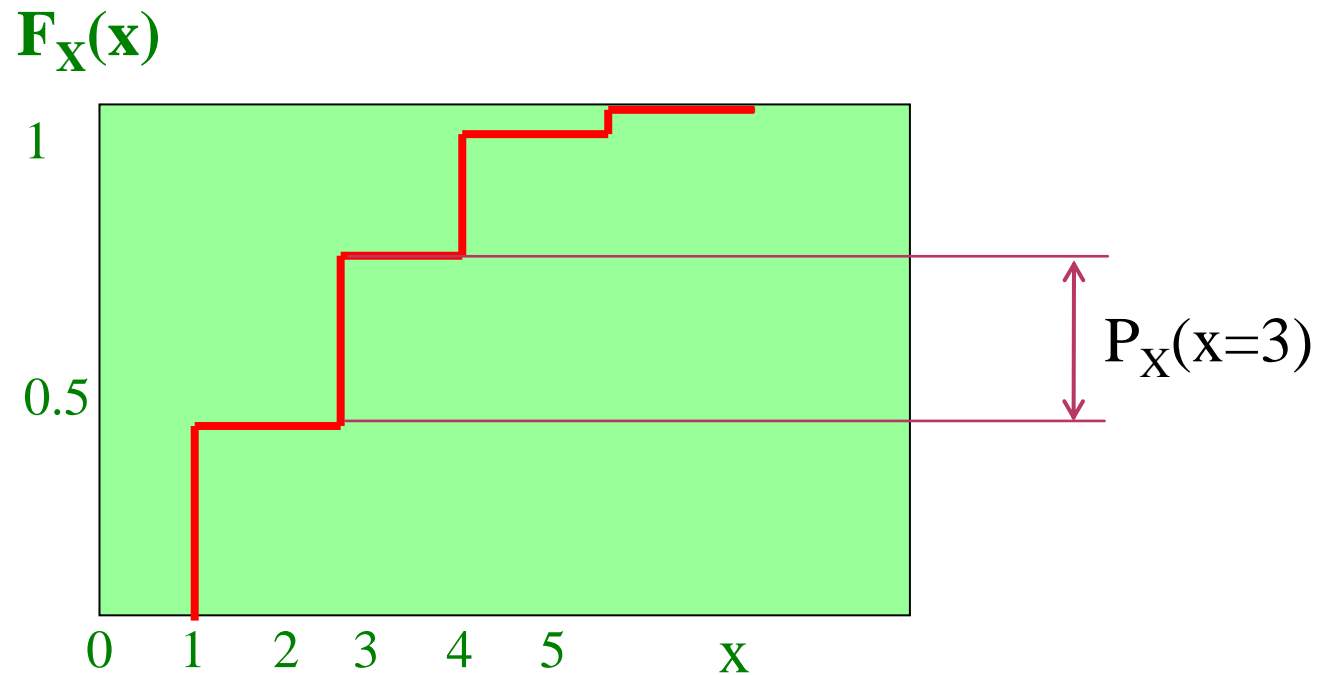
Lecture #7

Continuous Random Variable

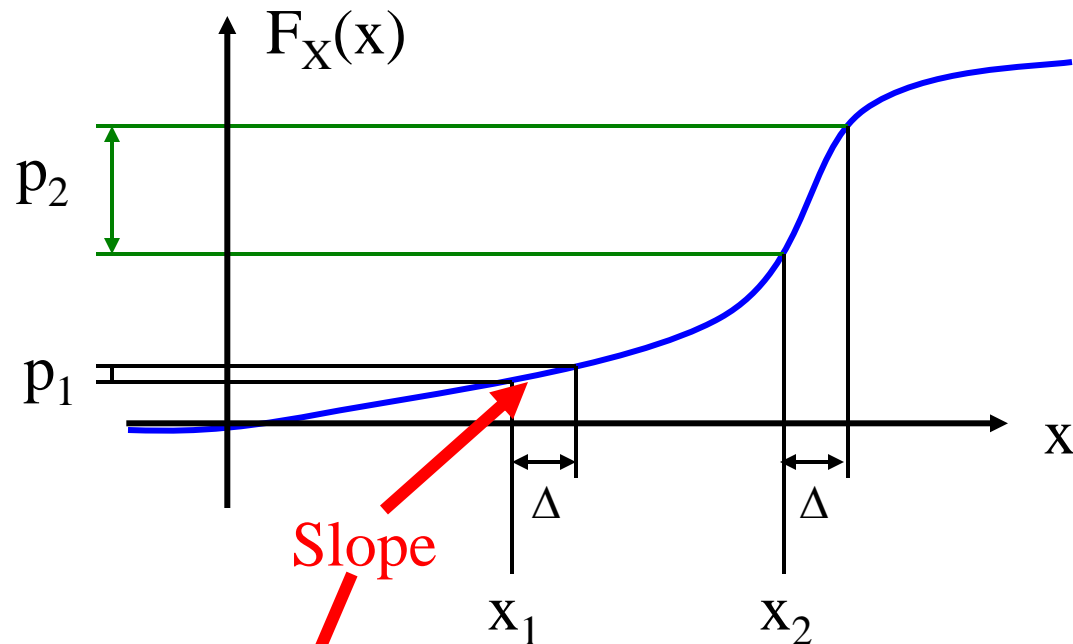
Assoc. Prof. Anan Phonphoem, Ph.D.
<http://www.cpe.ku.ac.th/~anan>

Department of Computer Engineering, Faculty of Engineering,
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CDF of Discrete RV



Probability Density Function



$$\begin{aligned} p_1 &= P[x_1 < X \leq x_1 + \Delta] \\ &= F_X(x_1 + \Delta) - F_X(x_1) \\ &= \frac{F_X(x_1 + \Delta) - F_X(x_1)}{\Delta} \Delta \end{aligned}$$

$$\begin{aligned} p_2 &= P[x_2 < X \leq x_2 + \Delta] \\ &= F_X(x_2 + \Delta) - F_X(x_2) \end{aligned}$$

For $\Delta \Rightarrow 0$,
Slope $\Rightarrow \frac{dF_X(x)}{dx}$ at x_1

Probability Density Function (PDF)

- The slope of CDF in a region near x
 - Probability of random variable X near x
 - The prob. in a small region(Δ) = slope * Δ
- Slope of CDF \Rightarrow PDF

Definition:

Probability Density Function (PDF) is

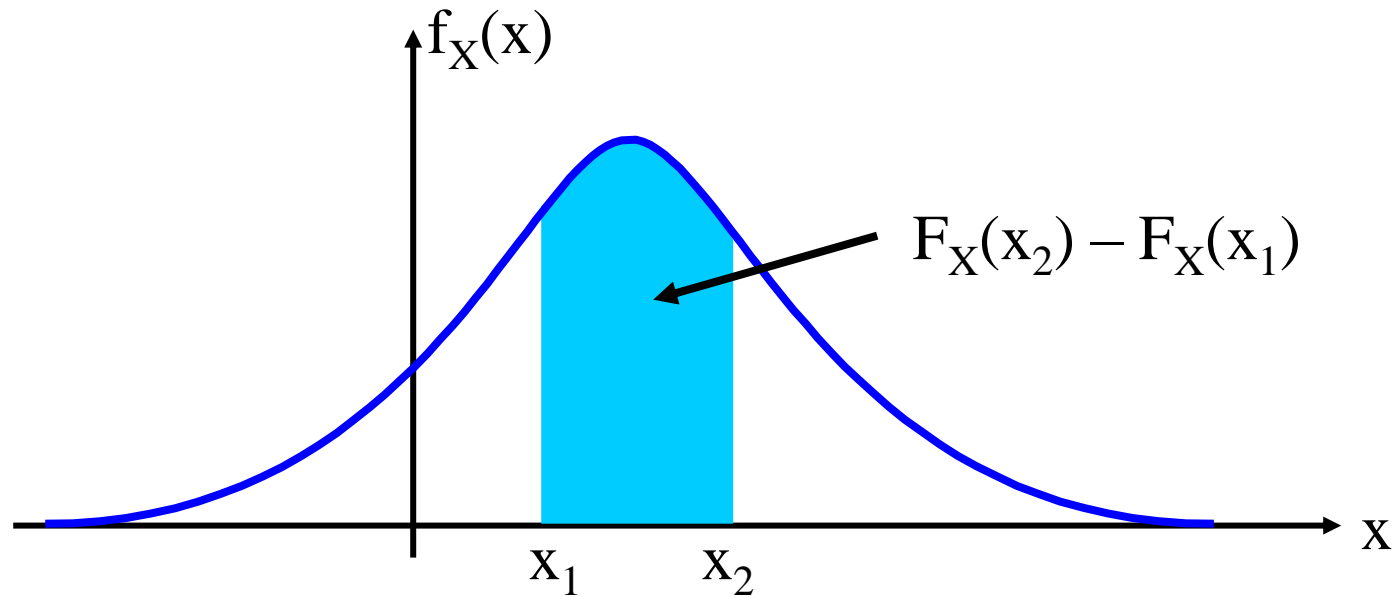
$$f_X(x) = \frac{dF_X(x)}{dx}$$

PDF Theorem

Theorem:

- $f_X(x) \geq 0$ for all x
- $F_X(x) = \int_{-\infty}^x f_X(u) du$
- $\int_{-\infty}^{\infty} f_X(x) dx = 1$

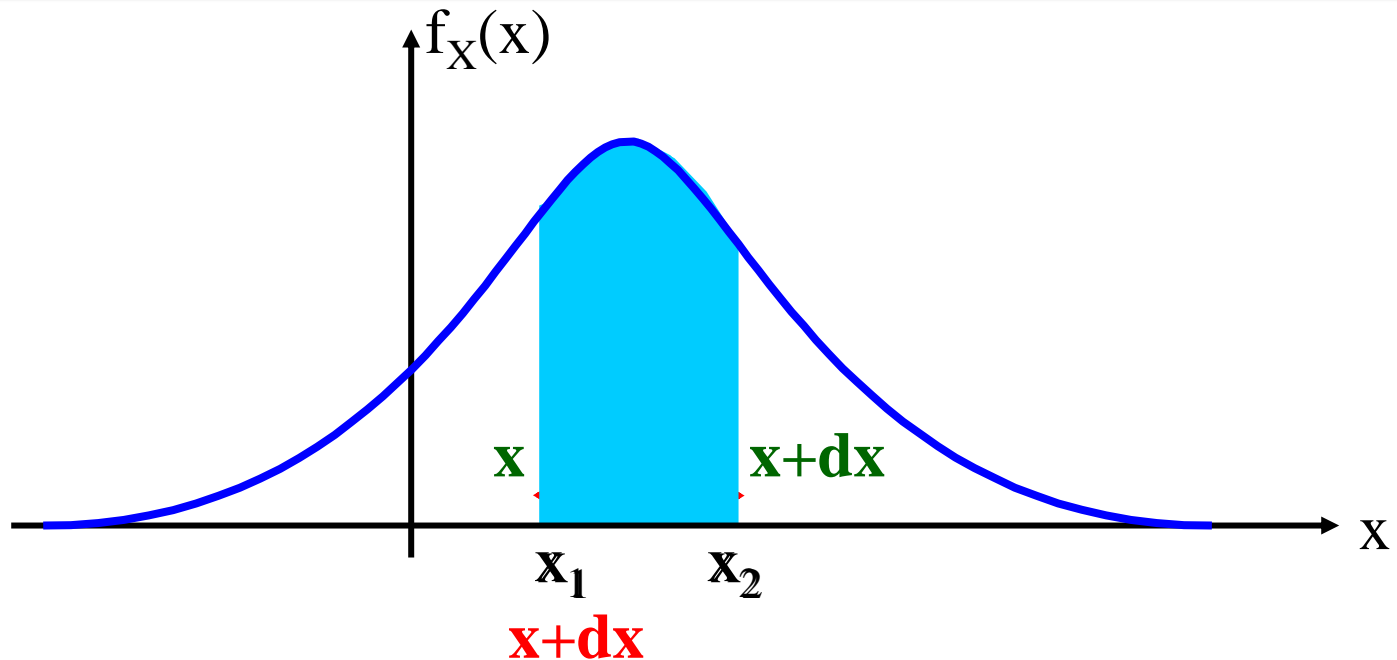
PDF and CDF



$$P[x_1 < X \leq x_2] = F_X(x_2) - F_X(x_1)$$

$$P[x_1 < X \leq x_2] = \int_{x_1}^{x_2} f_X(x) dx$$

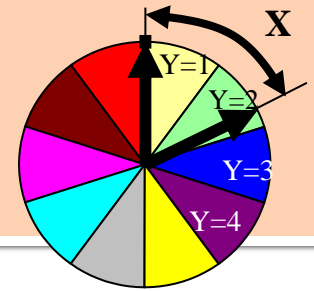
RV X & infinitesimal dx



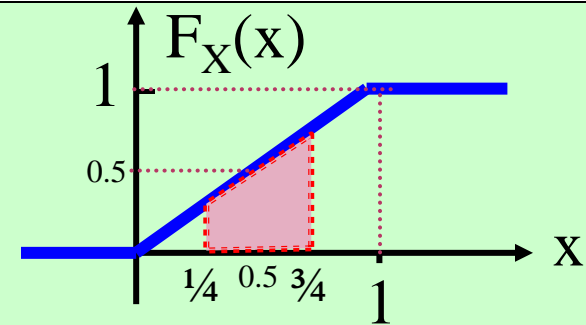
$$P[x_1 < X \leq x_2] = \int_{x_1}^{x_2} f_X(x) dx$$

$$\text{Approx: } P[x < X \leq x+dx] = f_X(x) dx$$

Example

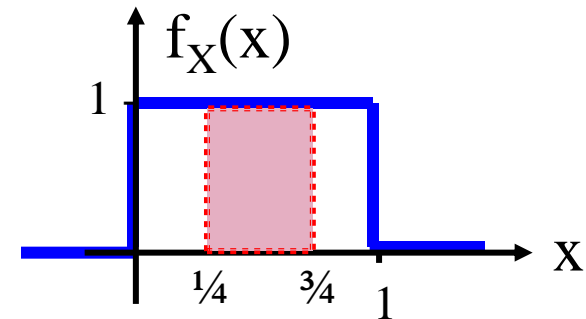


$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$



Find $f_X(x)$ and $P[1/4 < X \leq 3/4]$

$$f_X(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & \text{Otherwise} \end{cases}$$



$$P[1/4 < X \leq 3/4] = F_X(3/4) - F_X(1/4) = 3/4 - 1/4 = 1/2$$

$$P[1/4 < X \leq 3/4] = \int_{1/4}^{3/4} f_X(x) dx = \int_{1/4}^{3/4} dx = 1/2$$

Expected Values

For **Discrete** Random Variable:

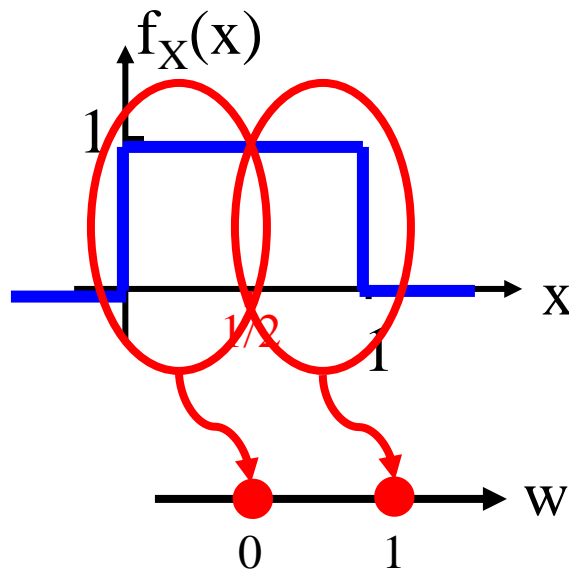
$$E[X] = \sum_{x \in S_X} x P_X(x)$$

For **Continuous** Random Variable:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

Function of RV

- A **function** of a continuous random variable is also a random variable
 - not necessary to be continuous
- Example



$$W = g(X) = \begin{cases} 0 & X \leq 1/2 \\ 1 & X > 1/2 \end{cases}$$

$W = \text{Discrete RV}$
 $S_W = \{0, 1\}$

Expected Values

For a function $g(X)$ of Random Variable X :

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Expected Value & Variance

- Find $E[X]$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Find $E[X^2]$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

- Find $\text{Var}[X]$

$$\text{Var}[X] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx$$

Theorem

- $E[X - \mu_X] = 0$
- $E[aX + b] = aE[X] + b$
- $\text{Var}[X] = E[X^2] - (E[X])^2$
- If X always takes value “a”, $\text{Var}[X] = 0$
- For $Y=X+b \rightarrow \text{Var}[Y] = \text{Var}[X]$
- For $Y=aX \rightarrow \text{Var}[Y] = a^2\text{Var}[X]$

Some Useful Continuous RVs

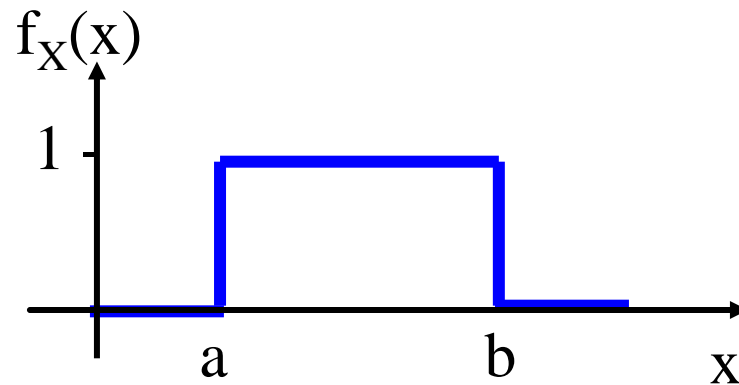
- Uniform
- Exponential
- Gaussian

Uniform Continuous RV

Definition:

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} 1/(\mathbf{b} - \mathbf{a}) & \mathbf{a} \leq \mathbf{x} < \mathbf{b} \\ \mathbf{0} & \text{Otherwise} \end{cases}$$

where $b > a$



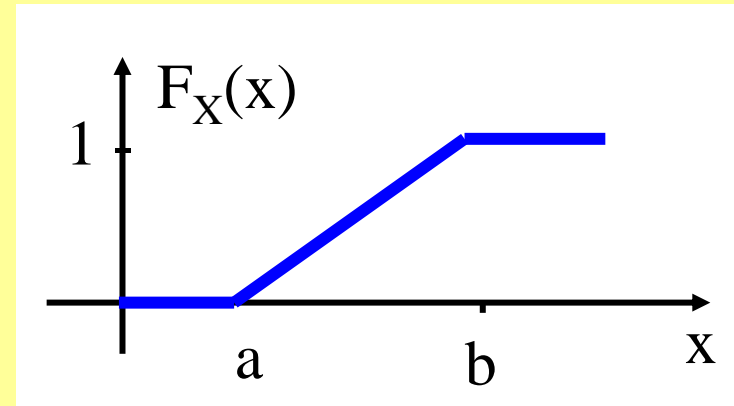
Uniform Continuous RV

Theorem:

- $$F_X(x) = \begin{cases} 0 & x \leq a \\ (x - a)/(b - a) & a < x \leq b \\ 1 & x > b \end{cases}$$

- $E[X] = (b + a)/2$

- $\text{Var}[X] = (b - a)^2/12$

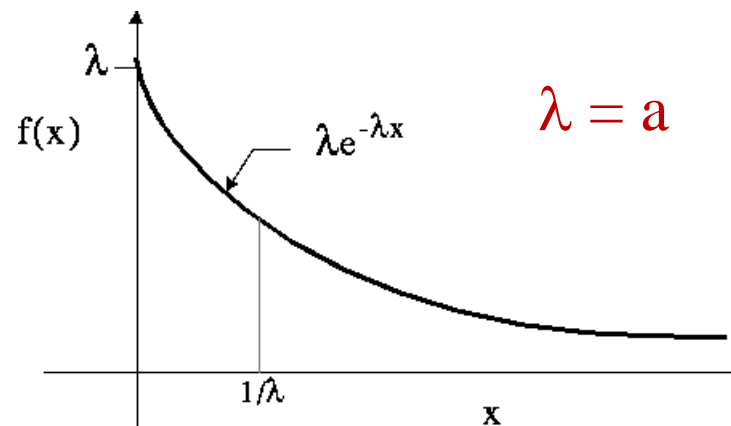


Exponential Continuous RV

Definition:

$$f_X(x) = \begin{cases} a e^{-ax} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

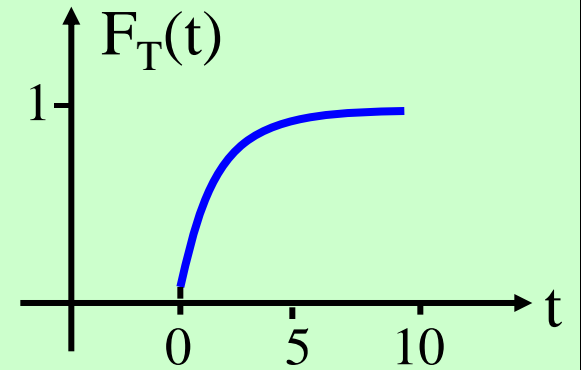
where $a > 0$



www.rzg.mpg.de/.../mc/node18.html

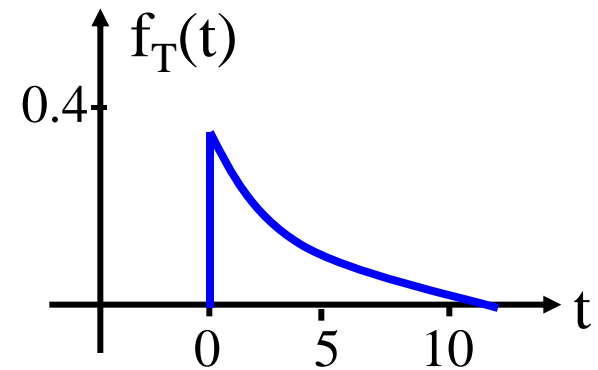
Exponential Example

$$F_T(t) = \begin{cases} 1 - e^{-t/3} & t \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$



Find PDF

$$\begin{aligned} f_T(t) &= \frac{dF_T(t)}{dt} \\ &= \begin{cases} (1/3) e^{-t/3} & t \geq 0 \\ 0 & \text{Otherwise} \end{cases} \end{aligned}$$



Exponential Example

Find E[T]

$$E[T] = \int_{-\infty}^{\infty} t f_T(t) dt$$

By Parts: $uv - \int v du$

Let $u = t$ $du = dt$

$dv = e^{-t/3} dt$ $v = -3e^{-t/3}$

$$= \int_0^{\infty} t (1/3)e^{-t/3} dt$$

$$= (1/3) \left[(t)(-3e^{-t/3}) \Big|_0^{\infty} - \int_0^{\infty} (-3e^{-t/3}) dt \right]$$

$$= -t e^{-t/3} \Big|_0^{\infty} - (1/3)(-3) \int_0^{\infty} e^{-t/3} dt$$

$$= 0 + \int_0^{\infty} e^{-t/3} dt$$

$$= 3$$

Exponential Example

Find Var[T] $\text{Var}[T] = E[T^2] - (E[T])^2$

$$\begin{aligned} E[T^2] &= \int_{-\infty}^{\infty} t^2 f_T(t) dt \\ &= (1/3) \int_0^{\infty} t^2 e^{-t/3} dt \\ &= -t^2 e^{-t/3} \Big|_0^{\infty} + \int_0^{\infty} (2t) e^{-t/3} dt \\ &= 2 \int_0^{\infty} t e^{-t/3} dt = 2(3E[T]) = 18 \end{aligned}$$

Exponential Example

$$\text{Var}[T] = E[T^2] - (E[T])^2$$

$$= 18 - 3^2 = 9 \text{ min}$$

$$\sigma_T = \sqrt{\text{Var}[X]} = 3 \text{ min}$$

Find Prob. that call duration is within 1 standard variation

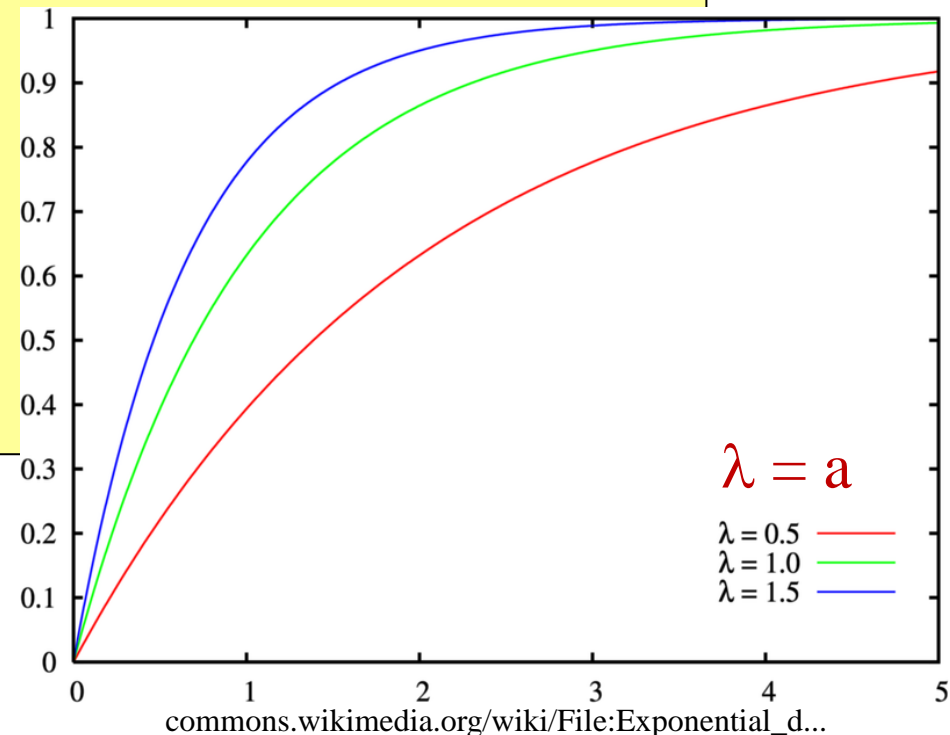
$$\begin{aligned} P[0 \leq T \leq 6] &= F_T(6) - F_T(0) \\ &= 1 - e^{-2} = 0.865 \end{aligned}$$

Exponential Continuous RV

Theorem:

$$F_X(x) = \begin{cases} 1 - e^{-ax} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

- $E[X] = 1/a$
- $\text{Var}[X] = 1/a^2$



Geometric & Exponential RV

Theorem:

If $X =$ Exponential RV with parameter a

Then $K = \lceil X \rceil$ is a Geometric RV
with parameter $p = 1 - e^{-a}$

$$P_K(k) = P[K=k] = P[k-1 \leq X < k]$$

$$= F_X(k) - F_X(k-1)$$

$$= 1 - e^{-ak} - (1 - e^{-a(k-1)})$$

$$= -e^{-ak} + e^{-a(k-1)}$$

$$= e^{-a(k-1)} \left(1 - \frac{e^{-ak}}{e^{-a(k-1)}}\right)$$

$$= e^{-a(k-1)} (1 - e^{-a})$$

$$= (1 - p)^{k-1} p \quad ; p = (1 - e^{-a})$$

Example

- Phone Company A:
 - 3 Baht / min.
 - With full min. charge
- Phone Company B:
 - 3 Baht / min.
 - With exact charge
- Let T = duration of call
- T : exponential with $\lambda = 1/3$

Example

- $E[T] = 1/a = 3$ min.
- R: money received per call
- For Company B:

$$E[R] = 3 E[T] = 9 \text{ Baht/Call}$$

- For Company A:

$$E[R] = 3 E[K]$$

where $K = \lceil T \rceil \rightarrow$ geometric with $p = 1 - e^{-1/3}$

$$E[R] = 3 (1/p) = 3 (3.53) = 10.59 \text{ Baht/Call}$$