LECTURE #5
MARKOV PROCESS

204528

Queueing Theory and Applications in Networks

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Outline

- Markov Processes
- Discrete Time Markov Chain
- Homogeneous, Irreducible, Transient/Recurrent, Periodic/Aperiodic
- Ergodic
- Stationary Probability
- Transient Behavior
- Birth-Death Process

Markov Processes

- X(t) is a Markov Process if it satisfies the **Markov (Memoryless) Property**

$$P\{X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \ldots, X(t_1) = x_1\}$$
$$= P\{X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n\}$$

Where $t_1 < t_2 < \ldots < t_{n-1} < t_n < t_{n+1}$

- X(t) only depends upon the current state
- The past history is summarized in the current state

Definition

- State Space (E)
  - Set of real numbers containing ranges of RV in a stochastic process

From Markov Processes …

- ***Discrete Time Markov Process***:
  State changes occur at integer points

- ***Continuous Time Markov Process***:
  State changes occur at arbitrarily time

From Markov Processes …

- ***Markov Chain***:
  Discrete state space Markov Process

- ***Discrete Time Markov Chain***:
  State (Discrete State) changes occur at integer points

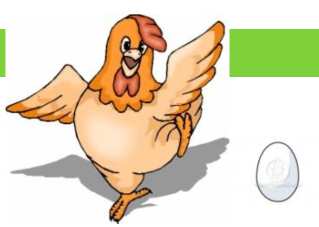
- ***Continuous Time Markov Chain***:
  State (Discrete State) changes occur at arbitrarily time

Markov Chain

- The easiest stochastic process
  - (IID) Independent and Identically distributed RV
- A discrete IID process
  - Markov Chain
  - Lack of memory (Future is independent of the past given the present)
- Named for probabilist "A.A.Markov"
  - 1907 finite state Markov Chain
  - 1930 infinite state by A.N.Kolmogorov

Andrey (Andrei) Andreyevich Markov Russia, Born:1856

Markov Chain Examples



- Egg game
  - Kai has 10 eggs, Ped has 10 eggs
  - Toss a die
    - If {1,2,3} occurs → Ped gives one egg to Kai
    - If {4,5,6} occurs → Kai gives one egg to Ped
  - Game ends when 20-0 or 0-20 (Kai or Ped has all eggs)
- Markov chain
  - The fifth play does not depend on the second play

Discrete Time Markov Chains

- One can stay in a ***Discrete state (position)*** and is permitted to change state at ***Discrete time***.

Discrete Time Markov Chains

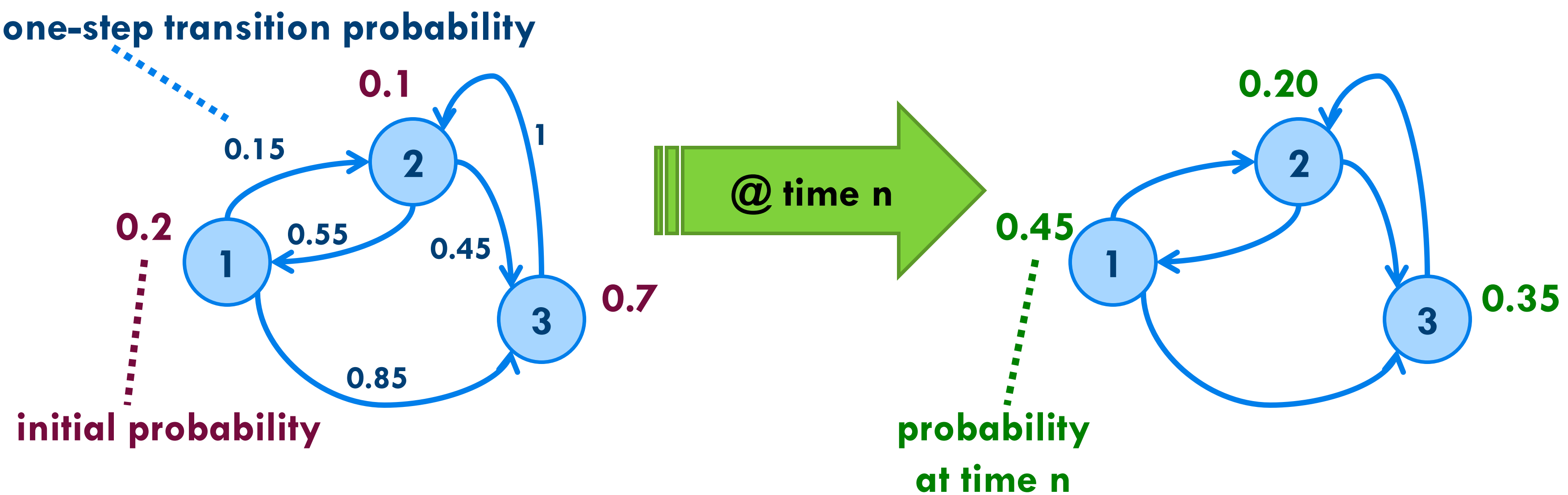
$$P\{X_n = j \mid X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}$$
$$= P\{X_n = j \mid X_{n-1} = i_{n-1}\}$$

Where n = 1,2,3,…

- X_n: The system is in state *j* at time *n*
- The system can begin at ***state 0*** with ***initial probability*** $P[X_0 = x]$
- $P\{X_n = j \mid X_{n-1} = i_{n-1}\}$ is the ***one-step transition probability***

Discrete Time Markov Chains

- From ***initial probability*** and ***one-step transition probability***,
  - we can find ***probability of being in various states at time n***



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Homogeneous Markov Chain

- If ***transition probabilities*** are independent of n, it is called ***Homogeneous Markov Chain.***
- Let $p_{ij} \equiv P[X_n = j \mid X_{n-1} = i]$
- We are in ***state i*** and going to be in ***state j*** in the next step
- The state transition prob. will only depend on the ***initial probability*** and ***transition probability***, regardless of transition time.

Markov Matrix

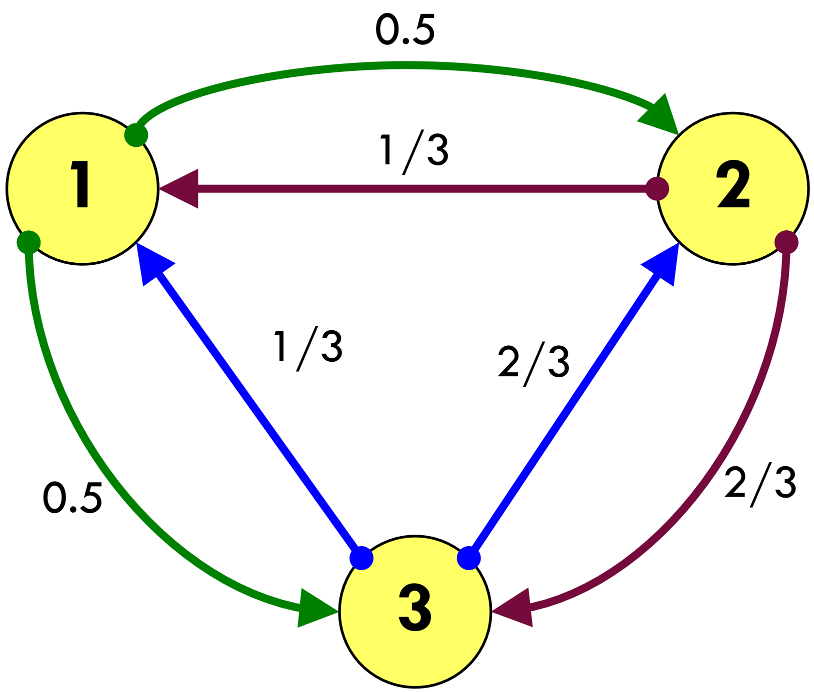
- A nonnegative square matrix
- Sum value of each row = 1

$$P = \begin{pmatrix} 0 & 0.3 & 0.7 \\ 0.1 & 0.2 & 0.7 \\ 0.8 & 0.2 & 0 \end{pmatrix} \begin{matrix} \rightarrow \text{Sum} = 1 \\ \rightarrow \text{Sum} = 1 \\ \rightarrow \text{Sum} = 1 \end{matrix}$$

Example 1: Jump Game

- A boy will jump among Plate#1, 2, & 3
- Every time he will change the plate
  - When he is in Plate #1, uses a coin tossing mechanism
    - If "head" → he jumps to Plate #2
    - If "tail" → he jumps to Plate #3
  - After that, uses a die tossing mechanism
    - If {1,2} → he will jump back to plate #1
    - Otherwise → he will jump to another plate
- Draw a Markov chain

Example 1: Jump Game



$$P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 1/3 & 0 & 2/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

Markov Matrix

State space E = {1,2,3}

Random variable X_n = 1 or 2 or 3

Example 2: Weather Forecast

- Observe the weather in an area
  - Typically longer periods of rainy or dry(sunshine) days
  - Rain and sunshine are same relative frequency over the entire year
- Sometimes claimed that the best way to predict tomorrow's weather
  - Guess that, tomorrow is as same as today
  - If assume predicting will **be correct in 75%** of the cases (regardless rain or sunshine)

O. Häggström (2002) *Finite Markov Chains and Algorithmic Applications*. CU Press, Cambridge

Example 2: Weather Forecast

- The weather can be easily modeled by **a Markov chain**
  - The state space consists of the two states (1= rain) and (2 = sunshine)
  - The transition matrix is

$$P = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}$$

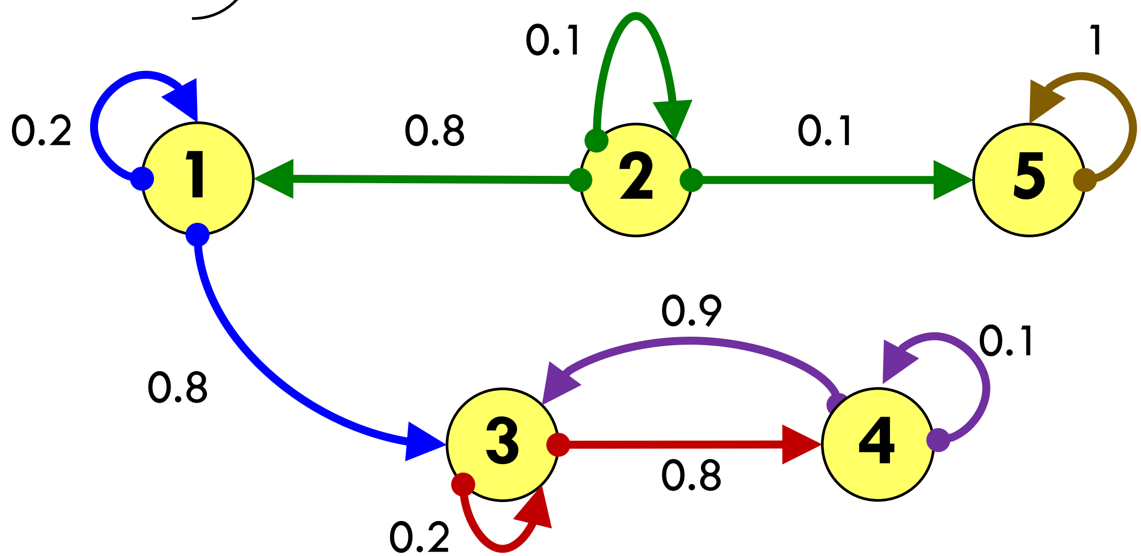
  - In areas where sunshine is much more common than rain such as Bangkok,
    - ore realistic transition matrix would be

$$P = \begin{pmatrix} 0.30 & 0.70 \\ 0.15 & 0.85 \end{pmatrix}$$

Example 3

$$P = \begin{pmatrix} 0.2 & 0 & 0.8 & 0 & 0 \\ 0.8 & 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Draw a Markov Chain



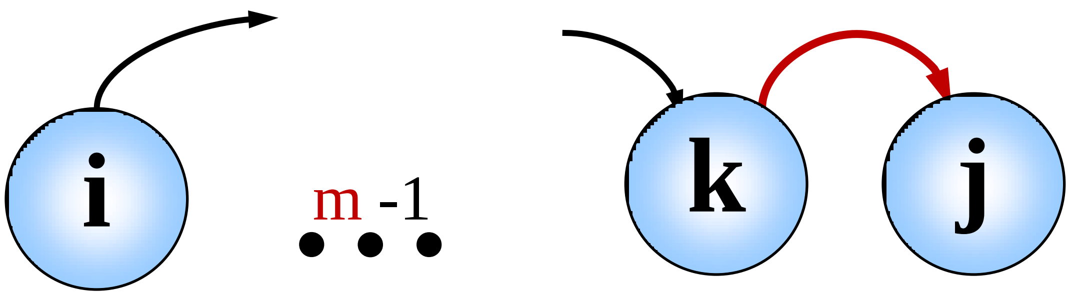
Multistep Transition

Homogeneous Markov Chain

- m-step transition probabilities are

$$p_{ij}^{(m)} \equiv P[X_{n+m} = j \mid X_n = i]$$

$$= \sum_{\forall k} p_{ik}^{(m-1)} \; p_{kj} \qquad m = 2,3,\ldots$$

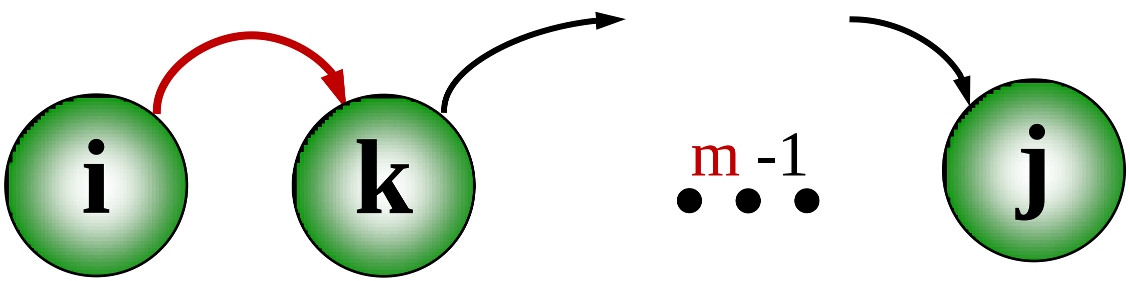


Multistep Transition

Homogeneous Markov Chain

$$p_{ij}^{(m)} \equiv P[X_{n+m} = j \mid X_n = i\ ]$$

$$= \sum_{\forall k} p_{ik}\ p_{kj}^{(m-1)} \qquad m = 2,3,\ldots$$



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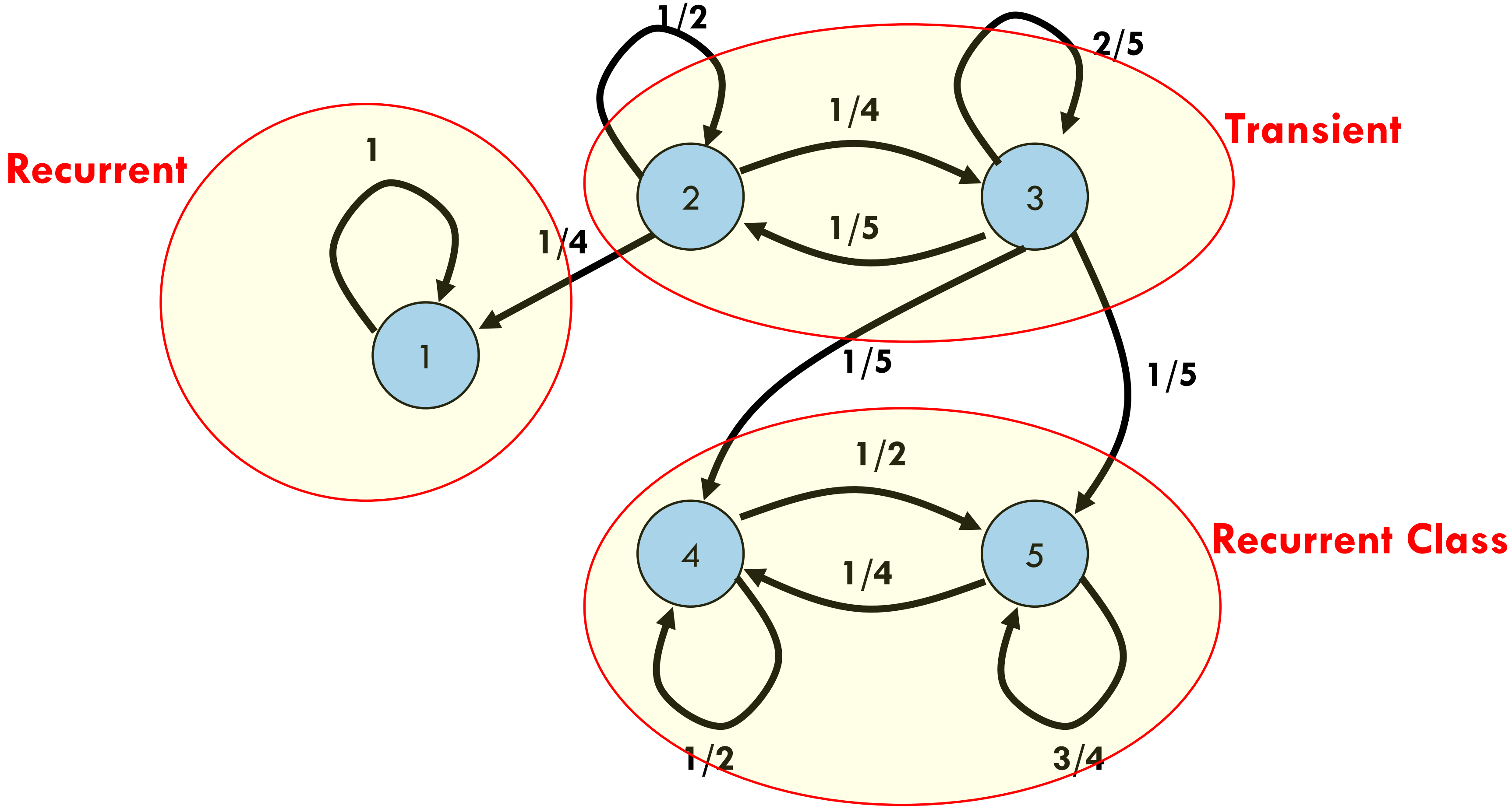
Type of States

- Transient State
  - Starting @ state E, it will eventually leave state E and never return
- Recurrent State
  - Starting @ state E, it will **continuously** reoccur (come back to state E)

Transient or Recurrent States



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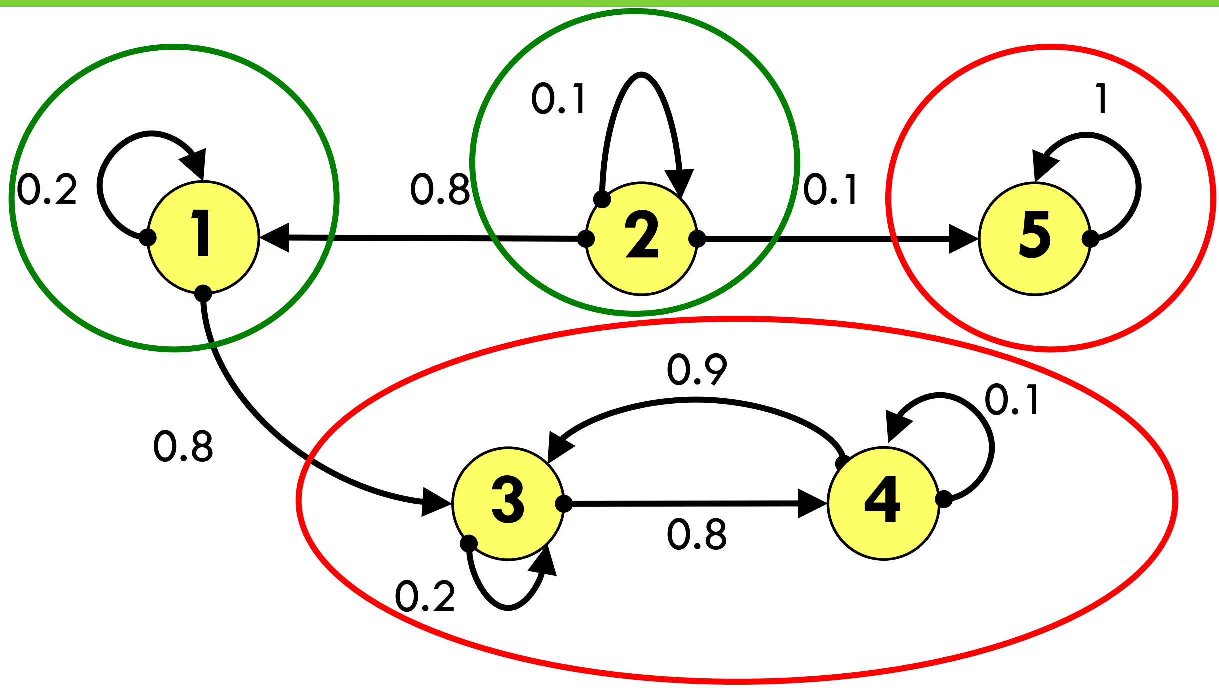
Example 3 (Revisit)



Identify

Transient states = ? {1} , {2}

Recurrent states = ? {5} , {3,4}

Transient or Recurrent States

- $f_j^{(n)}$ = P[the process first returns to state **j** after leaving state **j** in **n** steps]
- f_j = P[the process returns to state **j** after leaving state **j**]

$$f_j = \sum_{n=1}^{\infty} f_j^{(n)}$$

- M_j = Mean recurrence time of state **j**

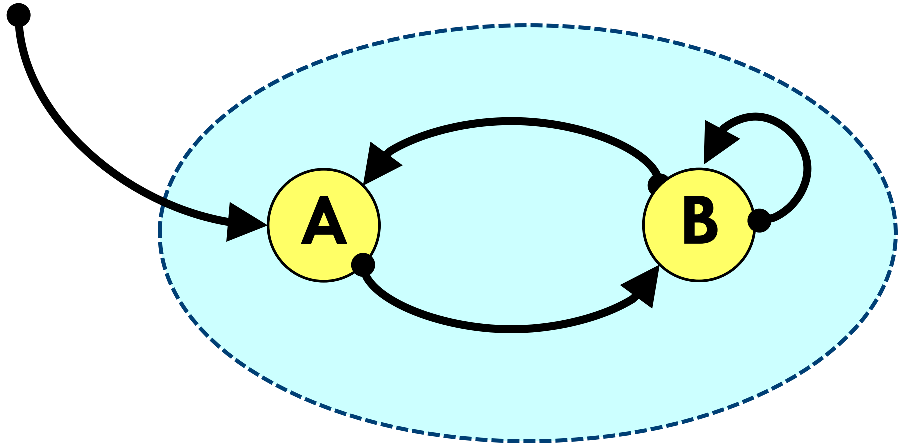
$$M_j = \sum_{n=1}^{\infty} n f_j^{(n)}$$

Transient or Recurrent States

- If $f_j < 1$
  - State E_j is called “**Transient State**”
- If $f_j = 1$
  - State E_j is called “**Recurrent State**”
  - If $M_j = \infty$
    - State E_j is called “**Recurrent Null State**”
  - If $M_j < \infty$
    - State E_j is called “**Recurrent Nonnull State**”

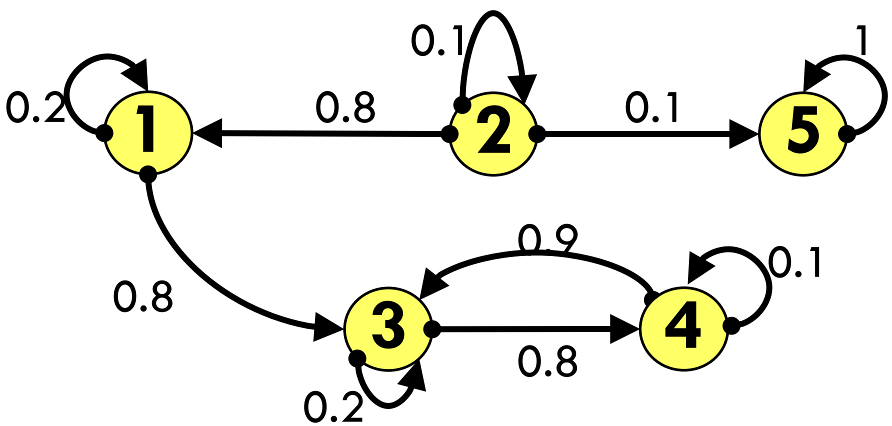
A Closed Set

- A set that once the Markov chain has entered the set → it cannot leave the set



Revisit: Example 3

$$P = \begin{pmatrix} 0.2 & 0 & 0.8 & 0 & 0 \\ 0.8 & 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



Identify is this a closed set ?

{1}	**Not Closed**
{1,2}	**Not Closed**
{1,2,3}	**Not Closed**
{2,3}	**Not Closed**
{4,5}	**Not Closed**
{3,4,5}	
{5}	
{1,2,3,4,5}	

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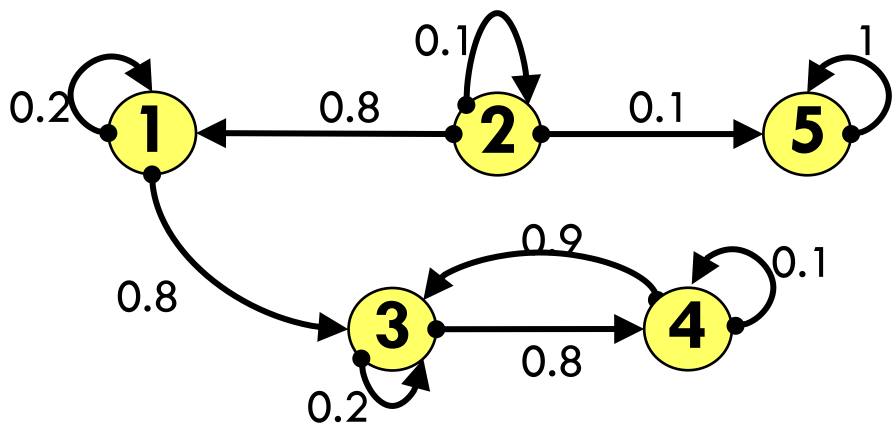
Revisit: Example 2

$$P = \begin{pmatrix} 0.2 & 0 & 0.8 & 0 & 0 \\ 0.8 & 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



Identify is this a closed set ?

{1}	**Not Closed**
{1,2}	**Not Closed**
{1,2,3}	**Not Closed**
{2,3}	**Not Closed**
{4,5}	**Not Closed**
{3,4,5}	**Closed**
{5}	**Closed**
{1,2,3,4,5}	**Closed**

A Closed Set

$$P = \begin{pmatrix} 0.2 & 0 & 0.8 & 0 & 0 \\ 0.8 & 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

{1,2,3,4,5} = Closed Set → It <u>can be reduced</u> to a smaller closed set {3,4,5}

{3,4,5} = Closed Set → It <u>can be reduced</u> to a smaller closed set {3,4} and {5}

{3,4} = Closed Set
{5} = Closed Set → It <u>can **NOT** be reduced</u> to any smaller closed set

Irreducible Set

$$P = \begin{pmatrix} 0.2 & 0 & 0.8 & 0 & 0 \\ 0.8 & 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Reducible Set

{1,2,3,4,5} = Closed Set → It can be reduced to a smaller closed set {3,4,5}

Reducible Set

{3,4,5} = Closed Set → It can be reduced to a smaller closed set {3,4} and {5}

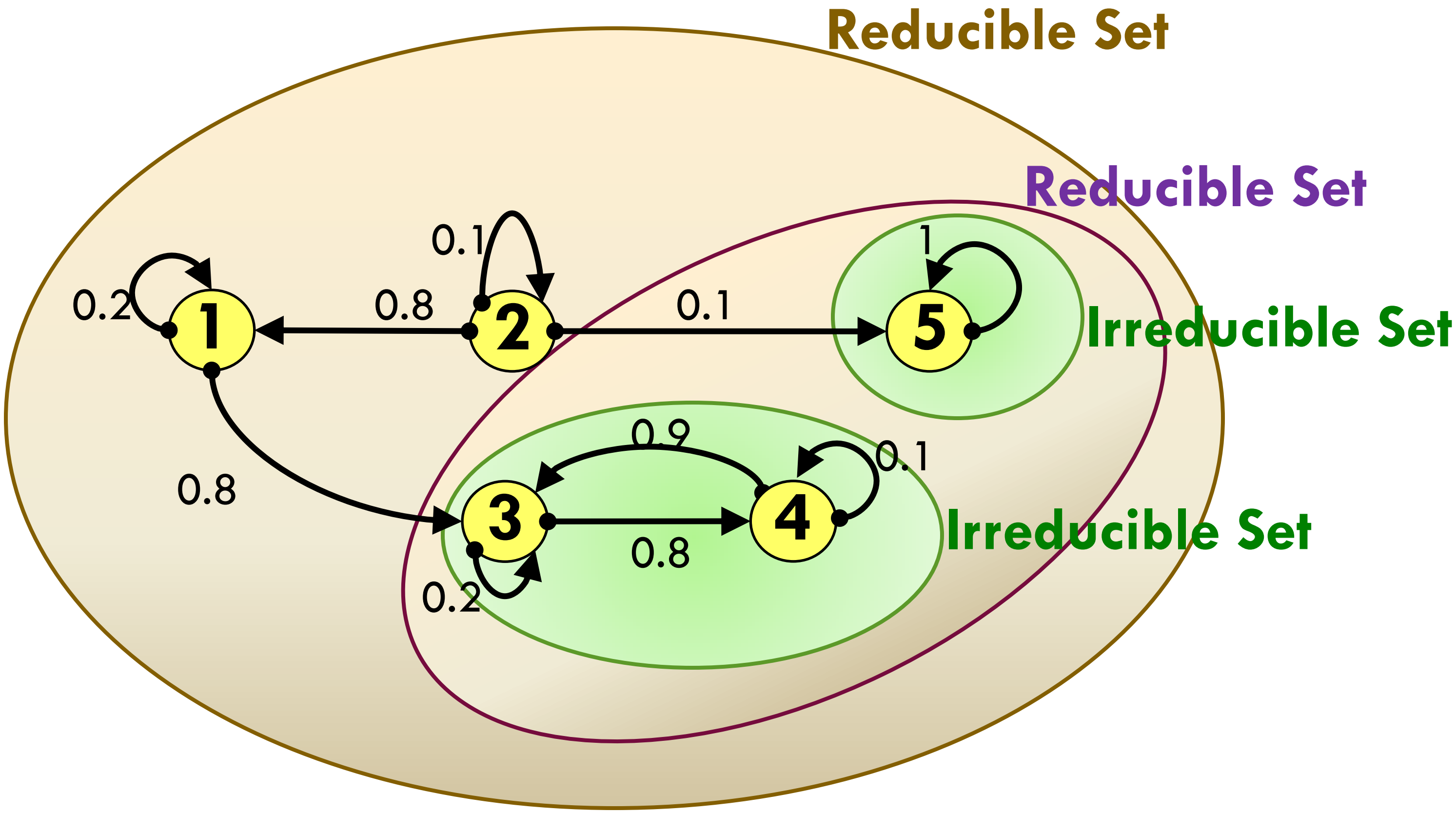
Irreducible Set

{3,4} = Closed Set

{5} = Closed Set → It can NOT be reduced to any smaller closed set

{5}: Absorbing State

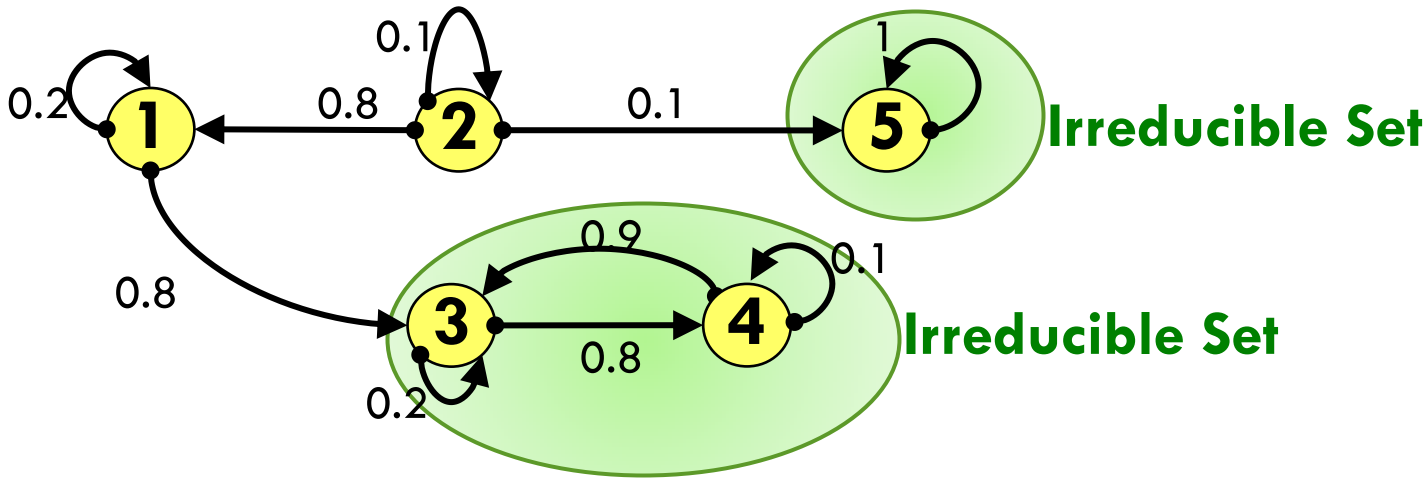
Irreducible Set



Irreducible Markov Chain

- A Markov Chain is ***irreducible*** if every state can be reached from every other state in a ***finite*** number of steps.

$$p_{ij}^{(m_0)} > 0 \qquad \text{for } m_0 = \text{integer}$$



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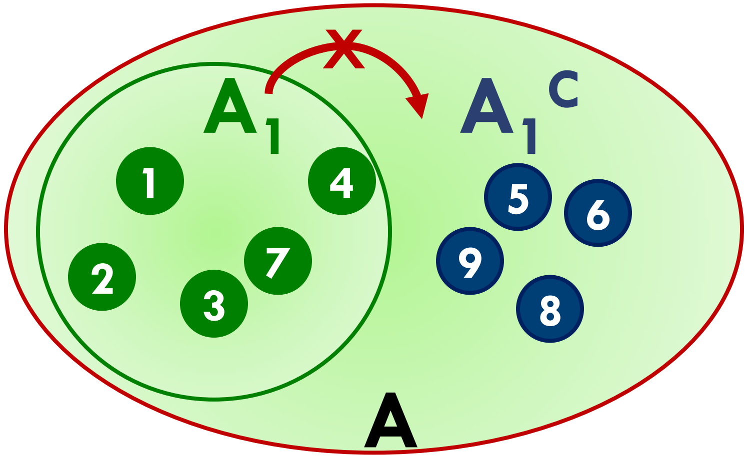
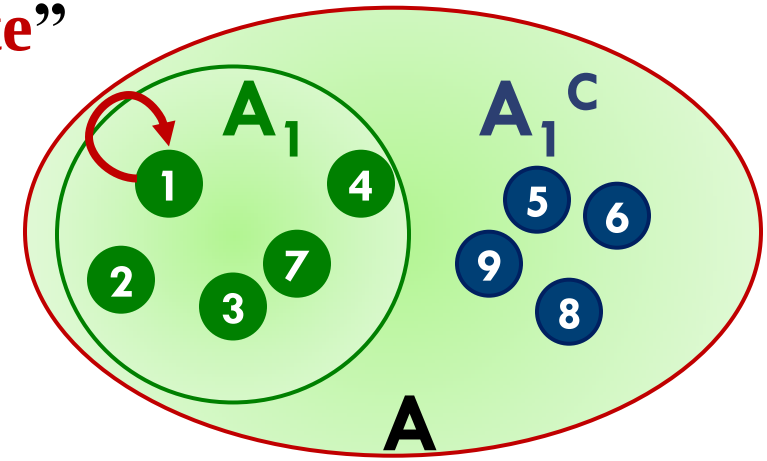
Not Irreducible Markov Chain

- Case 1
  - For A = set of all states in a Markov chain
  - $A_1 \subset A$
  - If <u>no</u> one-step transition from state A_1 to A_1^c
  - A_1 is defined as "**Closed**"



Not Irreducible Markov Chain

- Case 2
  - For A = set of all states in a Markov chain
  - $A_1 \subset A$
  - If A_1 consists of one or more state E_i that once get in state E_i, the process cannot move to any other states
  - E_i is called "**Absorbing State**"
  - $p_{ii} = 1$



Note on Irreducible

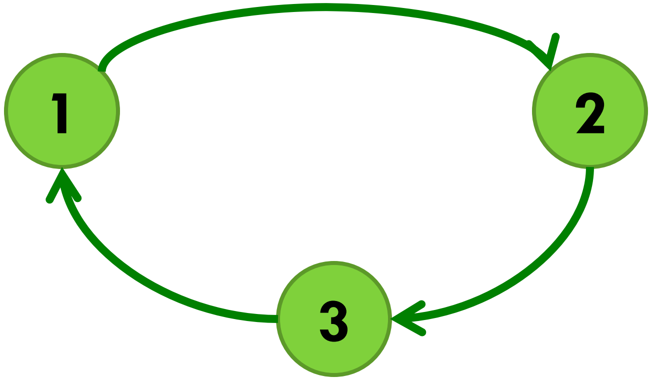
- All states within an irreducible set are of the same classification
  - If one state is transient → all transient
  - If one state is recurrent → all recurrent
- Irreducible sets
  - ➔ Communication between states
  - ➔ Communication must be both ways
  - ➔ But does not have to be in one step

Periodic or Aperiodic

- Recurrent State
- Let β = integer
- If the **only possible steps** that the process returns to state E_i are $\beta, 2\beta, 3\beta, \ldots$
  - If $\beta > 1$ and β is the largest integer
    - State E_i is called "**Periodic**"
    - The **recurrence time** for state E_j has period β
  - If $\beta = 1$
    - State E_i is called "**Aperiodic**"

Example

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

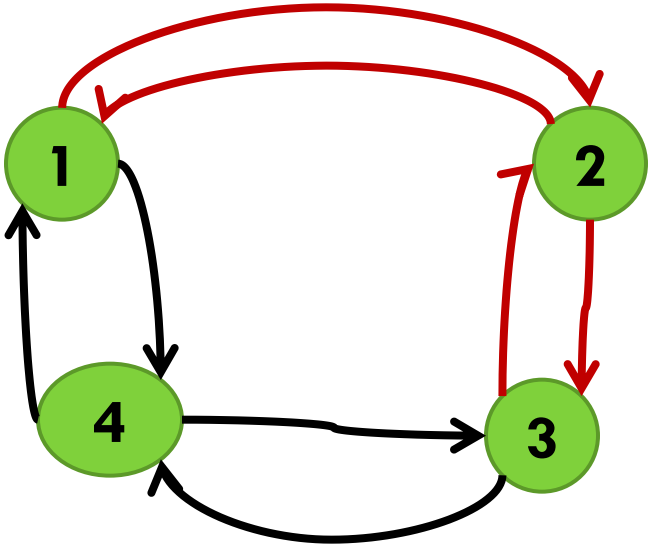


- Starting from state 1
  - It returns to state 1 for every 3 steps
  - Same as other two states.
- The chain is therefore **Periodic**

Modified from:
http://www.wikicoursenote.com/wiki/Again_on_Markov_Chain

Example

$$P = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{pmatrix}$$



- Starting from state 1
  - It returns to state 1 for every **multiple of 2 steps**
  - Same as other states.
- The chain is therefore **Periodic**

Modified from:
http://www.wikicoursenote.com/wiki/Again_on_Markov_Chain

Ergodicity

- E_j = **Ergodic** if
  - E_j = ***Aperiodic*** and ***Recurrent Nonnull***

    (recurrent nonnull)
  - $f_j = 1$, $M_j < \infty$, and $\beta = 1$

    (recurrent) (Aperiodic)
- A Markov Chain is **ergodic**
  - If **all** states of a Markov Chain are **ergodic**
  - If number of states is **finite** and **all** states of a Markov Chain are **aperiodic**, and **irreducible**

Theorem 1

- The states of **an irreducible** Markov Chain are either
  - all transient or
  - all recurrent nonnull or
  - all recurrent null
- If periodic, then all states have the same period β

Definition

- Let $\pi_j^{(n)}$ = P[finding the system in state E_j at the n^{th} step]

$$\pi_j^{(n)} = P[X_n = j]$$

- Let π_j = Stationary Probability

  = P[being in state j at arbitrarily time]

  = The limiting state probabilities

Theorem 2

- In an irreducible and aperiodic, homogeneous Markov Chain,
- the limiting state probabilities [π_j] always exist and are independent of the initial state probability distribution [$\pi_j^{(0)}$]

$$\pi_j = \lim_{n \to \infty} \pi_j^{(n)}$$

Theorem 2

- **Either**
- Case (a)
  - All states are transient or
  - All states are recurrent null

  ➔ $\pi_j = 0 \quad \forall j$

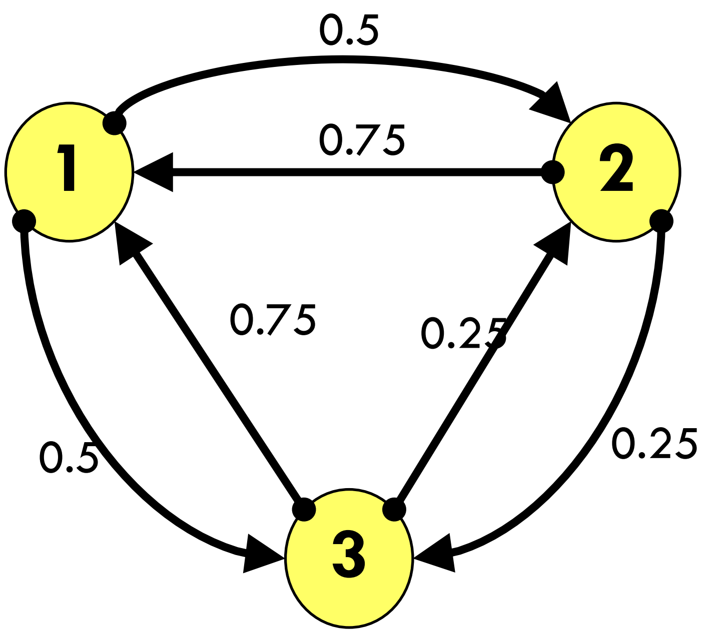
  ➔ No stationary distribution exist
- **Or** Case (b)
  - All states are recurrent nonnull

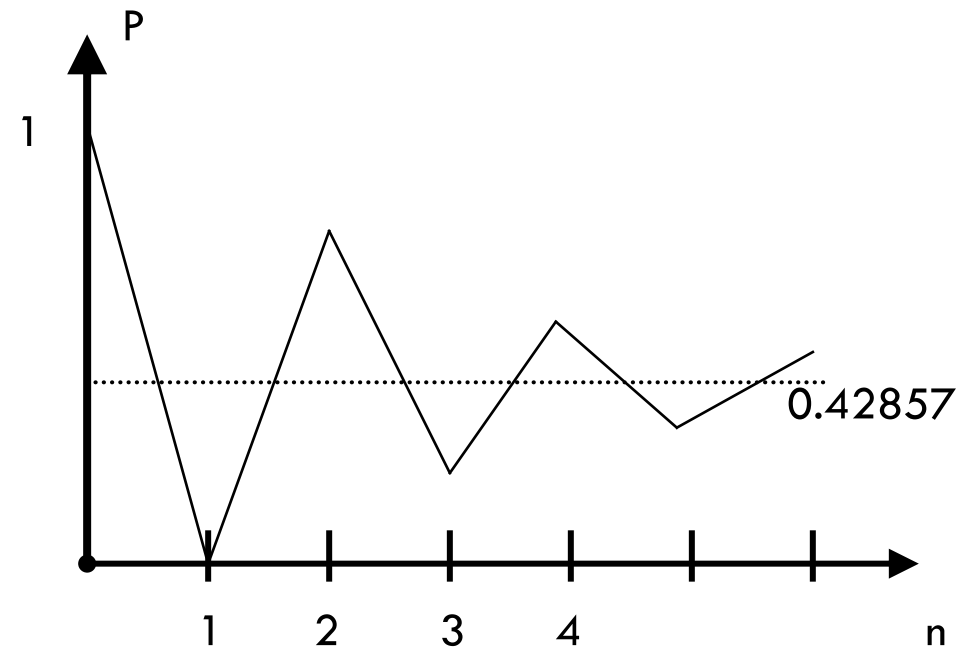
  ➔ $\pi_j > 0 \quad \forall j$

  ➔ Stationary distribution exist ➔ $\pi_j = 1 / M_j$

Steady-State Behavior
(Limiting condition)

P[be in state 1 @ time n | in state 1 @ t =0]





$$\lim_{n \to \infty} Pr\{ X_n = 1 \mid X_0 = 1 \} = 0.42857$$

Same as

$$\lim_{n \to \infty} Pr\{ X_n = 1 \mid X_0 = 2 \} = 0.42857$$

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To solve for π_j

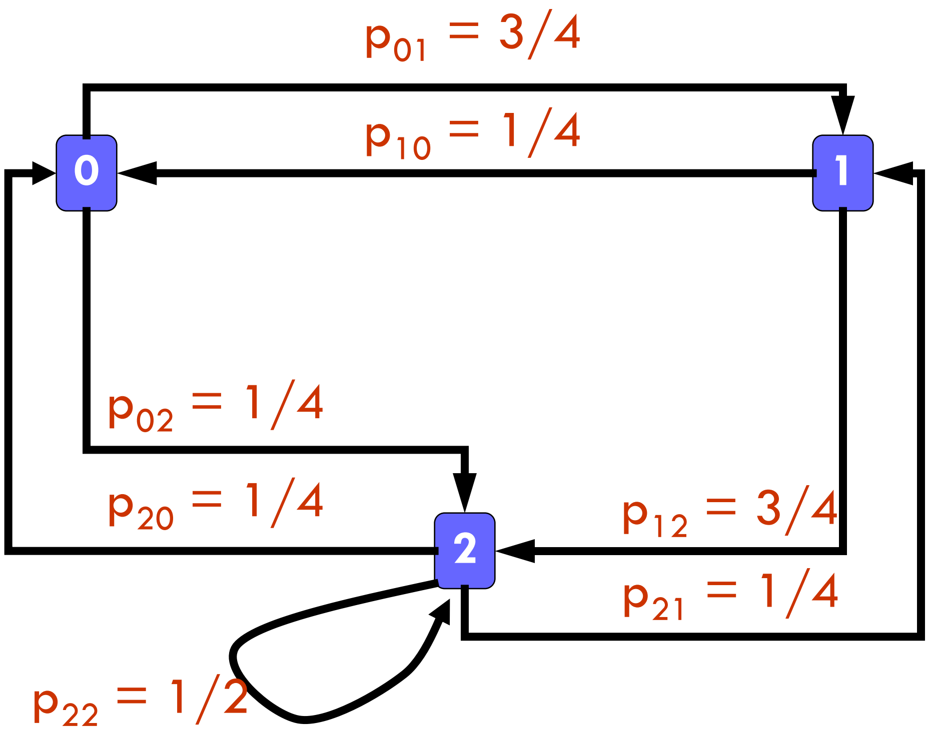
Balance Equations: $$\pi_j = \sum_i \pi_i p_{ij}$$

(Linear dependency)

Normalization condition: $$1 = \sum_i \pi_i$$

Markov Chain Example

- Driving from town to town



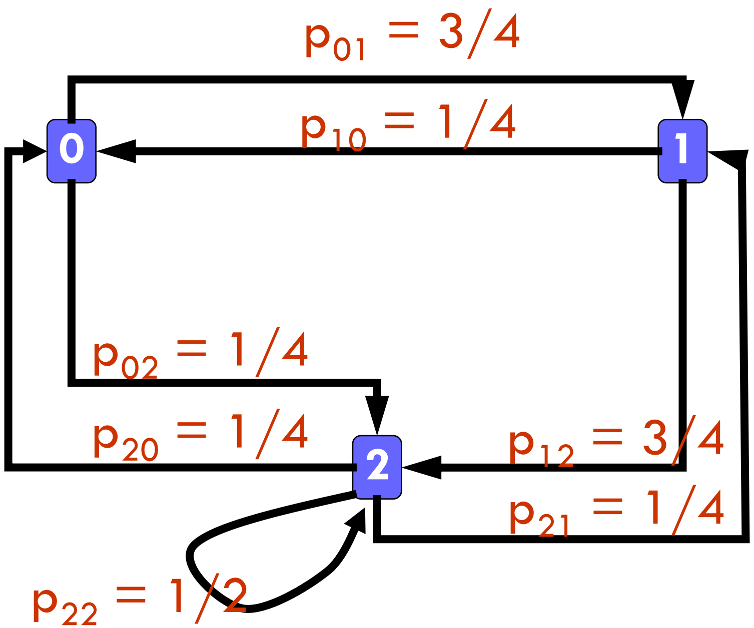
Markov Chain Example

- Let P = Transition probability matrix

  $= [p_{ij}]$
- Let $\pi = [\pi_0, \pi_1, \pi_2, \ldots]$
- From Balance equation

  $\pi = \pi P$

Markov Chain Example



$$P = \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/4 & 0 & 3/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

Markov Chain Example

$$\pi = \pi P$$

$$\pi_0 = 0 \quad \pi_0 + 1/4\ \pi_1 + 1/4\ \pi_2$$

$$\pi_1 = 3/4\ \pi_0 + 0 \quad \pi_1 + 1/4\ \pi_2$$

$$\pi_2 = 1/4\ \pi_0 + 3/4\ \pi_1 + 1/2\ \pi_2$$

$$1 = \pi_0 + \pi_1 + \pi_2$$

Markov Chain Example

Solution:

$$\pi_0 = 0.20$$
$$\pi_1 = 0.28$$
$$\pi_2 = 0.52$$

- This is the stationary (equilibrium) state probability
- This is the ergodic Markov Chain
  - Finite number of states
  - Irreducible (recurrent)

Transient Behavior

- We want to know the probability of finding the process in state E_j at time n
- $\pi^{(n)} = [\pi_0^{(n)}, \pi_1^{(n)}, \pi_2^{(n)}, \ldots]$
- From Transition Probability **P**
  - We can calculate: $\pi^{(1)} = \pi^{(0)}P$

    $\pi^{(n)} = \pi^{(n-1)}P$
  - By recursive: $\pi^{(n)} = \pi^{(0)}P^n$

Transient Behavior

- From stationary probability:

$$\pi = \lim_{n \to \infty} \pi^{(n)}$$

- From $\pi^{(n)} = \pi^{(n-1)}P$

$$\lim_{n \to \infty} \pi^{(n)} = \lim_{n \to \infty} \pi^{(n-1)} P$$

$$\pi = \pi P$$

- Note: The solution $\boldsymbol{\pi}$ is independent of $\pi^{(0)}$

Transient Behavior

$$\pi^{(0)} = [\ 1, 0, 0]$$

n	0	1	2	3	∞
$\pi_0^{(n)}$	1	0	0.25	0.187	0.20
$\pi_1^{(n)}$	0	0.75	0.062	0.359	0.28
$\pi_2^{(n)}$	0	0.25	0.688	0.454	0.52

$$\pi^{(0)} = [\ 0, 0, 1]$$

n	0	1	2	3	∞
$\pi_0^{(n)}$	0	0.25	0.187	0.203	0.20
$\pi_1^{(n)}$	0	0.25	0.313	0.266	0.28
$\pi_2^{(n)}$	1	0.50	0.500	0.531	0.52

Birth-Death Process

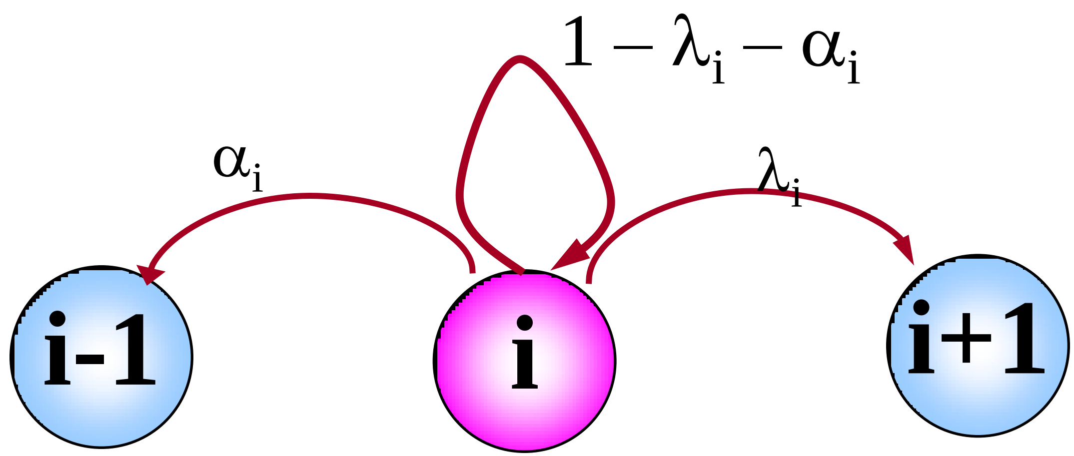
- A Markov Process
- Homogeneous, aperiodic, and irreducible
- Discrete time / Continuous time
- State changes can only happen between neighbors

Birth-Death Process

- Size of population
  - System is in state E_k when consists of ***k*** members
  - Changes in population size occur by at most one
  - Size increased by one ➔ "*Birth*"
  - Size decreased by one ➔ "*Death*"
- Transition probabilities p_{ij} do not change with time

Birth-Death Process

$$p_{ij} = \begin{cases} \alpha_i & j = i - 1 \\ 1 - \lambda_i - \alpha_I & j = i \\ \lambda_i & j = i + 1 \\ 0 & \text{Otherwise} \end{cases}$$



Birth-Death Process

- α_i = death (less one in population size)
- $\alpha_0 = 0$ (no population → no death)
- λ_i = birth (increase one in population)
- $\lambda_i > 0$ (birth is allowed)
- Pure Birth = no decrement, only increment
- Pure Death = no increment, only decrement

Queueing Theory Model

- Population = customers in the queueing system
- Death = a customer departure from the system
- Birth = a customer arrival to the system

Transition matrix

$$P = \begin{bmatrix} 1 - \lambda_0 & \lambda_0 & 0 & 0 & 0 & 0 & \cdots \\ \alpha_1 & 1 - \lambda_1 - \alpha_1 & \lambda_1 & 0 & 0 & 0 & \\ 0 & \alpha_2 & 1 - \lambda_2 - \alpha_2 & \lambda_2 & & & \\ 0 & & \cdots & & & & \\ 0 & & & & & & \\ & & & \alpha_i & 1 - \lambda_i - \alpha_i & \lambda_i & \\ \cdots & & & & & & \end{bmatrix}$$