LECTURE #2
PROBABILITY REVIEW (I)

204528

Queueing Theory and Applications in Networks

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Outline

- Probability meaning
- Random Variable
- Discrete RV
- Continuous RV
- Multiple RVs

What is Probability ?

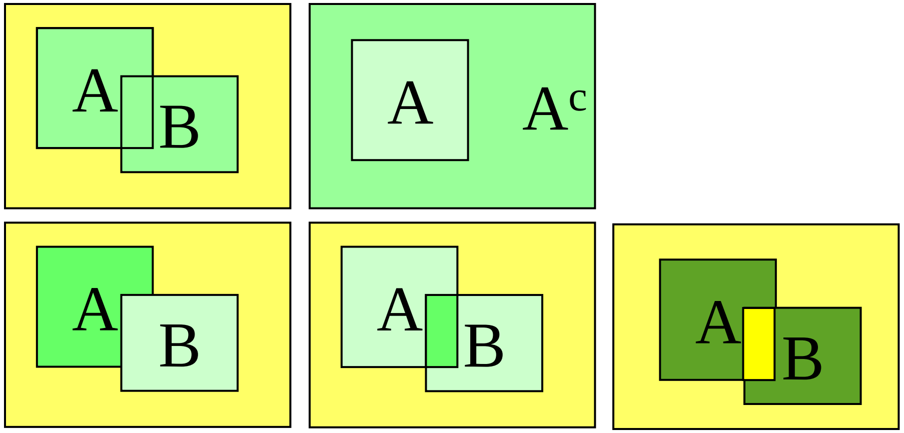
- Physical Property
  - Lottery
- Knowledge
  - Snow in Thailand
- Probability meaning
  - Situation cannot exactly replicate
  - But not chaotic (have a pattern)

Probability

- Definition
  - Logic of probability
- Axiom
  - Fact without proof
- Theorem
  - Derived from Definition, Axiom, or other Theorems

Probability Mathematics

- Set Theory
  - Set operation
  - Set properties

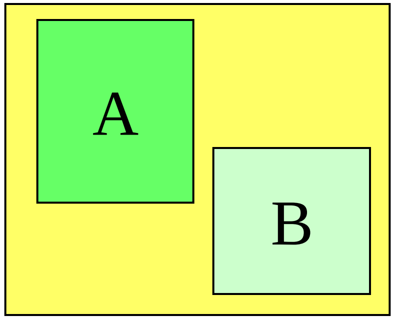


Important Set Properties

1. Mutually Exclusive

$\mathbf{A_i \cap A_j = \varnothing}$ for $i \neq j$

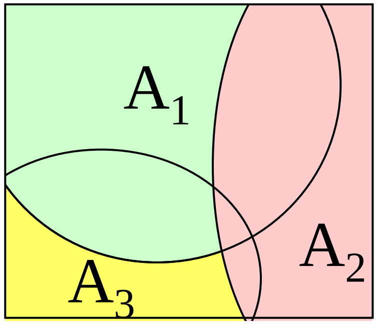
$\mathbf{A \cap B = \varnothing}$ → called **Disjoint** for only 2 sets



2. Collectively Exhaustive

$$\mathbf{A_1 \cup A_2 \cup \ldots \cup A_n = S}$$

$$\bigcup_{i=1}^{n} \mathbf{A_i = S}$$



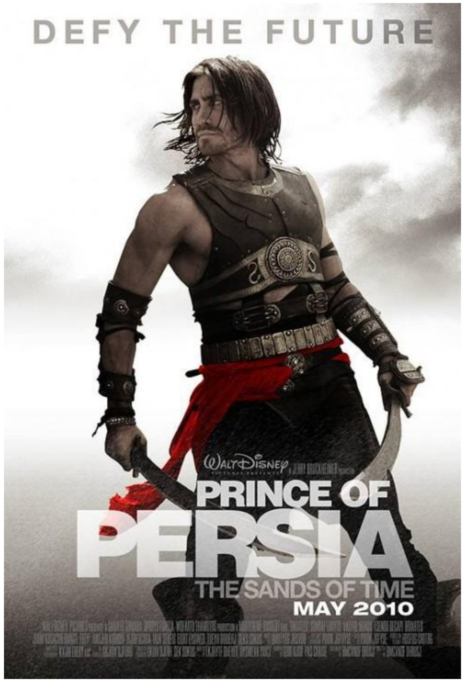
Experiment

What is an Experiment?

- Method for finding some facts/conclusions

Give an example?

- For movie “**Prince of Persia**”, is it fun?
- Stand in front of the theatre
- Ask audiences, fun or not?



Composition of an experiment

- Procedure
- Observation

Why experiment is needed?

- Uncertainty

Experiment

Concern about movie "Prince of Persia"

- Should I ask a man, woman, teenager or kid?
- Experience of the audiences
- Knowledge of the audiences

Complicated experiment →need Model

- Real experiments: too complicate
- Capture only the important part
- Model Example:
  - Treat all audiences the same
  - Answer will only be like/dislike

Experiment

Same Procedure
but different Observations
→ Different Experiments

Example:

1. Flip a coin 3 times, Observe the sequence of heads/tails
2. Flip a coin 3 times, Observe # of heads

Definition in Probability

- **Outcome**
  - Any possible observation
- **Sample Space**
  - ***Finest-grain****:* each outcome is different
  - ***Mutually exclusive****:* if one outcome occurs, other will not occur
  - ***Collectively exhaustive****:* every outcome must be in the sample space
- **Event**
  - Set of outcomes (Must know all outcomes)
  - Event $\subset$ Sample Space

Event Examples

For an experiment:

Roll a dice, observe the shown numbers

Outcomes:

number = 1,2,3,4,5,6

Sample space:

S = {1,2,3,…,6}

Event examples:

E_1 = {number < 3} = {1,2}

E_2 = {number is odd} = {1,3,5}

Set VS. Probability

Set Algebra	Probability
Set	Event
Universal set	Sample space
Element	Outcome

Probability of Event, P[☺]

P[☺]
is a function that maps event
in the sample space to real number

From experiment: Roll a dice

Outcomes:

number = 1,2,3,4,5,6

Sample space:

S = {1,2,3,…,6}

Event examples:

E_1 = {number < 3} = {1,2}

E_2 = {number is odd} = {1,3,5}

P[E_1] = 2/6 = 1/3

P[E_2] = 3/6 = 1/2

Probability Axioms

Axiom 1: For any event A, $\mathbf{P[A] \geq 0}$

Axiom 2: $\mathbf{P[S] = 1}$

Axiom 3: For events A_1, A_2,…, A_n of mutual exclusive events

$$\mathbf{P[A_1 \cup A_2 \cup \ldots \cup A_n] = P[A_1] + P[A_2] + \ldots + P[A_n]}$$

Example Theorems

- **Theorem:** If A and B are disjoint, then

$$\mathbf{P[A \cup B] = P[A] + P[B]}$$

- **Theorem:** If $B = B_1 \cup B_2 \cup \ldots \cup B_n$ and $B_i \cap B_j = \varnothing$ for $i \neq j$, then

$$\mathbf{P[B] = \sum_{i=1}^{n} P[B_i]}$$

Equally Likely

Theorem:

For an experiment with sample space $S=\{s_1,\ldots, s_n\}$ if each outcome is **equally likely**,

$$\mathbf{P[s_i] = 1/n \qquad 1 \le i \le n}$$

Consequences of Axioms

Theorem:

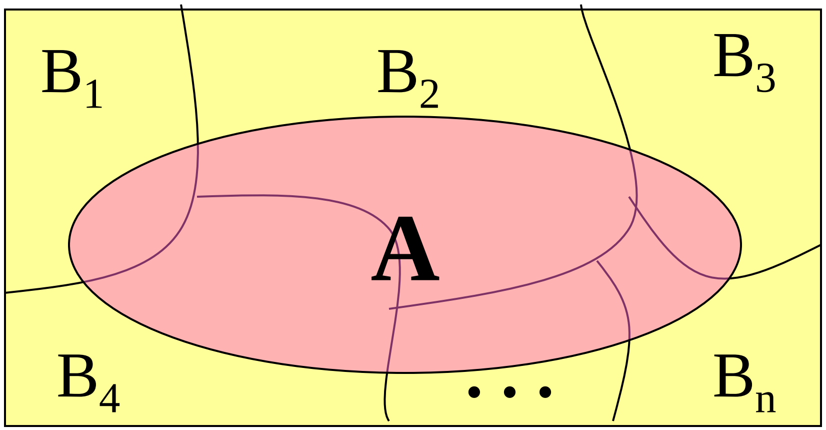
- $P[\varnothing] = 0$
- $P[A^c] = 1 - P[A]$
- For any A and B (not necessary disjoint)

  $P[A \cup B] = P[A] + P[B] - P[A \cap B]$
- If $A \subset B$, then $P[A] \leq P[B]$

A Useful Theorem

Let $B_1, B_2, \ldots, B_n$ be mutual exclusive events whose union equals sample space S

→ partition of S



For any event A

$$A = A \cap S = A \cap (B_1 \cup B_2 \cup \ldots \cup B_n)$$

$$P[A] = P[A \cap B_1] + P[A \cap B_2] + \ldots + P[A \cap B_3]$$

Theorem:

$$\mathbf{P[A] = \sum_{i=1}^{n} P[A \cap B_i]}$$

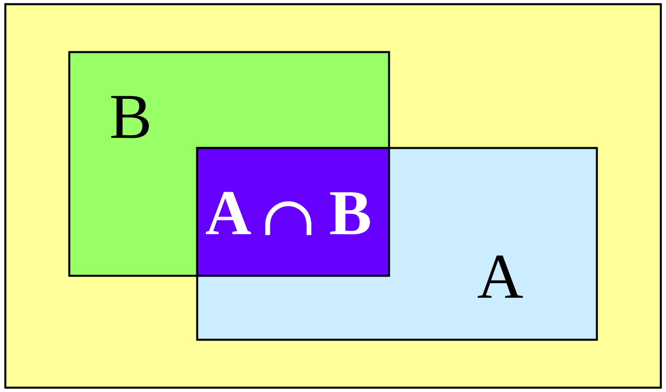
Conditional Probability

- In practice, it maybe impossible to find the precise outcome of an experiment
- However, if we know that Event B has occurred (the outcome of Event A is in set B)
  - Probability of A when B occurs can be described
  - Still don't know P[A]

Conditional Probability

- **Notation:** **P[A|B]**
  - "Probability of A given B"
  - The condition probability of the event A given the occurrence of the event B
- **Definition:**

$$P[A|B] = \frac{P[AB]}{P[B]}$$

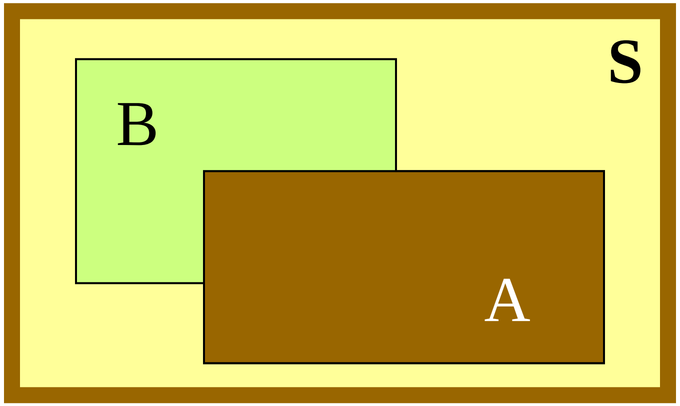


- **Example:**

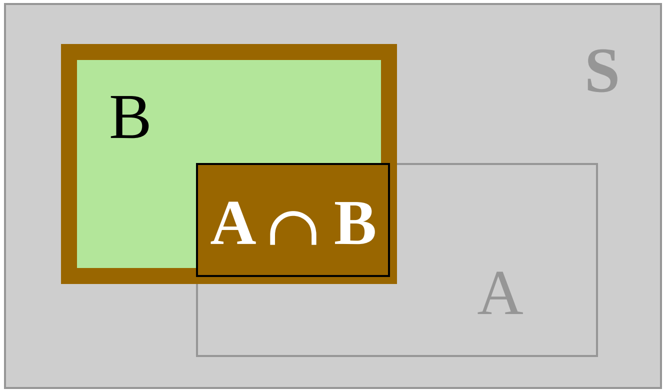
Tiger Woods hits Hole-in-one

More Explanation

$$P[A|S] = \frac{P[AS]}{P[S]} = \frac{P[A]}{1} = P[A]$$

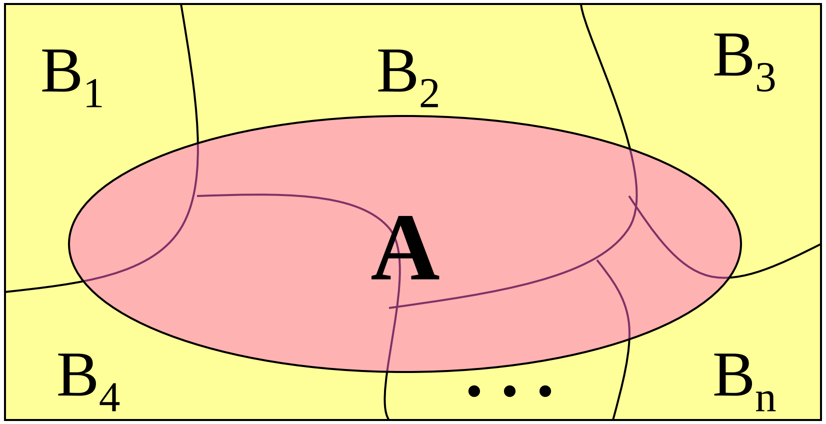


$$P[A|B] = \frac{P[AB]}{P[B]}$$



Law of Total Probability

- Let $B_1, B_2, \ldots, B_n$ be mutual exclusive events whose union equals sample space S
- $P[B_i] > 0$



Theorem: $P[A] = \sum_{i=1}^{n} P[A \cap Bi]$

$$P[A] = P[A \cap B_1] + P[A \cap B_2] + \ldots$$

$$P[A] = P[A|B_1]P[B_1] + P[A|B_2]P[B_2] + \ldots$$

Theorem: $P[A] = \sum_{i=1}^{n} P[A|B_i]P[B_i]$

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Bayes' Theorem

$$P[B|A] = \frac{P[BA]}{P[A]}$$

$$P[A|B] = \frac{P[AB]}{P[B]}$$

$$= \frac{P[A|B]P[B]}{P[A]}$$

Theorem: $$\mathbf{P[B|A] = \frac{P[A|B]P[B]}{P[A]}}$$

2 Independent Events

Definition: Event A and B are independent iff

$$\mathbf{P[AB] = P[A]P[B]}$$

$$P[A|B] = \frac{P[AB]}{P[B]}$$

$$= \frac{P[A]P[B]}{P[B]}$$

$$\mathbf{P[A|B] = P[A]}$$

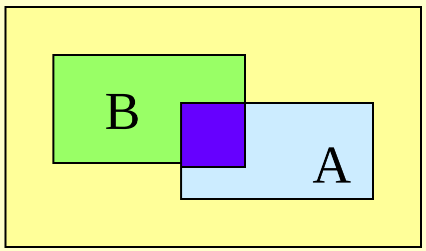
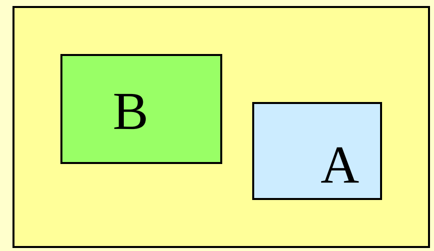
$$\mathbf{P[B|A] = P[B]}$$

Independent Interpretation

$$P[A] = 0.3$$
$$P[A|B] = 0.3$$

No matter event B occurs or not, event A is not affected

Independent VS. Disjoint

Independent	Disjoint
	
$P[AB] \neq 0$	$P[AB] = 0$
$P[A \cap B] = P[A] * P[B]$	$P[A \cup B] = P[A] + P[B]$

Note: Independent = Disjoint iff P[A]=0 or P[B]=0

3 Independent Events

Definition: Event A_1,A_2 and A_3 are independent iff

1) A_1 and A_2 are independent
2) A_2 and A_3 are independent
3) A_1 and A_3 are independent
4) $P[A_1 \cap A_2 \cap A_3] = P[A_1]\,P[A_2]\,P[A_3]$

Why only number 4 is insufficient ?

Definition: Event A and B are independent iff

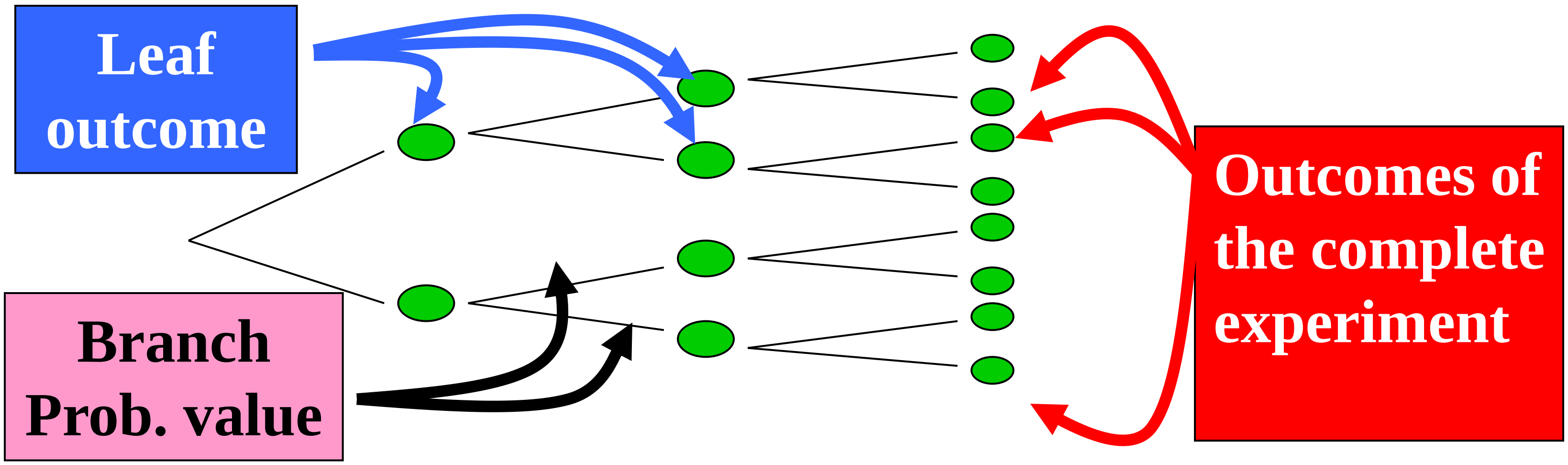
$$P[AB] = P[A]P[B]$$

Most Common Application

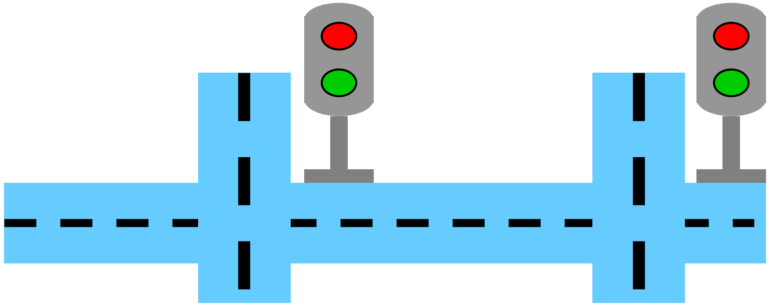
- **Assume that the events of separate experiments are independent**
- Example:
  - Assume that outcome of a coin toss is independent of the outcomes of all prior and all subsequent coin tosses
  - P[H] = P[T] = ½
  - P[HTH] = P[H]P[T]P[H] = 1/2*1/2*1/2 = 1/8

Sequential Experiments

- Experiment: in sequence
  subexperiments → subexperiments
- Each subexp. may depend on the previous one
- Represented by a **Tree Diagram**
- **Model Conditional Prob. → Sequential Experiment**



Sequential Example

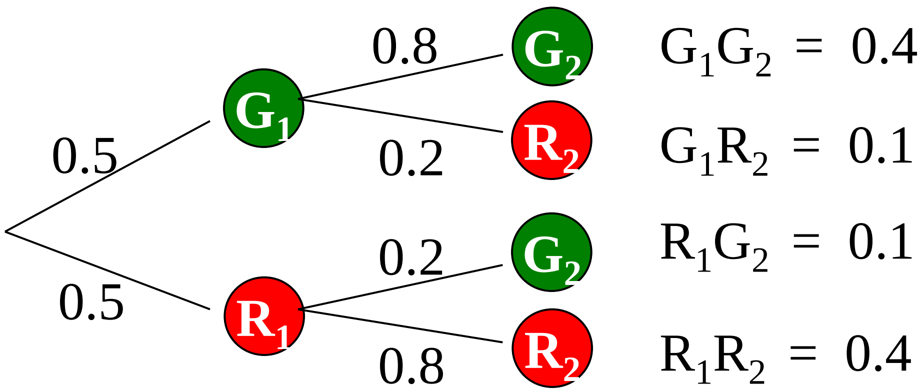


Timing coordination of 2 traffic lights

- P[the 2nd light is the same color as the 1st] = 0.8
- Assume 1st light is equally likely to be green or red

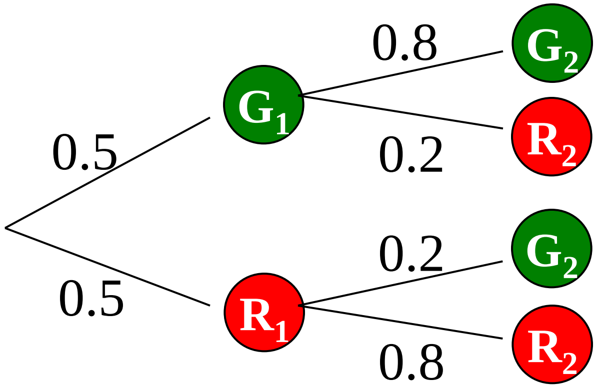
Find P[The second light is green] ?

Sequential Example



- $P[G_1] = P[R_1] = 0.5$
- $P[G_2G_1] = P[G_2|G_1]P[G_1] = (0.8)(0.5) = 0.4$

Sequential Example



$G_1G_2 = 0.4$

$G_1R_2 = 0.1$

$R_1G_2 = 0.1$

$R_1R_2 = 0.4$

P[The second light is green] ?

$$P[G_2] = P[G_2G_1] + P[G_2R_1] = 0.4 + 0.1 = 0.5$$

$$P[G_2] = P[G_2|G_1]P[G_1] + P[G_2|R_1]P[R_1]$$

Counting Method

Principle of Counting Method

If experiment A has **n** possible outcomes,
and experiment B has **k** possible outcomes,

→Then there are **nk** possible outcomes
when you perform both experiments

k-permutations

Theorem:

The number of **k**-permutations (ordered sequence) of **n** distinguishable objects is

$$(n)_k = \frac{n!}{(n-k)!}$$

Choose with replacement

Theorem: Given **n** distinguishable objects,
There are $\mathbf{n^k}$ ways to choose with replacement a sample of **k** objects

k-combination

Theorem:

The number of ways to choose **k** objects out of **n** distinguishable objects is

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}$$

Independent Trials

- Perform repeated trials
- **p** = a success probability
- **(1-p)** = a failure probability
- Each trial is independent
- $\mathbf{S_{k,n}}$ = the event that **k** successes in **n** trials

$$P[S_{k,n}] = \binom{n}{k} p^k (1-p)^{n-k}$$

Independent Trials: Example

- 3 trials with 2 successes
- 000 001 010 011 100 101 110 111
- How many way to choose 2 out of 3

$$= \binom{n}{k} = 3$$

- What is the probability of success for each way ?
- $p^2 * (1-p)$

$$P[S_{2,3}] = \binom{3}{2} p^2(1-p)^{3-2}$$

Independent Trials

Example: In the first round of a food contest, probability that a dish will pass the test is 0.8 .

From 10 candidates, what is the probability that x candidates will pass? P[x = 8]?

Solution:

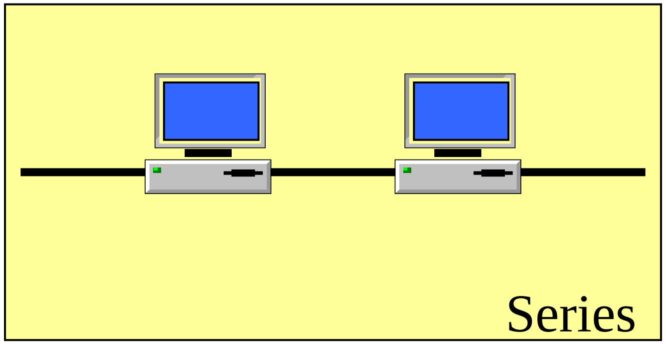
A = {a dish passes the test}, P[A] = 0.8

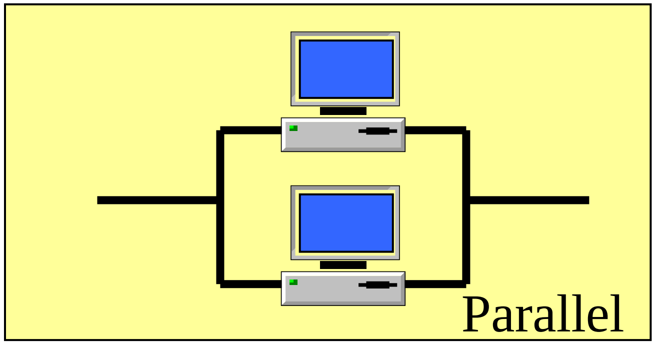
Testing a dish is an independent trial

$$P[A_{x,10}] = \binom{10}{x} (0.8)^{x}(1-0.8)^{10-x}$$

$$P[A_{8,10}] = (45)(0.1678)(0.04) = 0.3$$

Independent Trials: Reliability





Let probability that a computer works = p

Series: $P[A] = P[A_1A_2] = p^2$

Parallel: P[B] = ?

$$P[B] = 1 - P[B^c]$$
$$= 1 - P[B_1^cB_2^c]$$
$$= 1 - (1 - p)^2$$

Outline

- Probability meaning
- Random Variable
- Discrete RV
- Continuous RV
- Multiple RVs

Random Variable

Random Variable

Experiment (Physical Model)

→ Compose of procedure & observation

→ From observation, we get outcomes

→ From all outcomes, we get a (mathematical) probability model called "Sample space"

→ From the model, we get P[A], $A \subset S$

Random Variable

From a probability model

- Ex.: 2 traffic lights, observe the seq. of light

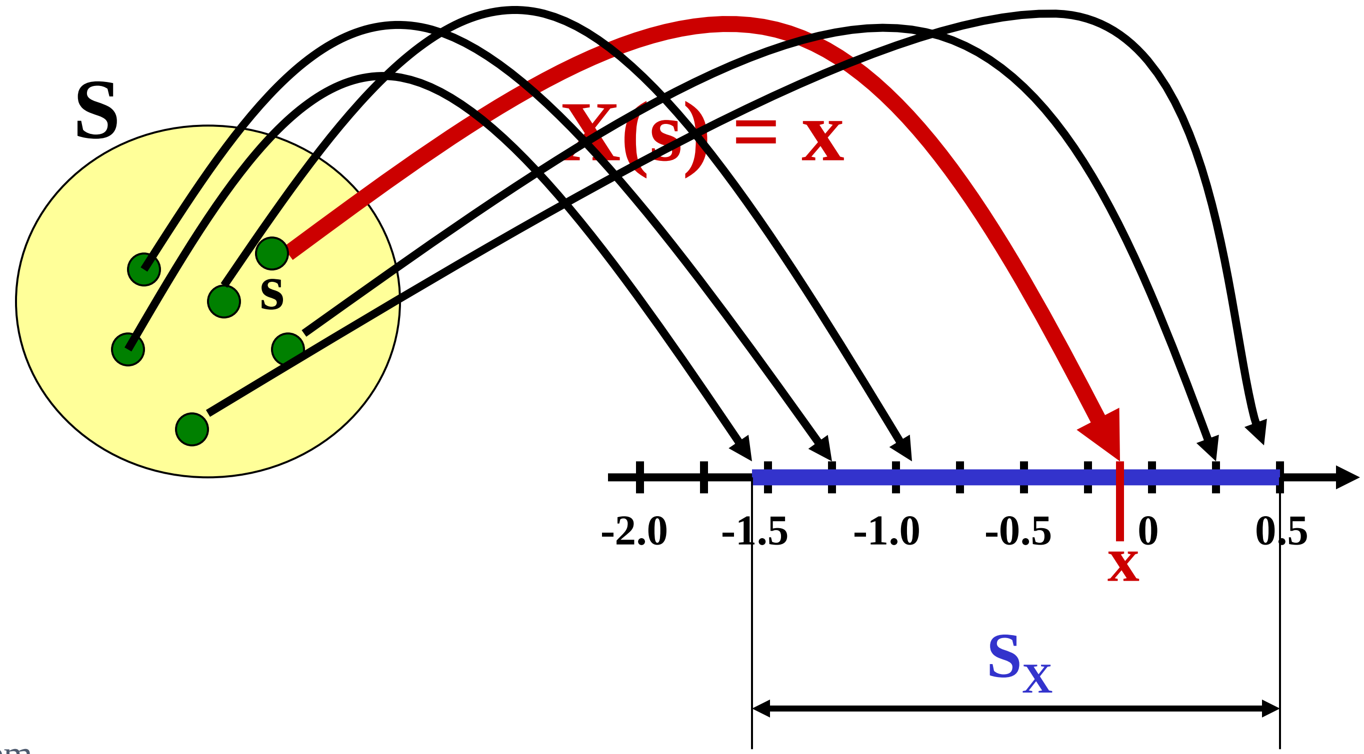
  $S = \{R_1R_2, R_1G_2, G_1R_2, G_1G_2\}$

- If assign a number to each outcome in S, each number that we observe is called "**Random Variable**"
- Observe the number of red light

  $S_X = \{0,1,2\}$

Random Variable

X is a function that maps each outcome, s, in S to a real number X(s), x



2 types of Random Variable

- Discrete Random Variable

  Example:

  X = # of shuttle-cocks used in one badminton game

  Y = # of people in a stand for a world cup soccer match

- Continuous Random Variable

  Example:

  Z = # of minutes for opening a web page

Discrete Random Variable

Definition:

- **X** is a **discrete random variable** if the range of **X** is countable

$$S_{\mathbf{x}} = \{x_1, x_2, \ldots\}$$

- **X** is a **finite random variable** if all values with nonzero probability are in the finite set

$$S_{\mathbf{x}} = \{x_1, x_2, \ldots, x_n\}$$

Why do we need a RV?

- For a probability model (experiment),
  the outcome in S can be in arbitrary form

- If we implement a Random Variable,
  we can calculate the average !

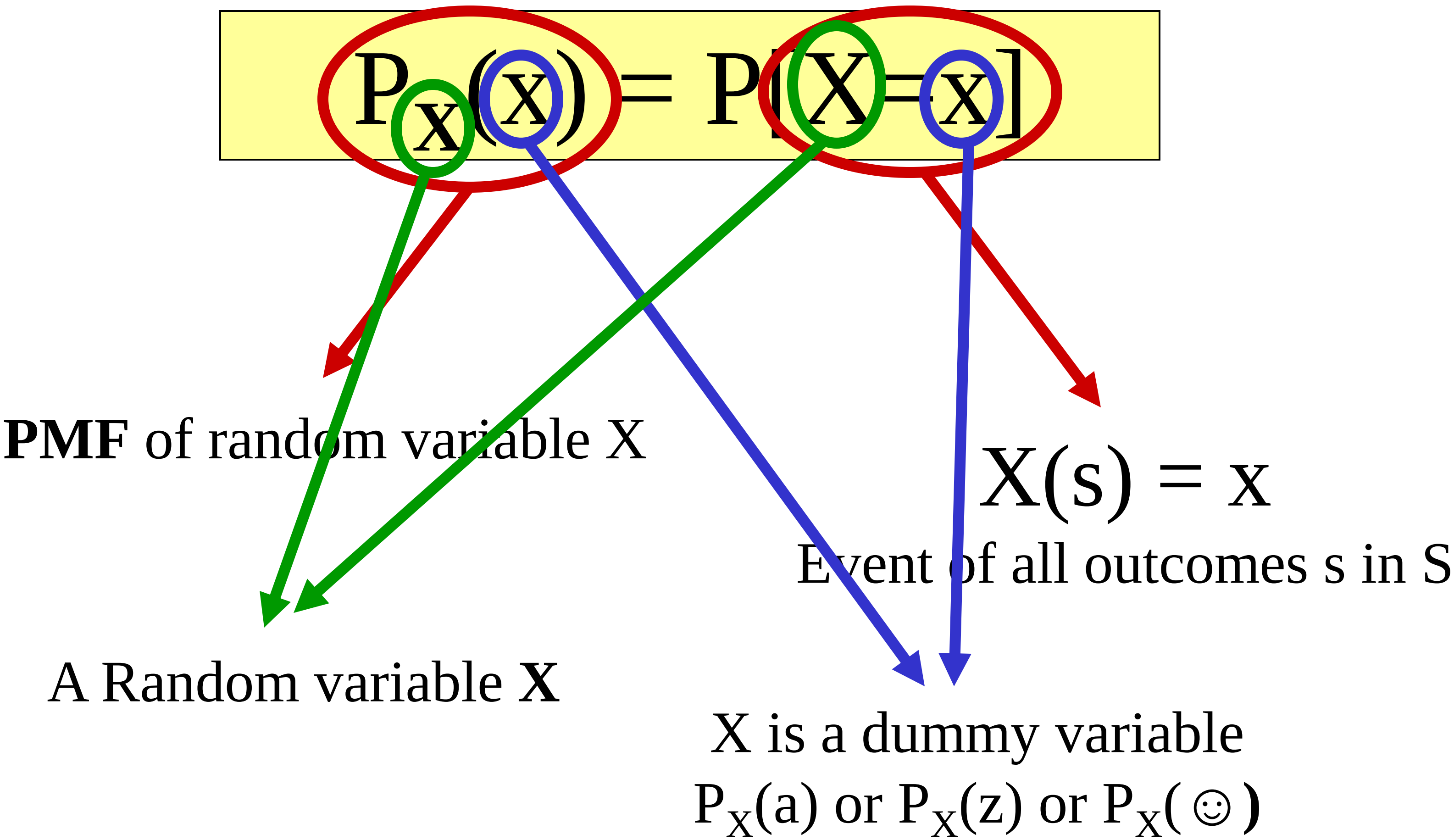
- In Probability, the average is called "**expected value**" of a random variable

Probability Mass Function

- For a (discrete) probability model, **P[A]** = [0,1]

- For a discrete random variable, the probability model is called a "**Probability Mass Function (PMF)**"

Probability Mass Function



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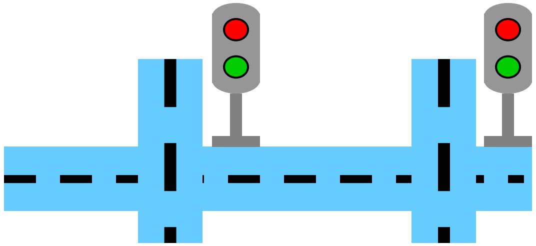
PMF Example



Example:

- 2 traffic lights, observe the seq. of light

  $S = \{ R_1R_2 , R_1G_2 , G_1R_2 , G_1G_2 \}$
- **Find PMF of T, the number of red light**

PMF Example

- T is a random variable of # of red light
  - → Find $P_T(t)$
  - → $P_T(t) = P[T = t]$
  - → First, find probability for each t
  - → Each outcome is equally likely → 1/4

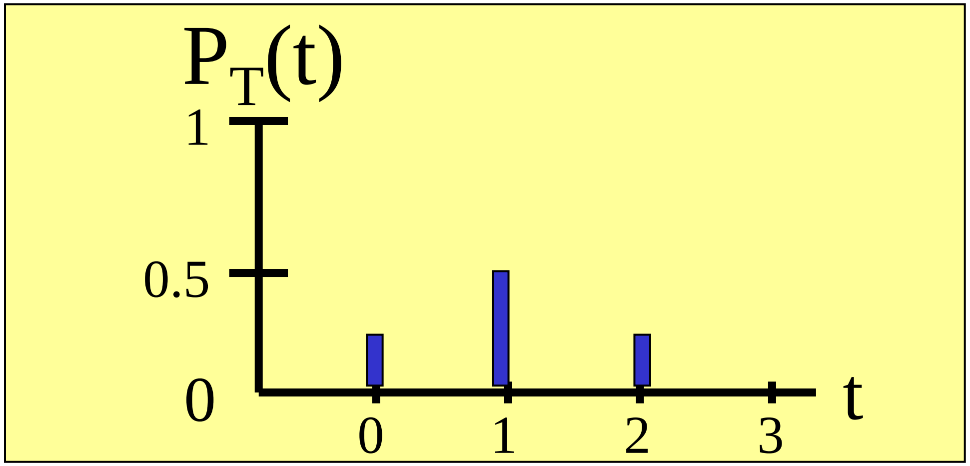
$$P[T=0] = P[\{G_1G_2\}] = 1/4$$

$$P[T=1] = P[\{R_1G_2 , G_1R_2 \}] = 2/4 = 1/2$$

$$P[T=2] = P[\{R_1R_2\}] = 1/4$$

PMF Example

$$P_T(t) = \begin{cases} 1/2 & t = 1 \\ 1/4 & t = 0, 2 \\ 0 & \text{Otherwise} \end{cases}$$



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PMF Theorem

Theorem: For a discrete random variable X with PMF $P_X(x)$ and Range S_X:

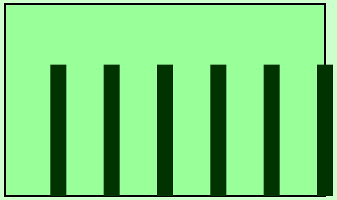
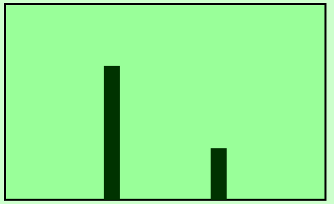
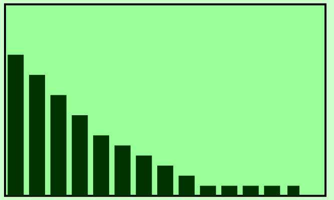
1) For any x, $P_X(x) \geq 0$
2) $\sum_{x \in S_X} P_X(x) = 1$
3) For event $B \subset S_X$, P[B], the probability that X is in the set B is

$$\mathbf{P[B] = \sum_{x \in B} P_X(x)}$$

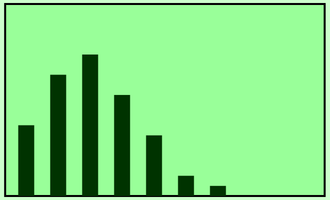
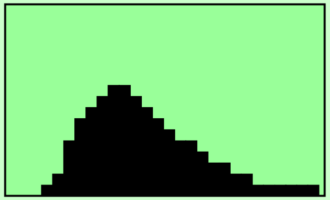
Useful Discrete RV

- Discrete Uniform Random Variable
- Bernoulli Random Variable
- Geometric Random Variable
- Binomial Random Variable
- Pascal Random Variable
- Poisson Random Variable

Useful Discrete RV

Uniform Equiprobable outcomes	$\begin{cases} 1/(j-k+1) & x = k,k+1,k+2,\ldots,j \\ 0 & \text{Otherwise} \end{cases}$	
Bernoulli Pass/Fail	$\begin{cases} 1-p & x = 0 \\ p & x = 1 \\ 0 & \text{Otherwise} \end{cases}$	
Geometric # tests until fail	$\begin{cases} p(1-p)^{x-1} & x = 1,2,3,\ldots \\ 0 & \text{Otherwise} \end{cases}$	

Useful Discrete RV

Binomial # fails in n tests	$\begin{cases} \binom{n}{x} p^x(1-p)^{n-x} & x=1,2,\ldots,n \\ 0 & \text{Otherwise} \end{cases}$	
Pascal # tests until k fails	$\begin{cases} \binom{x-1}{k-1} p^k(1-p)^{x-k} & x=k,k+1,\ldots \\ 0 & \text{Otherwise} \end{cases}$	
Poisson occurrence in a period	$\begin{cases} \dfrac{(\lambda T)^x e^{-(\lambda T)}}{x!} & x=0,1,2\ldots \\ 0 & \text{Otherwise} \end{cases}$