

LECTURE #4

STOCHASTIC PROCESS

204528

Queueing Theory and
Applications in Networks

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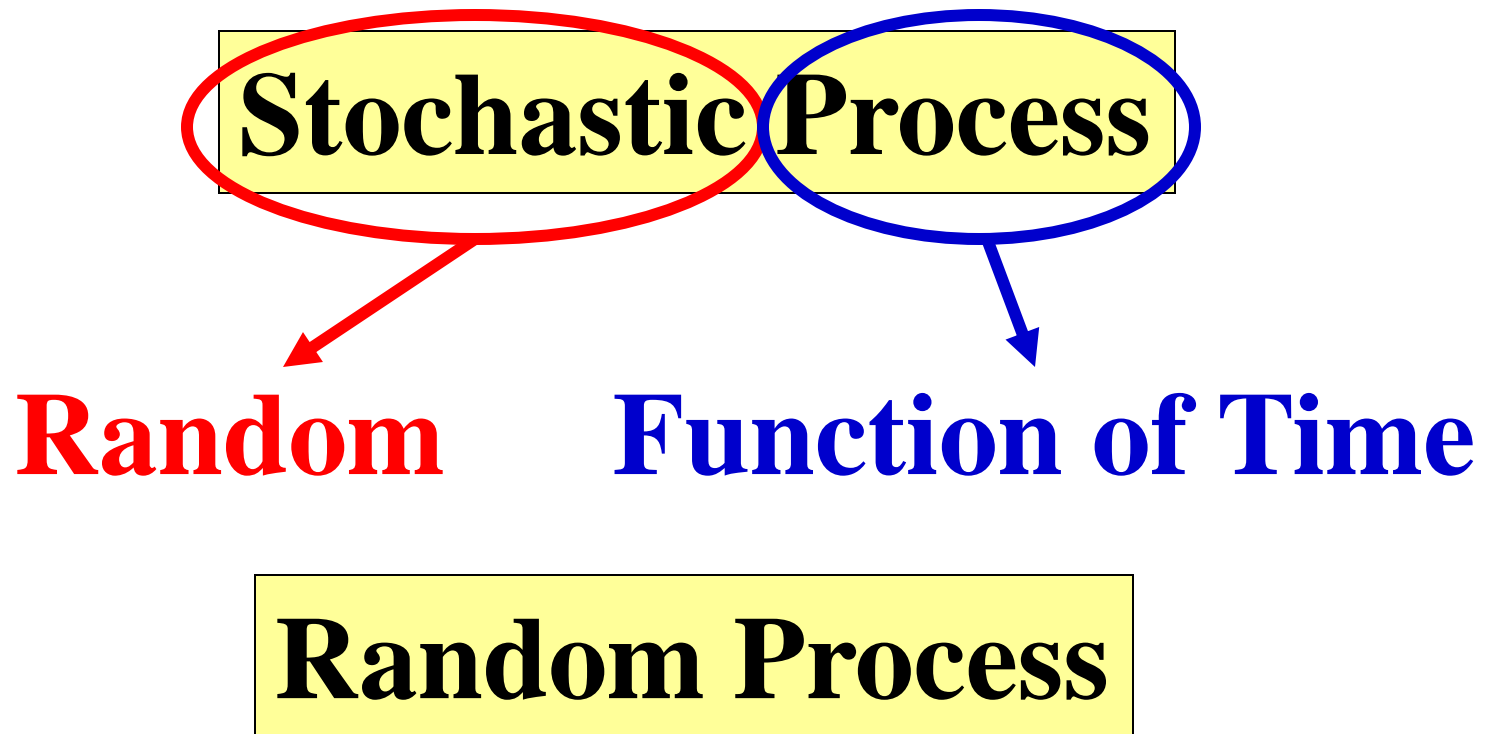
Outline

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- Stochastic Process
- Counting Process
- Poisson Process
- Brownian Motion Process
- Autocovariance and Autocorrelation
- Random Sequence
- Stationary Process
- Wide-sense Stationary Process

Definition

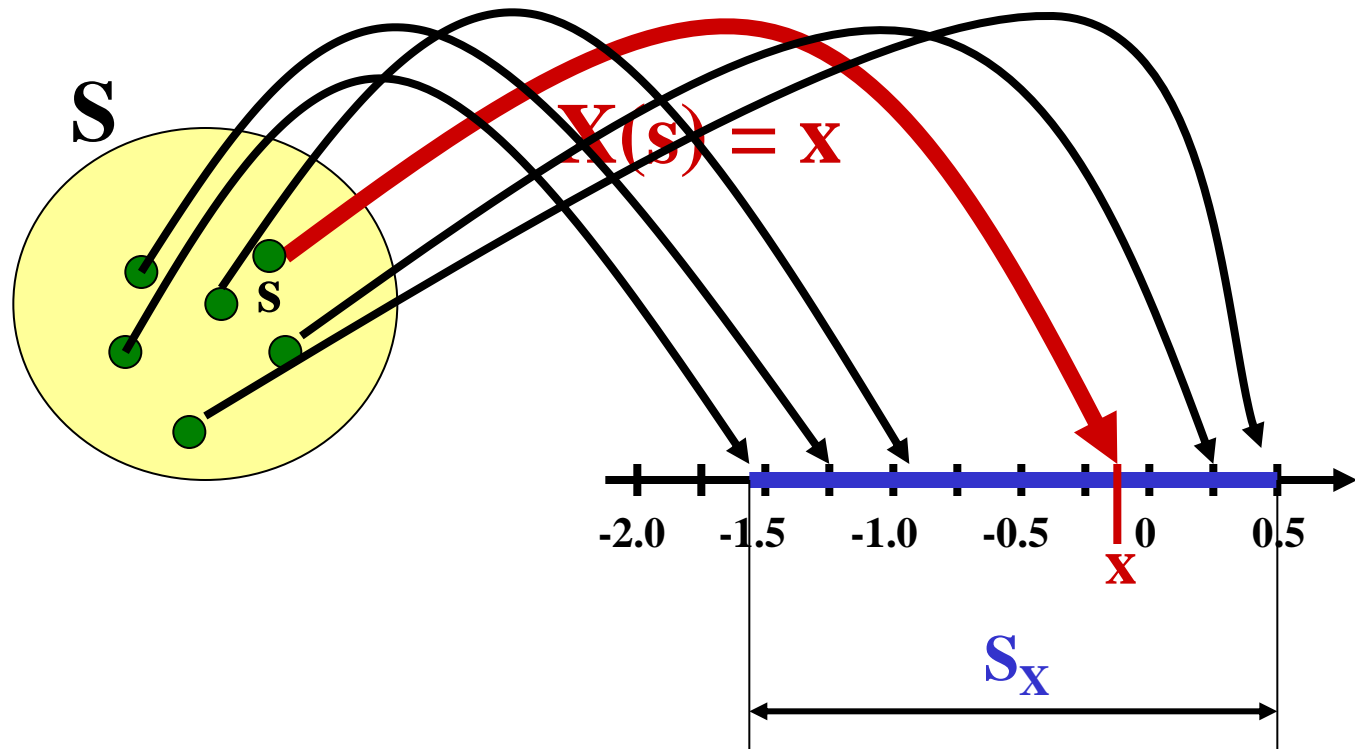
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Random Variable

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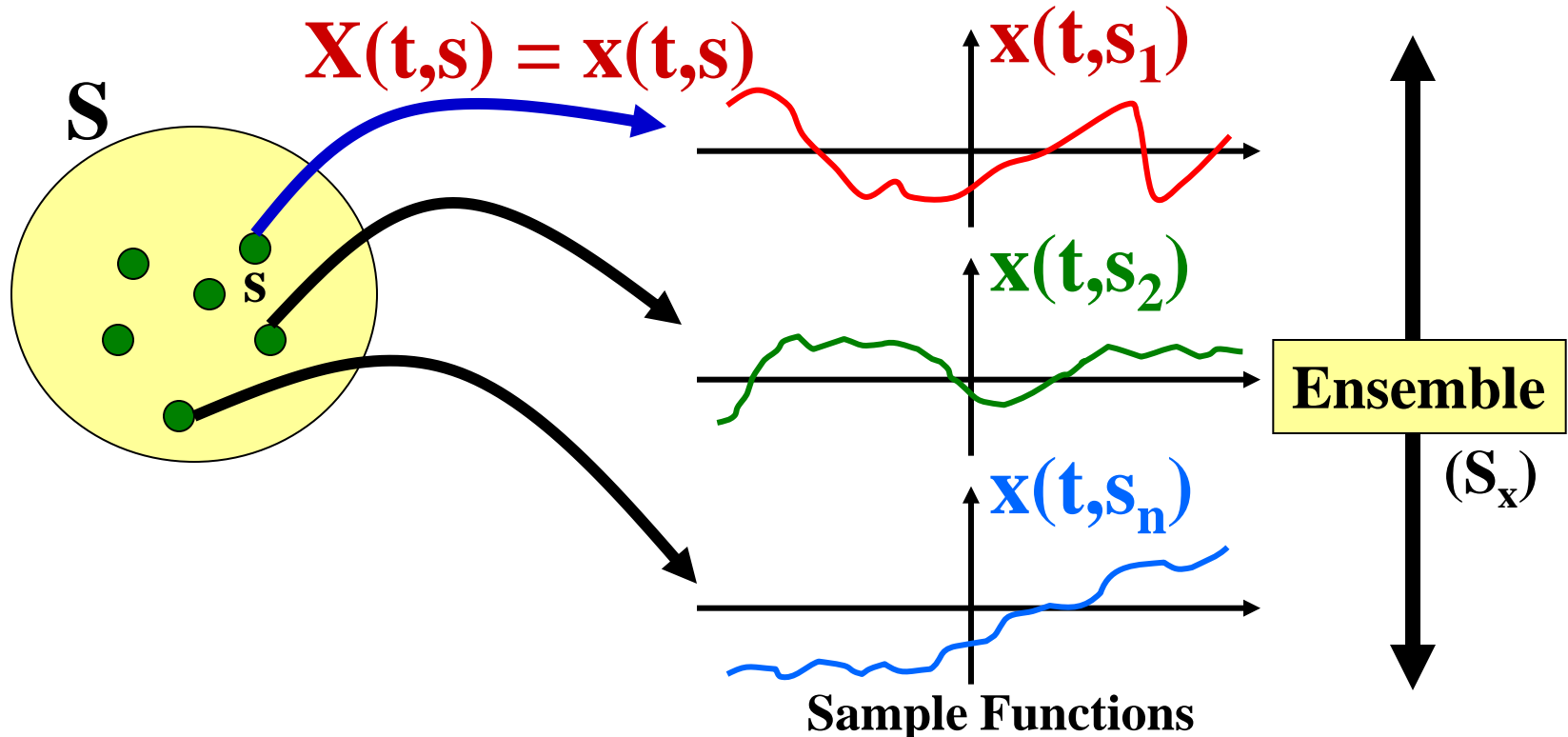
X is a function that maps each outcome, s , in S to a real number $X(s)$, x



Random Process

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$X(t)$ is a function that maps each outcome, s , in S to a time function $x(t,s)$



Example 1

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- Taking temperature at the surface of a space shuttle
- Starting at launch time $t = 0$
- $X(t)$ = temp in degree Celsius on the surface
- Each launch, record $x(t,s)$



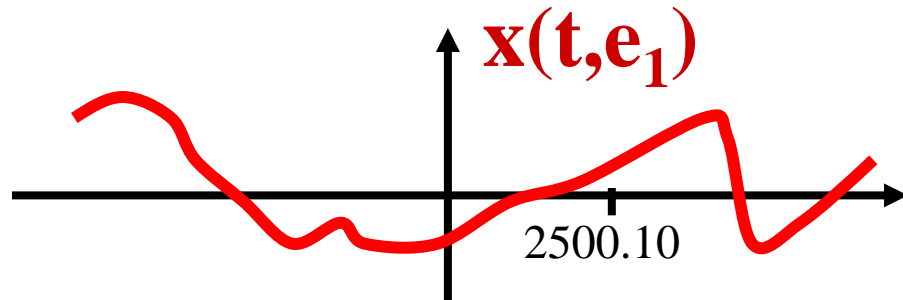
<http://spaceflight.nasa.gov/history/shuttle-mir/spacecraft/s-orb-sscomponents-main.htm>

Example 1

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<http://sites.indianriverschools.org/srhs/teachers/nyberg/main.htm>



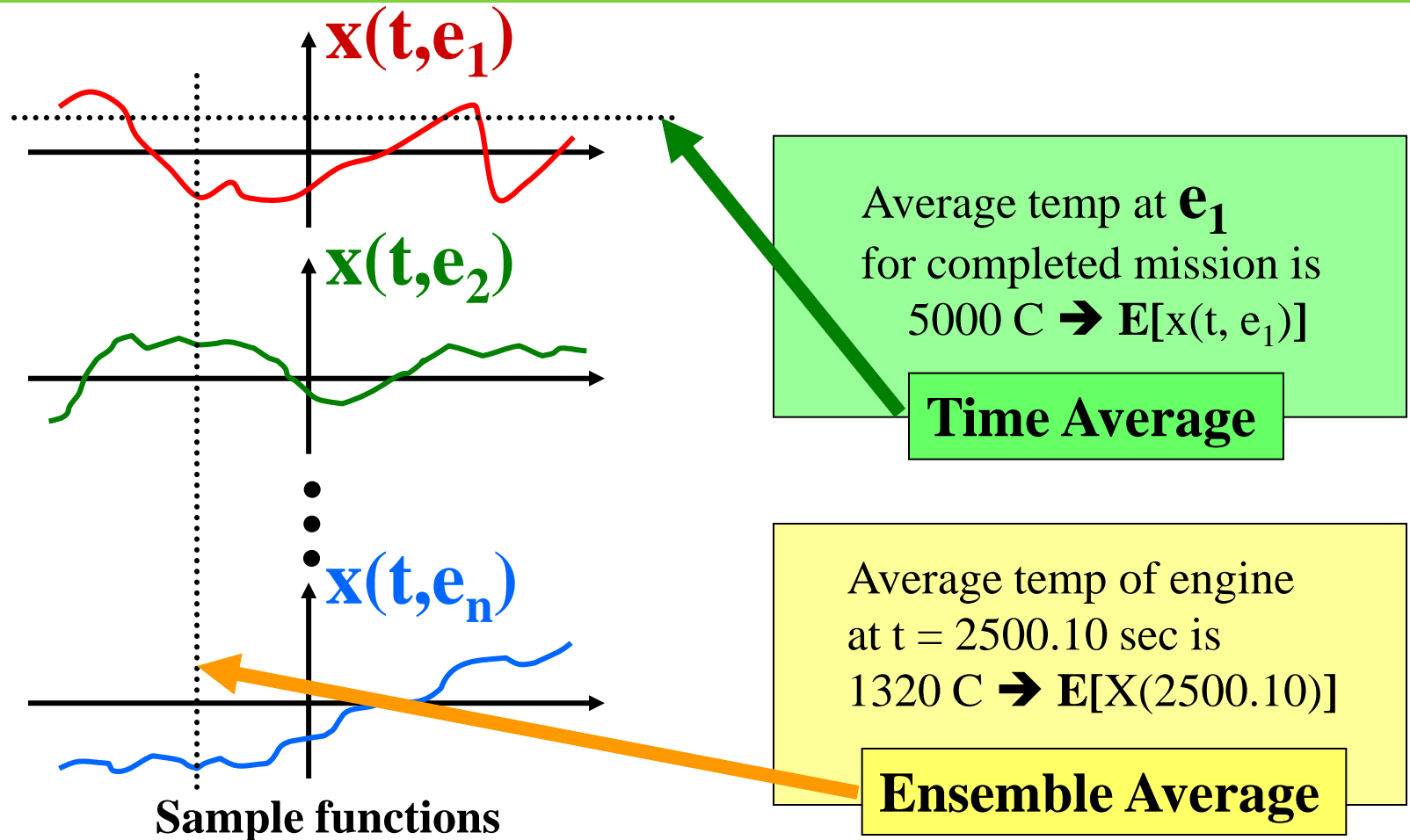
At time $t = 2500.10$ sec

$$x(2500.10, e_1) = 1200 \text{ }^\circ\text{C}$$

$e_1 = 1^{\text{st}}$ launch measuring

Example 1

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Example 2

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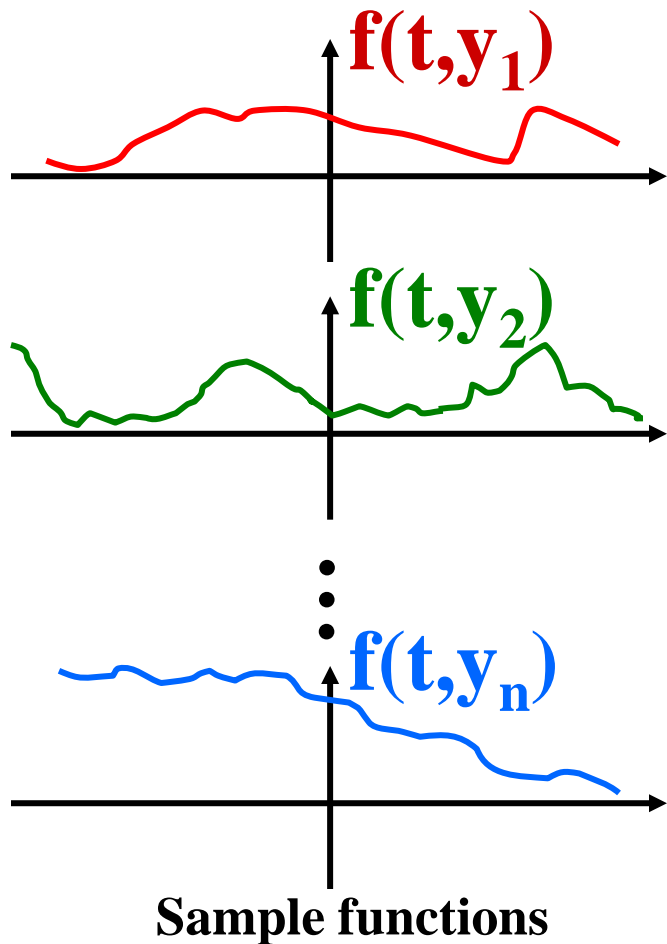
- Measure the rain fall in a day @Bangkhen every day
- Let $F(t)$ = random process
- $f(t,y)$ = a sample function for measuring at day “t” of the year “y”



http://snap.newsadvance.com/Ina/snap/media_view/94

Example 2

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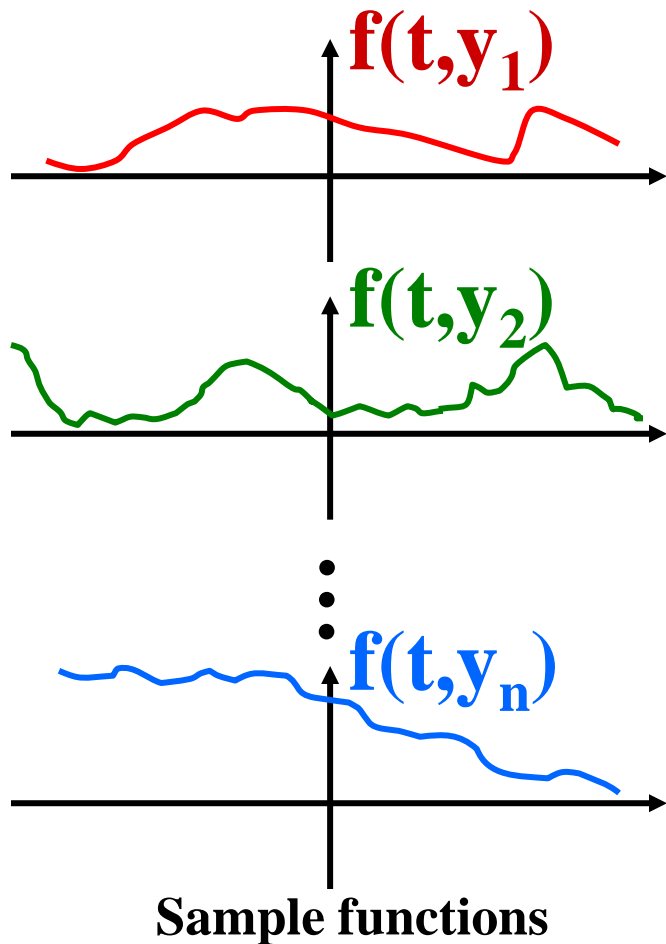
A sample function of rain fall
in $y_1 = \text{year 2002}$ ($1 \leq t \leq 365$)

A sample function of rain fall
in $y_2 = \text{year 2003}$ ($1 \leq t \leq 365$)

A sample function of rain fall
in $y_n = \text{year 2009}$ ($1 \leq t \leq 365$)

Example 2

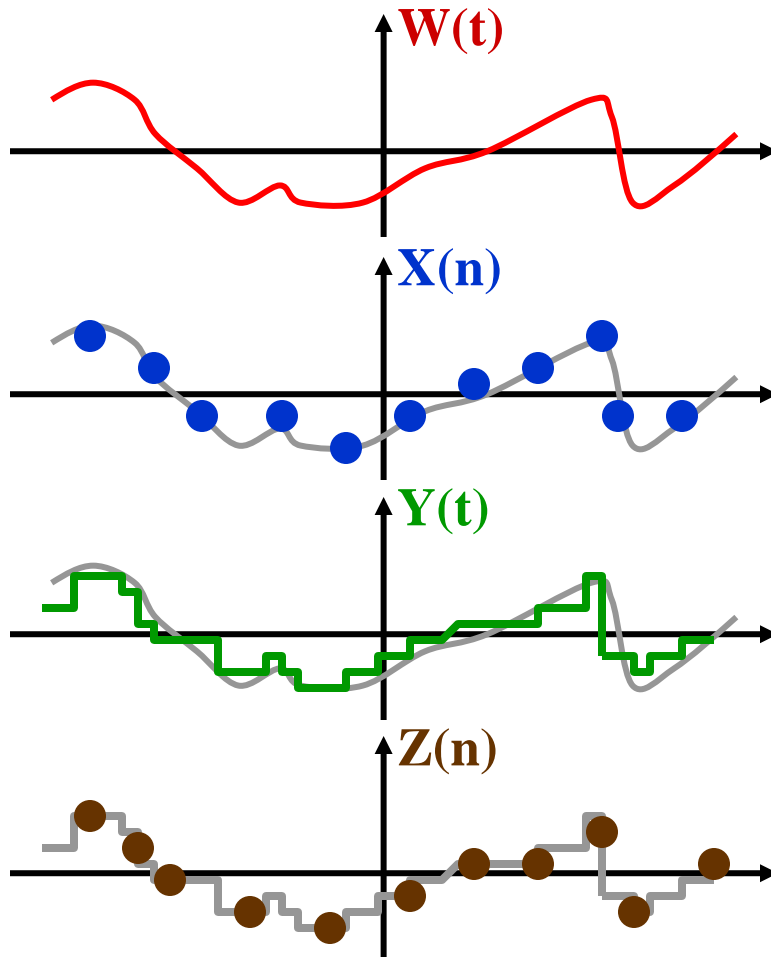
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- Therefore,
- we might want to know
- The average rain fall in year 2007
 - The average rain fall for June 29

Types of Stochastic Process

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**Continuous Time,
Continuous value Process**

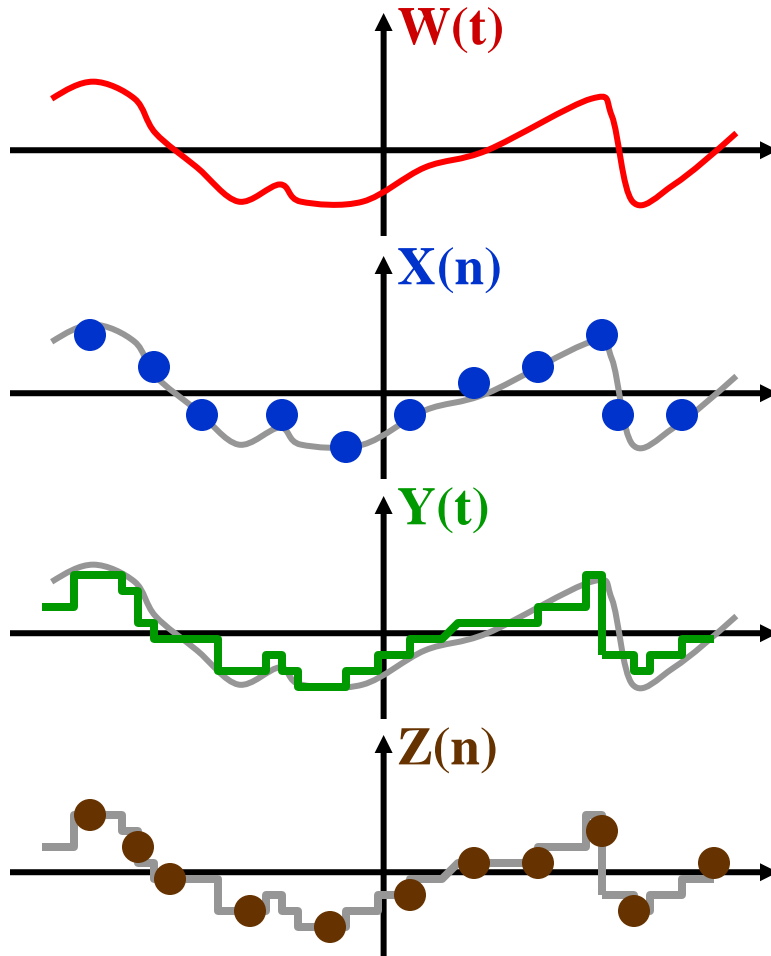
**Discrete Time,
Continuous value Process**

**Continuous Time,
Discrete value Process**

**Discrete Time,
Discrete value Process**

Stochastic Process Examples

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Record temperature as a continuous time

Record temperature every T seconds

Record round(temperature) as a continuous time

Record round(temperature) every T seconds

IID Random Sequence

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- Independent, **I**dentically **D**istributed Random Sequence
- Independent trials of an experiment at a constant rate
- Discrete / Continuous

Theorem:

$$P_{X_{n_1} \dots X_{n_k}}(x_1, \dots, x_k) = P_X(x_1) \dots P_X(x_k) = \prod_{i=1}^k P_X(x_i)$$

Counting Process

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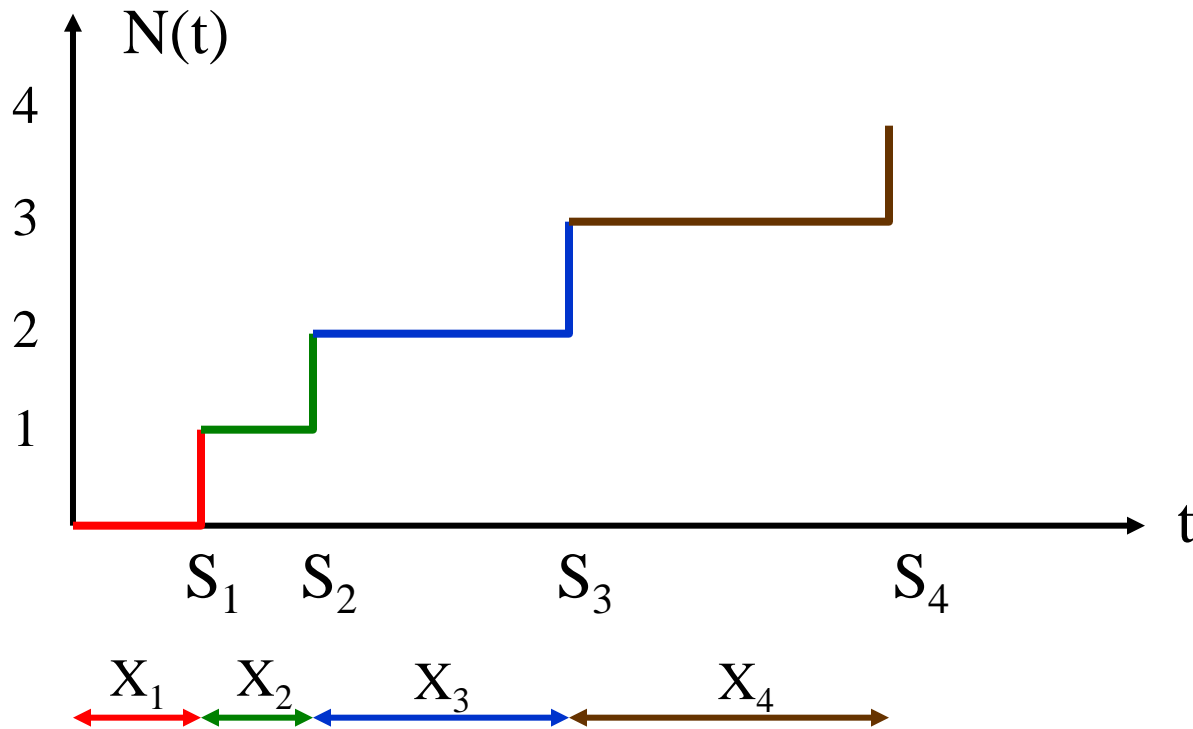
Definition: A Stochastic Process is a Counting Process $N(t)$ if

- $n(t,s) = 0$ for $t < 0$
- $n(t,s) =$ integer valued and non-decreasing

Counting Process

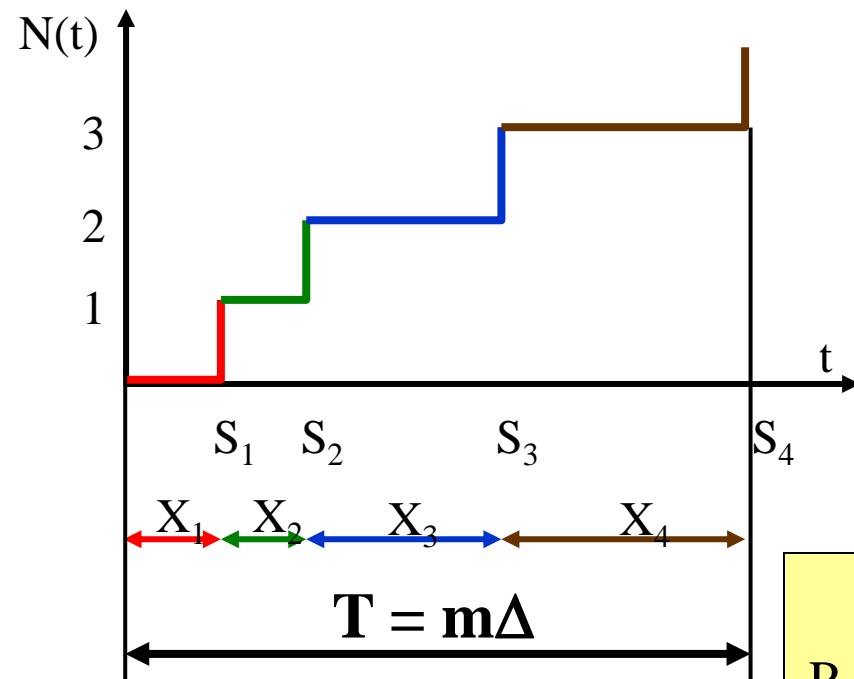
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of customers arrive at $(0,t]$



Counting Process

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- For a small step Δ , only one arrival ($X_n = 1$)
- Success Prob. of $X_n = \lambda \Delta$
 $= \lambda T/m$

Binomial PMF

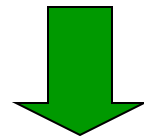
$$P_{Nm}(n) = \begin{cases} \binom{m}{n} (\lambda T/m)^n (1 - \lambda T/m)^{m-n} & n = 0, 1, 2, \dots \\ 0 & \text{Otherwise} \end{cases}$$

Counting Process

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Binomial Process

$$P_{N_m}(n) = \begin{cases} \binom{m}{n} (\lambda T/m)^n (1 - \lambda T/m)^{m-n} & n = 0, 1, 2, \dots \\ 0 & \text{Otherwise} \end{cases}$$

 $m \rightarrow \infty$ or $\Delta \rightarrow 0$

Poisson Process

$$P_{N(t)}(n) = \begin{cases} \frac{(\lambda T)^n e^{-\lambda T}}{n!} & n = 0, 1, 2, \dots \\ 0 & \text{Otherwise} \end{cases}$$

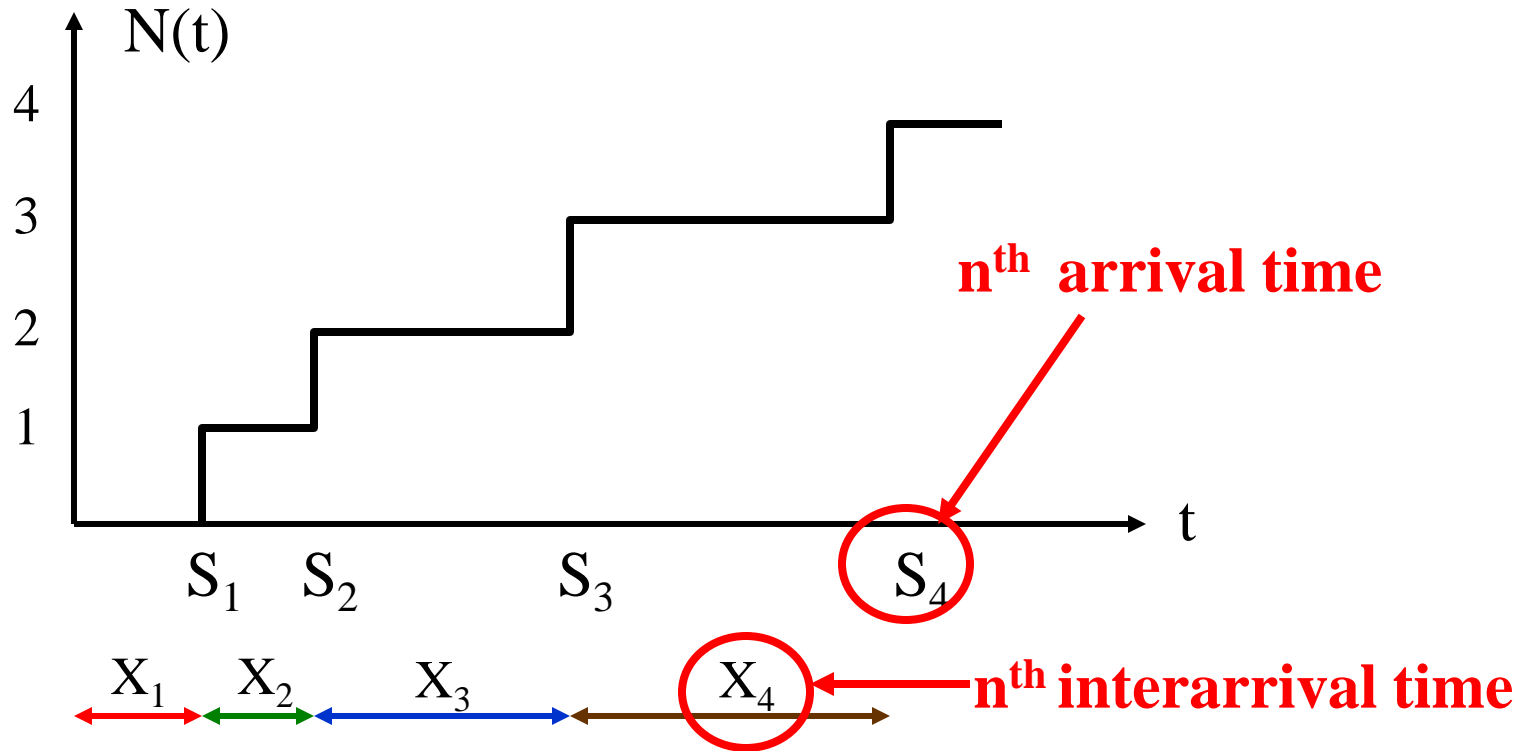
Poisson Process

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- Poisson Process is a Counting Process that the # of **Arrival** during any interval is Poisson RV
- An arrival during any instant is **independent** of the past history of the process → **Memoryless**
- X_n is called **Interarrival Time**

Poisson Process

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Poisson Process

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Definition:

A Counting Process $N(t)$ is a Poisson Process $N(t)$ if

- # of arrivals in $(t_0, t_1]$, $N(t_1) - N(t_0)$, is a Poisson RV with expected value $\lambda(t_1 - t_0)$
- # of arrivals in each interval are independent random variable

Poisson Process

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- Process rate (λ) = $E[N(t)] / t$
- $M = N(t_1) - N(t_0) = \text{Poisson RV}$

$$P_M(m) = \begin{cases} \frac{[\lambda(t_1-t_0)]^m e^{-\lambda(t_1-t_0)}}{m!} & m = 0, 1, 2, \dots \\ 0 & \text{Otherwise} \end{cases}$$

Joint PMF

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Theorem: Poisson Process $N(t)$ of rate λ ,
Joint PMF of $N(t_1), \dots, N(t_k)$, $t_1 < \dots < t_k$

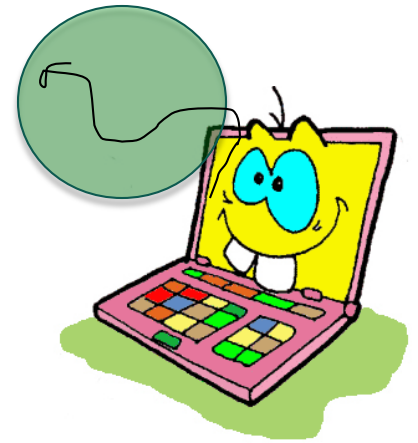
$$P_{N(t_1), \dots, N(t_k)}(n_1, \dots, n_k) = \begin{cases} \frac{\alpha_1^{n_1} e^{-\alpha_1}}{n_1!} \frac{\alpha_2^{(n_2 - n_1)} e^{-\alpha_2}}{(n_2 - n_1)!} \dots \frac{\alpha_k^{(n_k - n_{k-1})} e^{-\alpha_k}}{(n_k - n_{k-1})!} & 0 \leq n_1 \leq \dots \leq n_k \\ 0 & \text{Otherwise} \end{cases}$$

$$\alpha_i = \lambda(t_i - t_{i-1})$$

Example

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- A mobile station transmits data packet as Poisson process with rate 12 packets/sec
 - Find # of packets transmitted in the k^{th} hour
 - Find **Joint** PMF of # of packets transmitted in the k^{th} hour and z^{th} hour



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Example

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- Let $N_k = \#$ of packets transmitted in k^{th} hour
- # packets in each hour is IID

$$P_{N_i}(n) = \begin{cases} \frac{[12(3600-0)]^n e^{-12(3600-0)}}{n!} & n = 0, 1, 2, \dots \\ 0 & \text{Otherwise} \end{cases}$$
$$= \begin{cases} \frac{[43200]^n e^{-43200}}{n!} & n = 0, 1, 2, \dots \\ 0 & \text{Otherwise} \end{cases}$$

Example

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- Joint PMF of # of packets transmitted in the k^{th} hour and z^{th} hour

$$\begin{aligned}
 P_{N_k, N_z}(n_k, n_z) &= \begin{cases} \frac{\alpha_k^{n_k} e^{-\alpha_k}}{n_k!} \frac{\alpha_z^{n_z} e^{-\alpha_z}}{n_z!} & n_k = 0, 1, \dots \\ & n_z = 0, 1, \dots \\ 0 & \text{Otherwise} \end{cases} \\
 &= \begin{cases} \frac{\alpha^{(n_k+n_z)}}{n_k! n_z!} e^{-2\alpha} & n_k = 0, 1, \dots \\ & n_z = 0, 1, \dots \\ 0 & \text{Otherwise} \end{cases}
 \end{aligned}$$

$$\alpha = \alpha_k = \alpha_z = \lambda T = [12(3600-0)] = 43200$$

Interarrival Time

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Theorem:

Poisson Process of rate λ , the interarrival times X_1, X_2, \dots are an iid random sequence with **Exponential PDF**

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

Interarrival Time

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Theorem:

A Counting Process with *independent exponential interarrival time* X_1, X_2, \dots with $E[X_i] = 1/\lambda$ is a *Poisson Process* of rate λ

Brownian Motion Process

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- A continuous time, continuous value random process

Definition: A Brownian Motion Process $X(t)$ has the property that

- $X(0) = 0$
- For $\tau > 0$, $X(t + \tau) - X(t) = \text{Gaussian RV}$ with $\mathbf{E}[X(t)] = \mathbf{0}$ and $\mathbf{Var}[X(t)] = \alpha\tau$ that is independent of $X(t')$ for $t' \leq t$

Brownian Motion Process

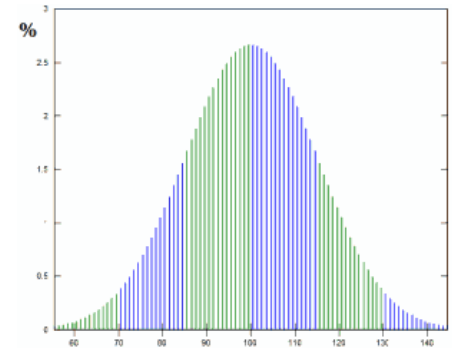
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Gaussian RV

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x - \mu)^2}{2\sigma^2}}$$

$E[X(t)] = 0$ and $\text{Var}[X(t)] = \alpha t$

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi\alpha t}} e^{-\frac{x^2}{2\alpha t}}$$



http://www.damninginteresting.net/content/gaussian_iq_distribution.gif

Joint PDF

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- **Theorem:** For the Brownian motion process $X(t)$, joint PDF of $X(t_1), \dots, X(t_k)$

$$f_{X(t_1), \dots, X(t_k)}(x_1, \dots, x_k) = \prod_{n=1}^k \frac{1}{\sqrt{2\pi\alpha(t_n - t_{n-1})}} e^{-\frac{(x_n - x_{n-1})^2}{2\alpha(t_n - t_{n-1})}}$$

Expected Value

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$$\mathbf{X}(t): X(t_1) \rightarrow f_{X(t_1)}(x) \rightarrow E[X(t_1)]$$

Definition: The expected value of a stochastic process $X(t)$ is the deterministic function

$$\mu_X(t) = E[X(t)]$$

Covariance of X and Y

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Definition: $\text{Cov}[X, Y] = E[(X - \mu_x)(Y - \mu_y)]$

$$\begin{aligned}\text{Cov}[X, Y] &= E[XY - \mu_x Y - \mu_y X + \mu_x \mu_y] \\ &= E[XY] - \mu_x E[Y] - \mu_y E[X] + \mu_x \mu_y \\ &= E[XY] - \mu_x \mu_y - \mu_x \mu_y + \mu_x \mu_y\end{aligned}$$

Theorem: $\text{Cov}[X, Y] = E[XY] - \mu_x \mu_y$

- High covariance = an observation of X provides an accurate indication of Y

Autocovariance

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- Auto = self = same process
- For a same process $X(t)$, in 2 different times t_1 and $t_2 = t + \tau$
- For high covariance = a sample function, is unlikely to change in τ interval
- For near zero covariance = rapid change

How much the sample function is likely to change in the τ interval after t

Autocovariance

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Definition:

The autocovariance function of a stochastic process $X(t)$ is

$$C_X(t, \tau) = \text{Cov}[X(t), X(t+\tau)]$$

Note

- For $\tau = 0 \rightarrow C_X(t, t) = \text{Var}[X(t)]$

Correlation

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Definition: The correlation of X and Y is $r_{X,Y}$

$$r_{X,Y} = E[XY]$$

Theorem: $\text{Cov}[X, Y] = r_{X,Y} - \mu_x \mu_y$

Autocorrelation

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- **Definition:** The autocorrelation function of a stochastic process $X(t)$ is

$$R_X(t, \tau) = E [X(t) X(t+\tau)]$$

Autocovariance & Autocorrelation

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Theorem: The autocovariance and autocorrelation functions of a process $X(t)$ satisfy

$$C_X(t, \tau) = R_X(t, \tau) - \mu_X(t) \mu_X(t + \tau)$$

Note:

Autocovariance \rightarrow use $X(t)$ to predict a future $X(t+\tau)$

Autocorrelation \rightarrow describe the power of a random signal

Random Sequence

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- For a discrete time process, the sample function is described by the ordered sequence of random variable $X_n = X(nT)$

Definition: A random sequence X_n is an ordered sequence of random variable X_0, X_1, \dots

Autocovariance & Autocorrelation of a Random Sequence

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Definition: The autocovariance of a random sequence X_n is

$$C_X[m,k] = \text{Cov}[X_m, X_{m+k}]$$

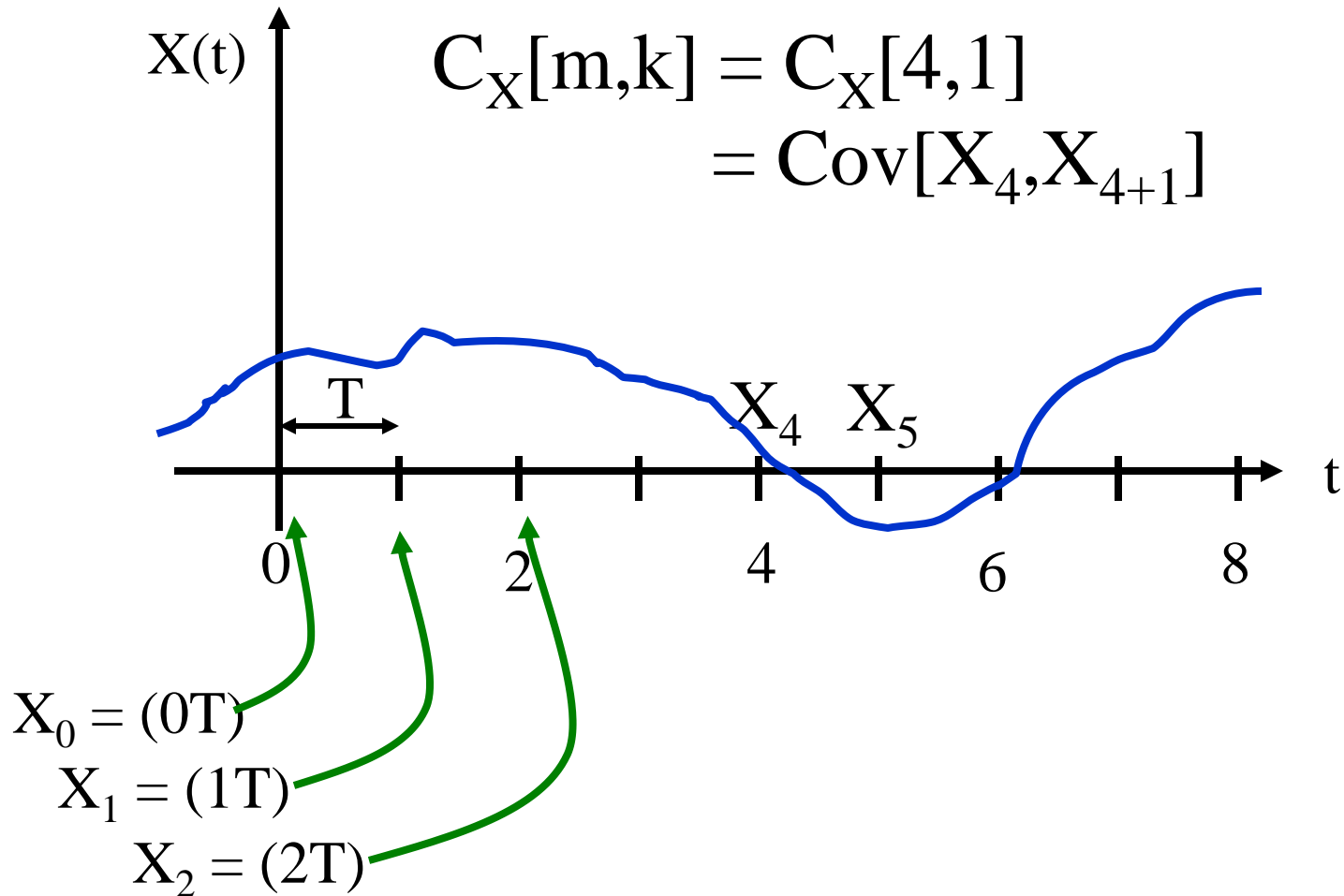
Definition: The autocorrelation of a random sequence X_n is

$$R_X[m,k] = E[X_m X_{m+k}]$$

m and k are integers

Autocovariance & Autocorrelation of a Random Sequence

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STATIONARY PROCESS



Stationary Process

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- For a random process $X(t)$, normally,
at t_1 : $X(t_1)$ has pdf = $f_{X(t_1)}(x)$ [depends on t_1]
- For a random process $X(t)$,
at t_1 : $X(t_1)$ has pdf = $f_{X(t_1)}(x)$ [not depend on t_1]

Stationary Process

= same random variable at all time

= no statistical properties change with time

$$f_{X(t_1)}(x) = f_{X(t_1 + \tau)}(x) = f_X(x)$$

Stationary Process

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Definition: A stochastic process $X(t)$ is stationary iif for all sets of time t_1, \dots, t_m and any time different τ ,

$$f_{X(t_1), \dots, X(t_m)}(x_1, \dots, x_m) = f_{X(t_1 + \tau), \dots, X(t_m + \tau)}(x_1, \dots, x_m)$$

Stationary Random Sequence

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Definition: A random sequence X_n is stationary iif for any finite sets of time instants n_1, \dots, n_m and any time different k ,

$$f_{X(n_1), \dots, X(n_m)}(x_1, \dots, x_m) = f_{X(n_1+k), \dots, X(n_m+k)}(x_1, \dots, x_m)$$

Stationary Process

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Theorem: A stationary process $X(t)$,

$$\mu_X(t) = \mu_X$$

$$R_X(t, \tau) = R_X(0, \tau) = R_X(\tau)$$

$$C_X(t, \tau) = R_X(\tau) - \mu_X^2 = C_X(\tau)$$

Stationary Random Sequence

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Theorem: A stationary random sequence X_n , for all m

$$E[X_m] = \mu_X$$

$$R_X[m,k] = R_X[0,k] = R_X[k]$$

$$C_X[m,k] = R_X[k] - \mu_X^2 = C_X[k]$$

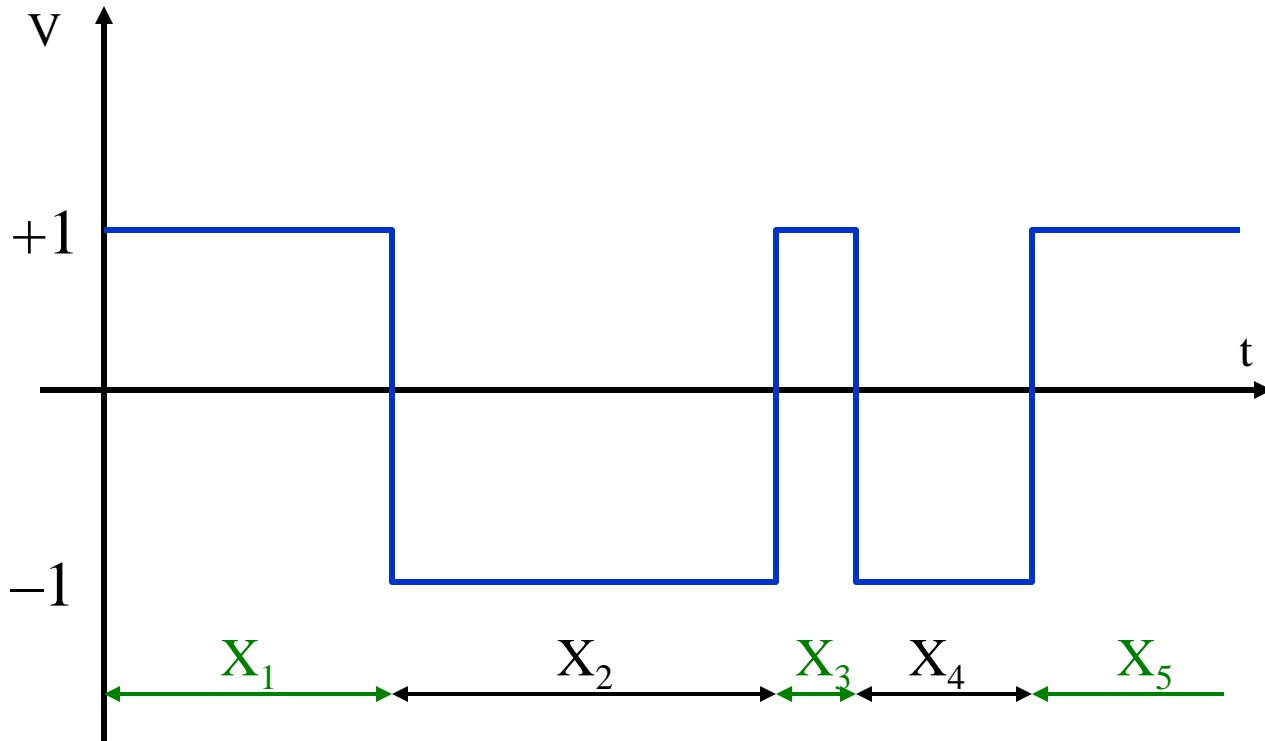
Example

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- Telegraph Signal, $X(t)$ take value ± 1
- $X(0) = \pm 1$ with probability = 0.5
- Let $X(t)$ toggles the polarity with each occurrence of an event in a Poisson process rate α

Example

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Example

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- Find PMF of $X(t)$, $f_{X(t)}(x)$
- $$P[X(t)] = P[X(t) | X(0) = 1] P[X(0) = 1] + P[X(t) | X(0) = -1] P[X(0) = -1]$$
- $$\begin{aligned} P[X(t) | X(0) = 1] &= P[N(t) = \text{even}] \\ &= \sum_{j=0}^{\infty} \frac{(\alpha t)^{2j}}{(2j)!} e^{-\alpha t} \\ &= e^{-\alpha t} (1/2) (e^{\alpha t} + e^{-\alpha t}) \\ &= (1/2) (1 + e^{-2\alpha t}) \end{aligned}$$

Example

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- $P[X(t) \mid X(0) = -1] = P[N(t) = \text{odd}]$
$$= \sum_{j=0}^{\infty} \frac{(\alpha t)^{2j+1}}{(2j+1)!} e^{-\alpha t}$$
$$= e^{-\alpha t} (1/2) (e^{\alpha t} - e^{-\alpha t})$$
$$= (1/2) (1 - e^{-2\alpha t})$$

Example

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- $P[X(t) = 1]$
= $P[X(t) | X(0) = 1] P[X(0) = 1]$
+ $P[X(t) | X(0) = -1] P[X(0) = -1]$
= $(1/2) (1 + e^{-2\alpha t})(1/2) + (1/2) (1 - e^{-2\alpha t})(1/2)$
= $1/2$
- $P[X(t) = -1]$
= $1 - P[X(t) = 1] = 1/2$

$$f_{X(t)}(x) = \begin{cases} 1/2 & X(t) = -1, 1 \\ 0 & \text{Otherwise} \end{cases}$$

Example

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- $\mu_X(t) = 1 (1/2) + (-1)(1/2) = 0$
- $\text{Var} [X(t)] = E[X^2(t)] = 1^2(1/2) + (-1)^2(1/2) = 1$

Autocovariance, $C_X(t, \tau) = e^{-2\alpha\tau}$

$$f_{X(t_1), \dots, X(t_m)}(x_1, \dots, x_m) = f_{X(t_1 + \tau), \dots, X(t_m + \tau)}(x_1, \dots, x_m)$$

Wide Sense Stationary

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Definition: $X(t)$ is a wide sense stationary random process iff for all t ,

$$E[X(t)] = \mu_X$$

$$R_X(t, \tau) = R_X(0, \tau) = R_X(\tau)$$

Definition: X_n is a wide sense stationary random sequence iff for all n ,

$$E[X_n] = \mu_X$$

$$R_X[n, k] = R_X[0, k] = R_X[k]$$

Wide Sense Stationary

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- For every **stationary** process or sequence, it is also **wide sense stationary**.
- However, if it is a **wide sense stationary** it may or may not be **stationary**.

Example

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- Let $X_n = \pm 1$ with prob = $1/2$ ($n = \text{even}$)
- For $n = \text{odd}$
 - $X_n = -1/3$ with prob = $9/10$
 - $X_n = 3$ with prob = $1/10$
- Stationary ?
 - No
- Wide sense stationary ?
 - Mean = 0 for all n
 - $C_X(t, \tau) = 0$ for $\tau > 0$
 - $C_X(t, \tau) = 1$ for $\tau = 0$
 - Yes , it's wide sense stationary

Wide Sense Stationary

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Theorem: For a wide sense stationary process $X(t)$,

$$R_X(0) \geq 0$$

$$R_X(\tau) = R_X(-\tau)$$

$$|R_X(\tau)| \leq R_X(0)$$

Wide Sense Stationary

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Theorem: For a wide sense stationary sequence X_n ,

$$R_X[0] \geq 0$$

$$R_X[k] = R_X[-k]$$

$$|R_X[k]| \leq R_X[0]$$