

# LECTURE #11

## M/G/1 WITH VACATION

204528

Queueing Theory and  
Applications in Networks

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# Outline

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- More on M/G/1
- Busy period and its duration
- M/G/1 with Vacations

# M/G/1

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$$W_q = \frac{\lambda \bar{X}^2}{2(1-\rho)}$$

$$N_q = \frac{\lambda^2 \bar{X}^2}{2(1-\rho)}$$

$$W_T = \bar{X} + \frac{\lambda \bar{X}^2}{2(1-\rho)}$$

$$N_T = \rho + \frac{\lambda^2 \bar{X}^2}{2(1-\rho)}$$

# The coefficient of variation ( $C_v$ )

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- A normalized measure of dispersion of a probability distribution
- Defined as the ratio of the standard deviation to the mean

$$C_v = \frac{\sigma}{\mu}$$

- Only defined for *non-zero* mean
- $C_v$  should only be computed for data measured on a **ratio scale**

dispersion = การกระจาย

# The coefficient of variation ( $C_V$ )

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- **Example** (from wikipedia)
  - For a group of temperatures
  - An object changes its temperature by 1 K also changes its temperature by 1 C
  - The standard deviation does not depend on whether the Kelvin or Celsius scale
  - However, the mean temp would differ in each scale by 273
  - So, the coefficient of variation would differ
- **Investment Dictionary** (<http://www.answers.com/topic/coefficient-of-variation>)
  - The lower the ratio, the better your risk-return tradeoff
  - The higher the ratio, the higher the risk

# The coefficient of variation ( $C_V$ )

- In queueing theory
  - Exponential Dist. is often more important than the Normal Dist.
- For  $C_V = 1$ 
  - E.g. Exponential distribution  $\rightarrow$  ( $\sigma = \mu$ )
- For  $C_V < 1 \rightarrow$  low-variance
  - E.g. Erlang distribution [**r-stage Erlangian server ( $E_r$ )**]
- For  $C_V > 1 \rightarrow$  high-variance
  - E.g. Hyper-Exponential distribution [**r-Stage Parallel Servers ( $H_r$ )**]
- Some formulas are expressed using **squared coefficient of variation (SCV)**

# More on P-K Formula

Squared coefficient of variation  $C_v^2$

$$C_v^2 = \frac{\text{Var}[X]}{(\bar{X})^2}$$

$$\begin{aligned}\overline{X^2} &= \text{Var}[X] + (\bar{X})^2 \\ &= (1 + C_v^2) \cdot (\bar{X})^2\end{aligned}$$

# More on P-K Formula

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$$W_q = \frac{\lambda \bar{X}^2}{2(1-\rho)}$$

$$= \frac{1 + C_v^2}{2} \cdot \frac{\rho}{1-\rho} \cdot \bar{X}$$

$$W_T = \bar{X} + \frac{\lambda \bar{X}^2}{2(1-\rho)}$$

$$= \left( 1 + \frac{1 + C_v^2}{2} \cdot \frac{\rho}{1-\rho} \right) \cdot \bar{X}$$



# More on P-K Formula

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$$N_q = \frac{\lambda^2 \bar{x}^2}{2(1-\rho)}$$

$$= \frac{1 + C_v^2}{2} \cdot \frac{\rho^2}{1-\rho}$$

$$N_T = \rho + \frac{\lambda^2 \bar{x}^2}{2(1-\rho)}$$

$$= \rho + \frac{1 + C_v^2}{2} \cdot \frac{\rho^2}{1-\rho}$$

# More on P-K Formula

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- Mean values depend only on the expectation  $E[X]$  and variance  $\text{Var}[X]$  of the service time distribution but not on higher moments.
- Mean values increase linearly with the variance.
- Randomness, ‘disarray’, leads to an increased waiting time and queue length.
- The formula are similar to those of the  $M/M/1$  queue;
  - the only difference is the extra factor  $\frac{1 + C_v^2}{2}$

# M/G/1 Steady-State

$$W_q = \frac{\lambda (\overline{X^2} + \sigma^2)}{2(1 - \rho)}$$

$$N_q = \frac{\lambda^2 (\overline{X^2} + \sigma^2)}{2(1 - \rho)}$$

$$W_T = \overline{X} + \frac{\lambda (\overline{X^2} + \sigma^2)}{2(1 - \rho)}$$

$$N_T = \rho + \frac{\lambda^2 (\overline{X^2} + \sigma^2)}{2(1 - \rho)}$$

$$p_0 = 1 - \rho$$

# M/G/1 Example

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- There are two workers competing for a job.
- **Dang** claims that an average service time is **faster** than **Jook**'s
- But **Jook** claims to be **more consistent**, if not as fast.
- The arrivals is a Poisson process at a rate of  $\lambda = 2$  per hour. (1/30 per minute).
- Dang's service statistics are an average service time of 24 minutes with a standard deviation of 20 minutes.
- Jook's service statistics are an average service time of 25 minutes, but a standard deviation of only 2 minutes.
- If the average length of the queue is the criterion for hiring, which worker should be hired?

# M/G/1 Example

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- For Dang,

$$\lambda = 1/30 \text{ (per min)}$$

$$1/\mu = 24 \text{ min.}$$

$$\rho = \lambda/\mu = 24/30 = 4/5$$

$$\sigma^2 = 20^2 = 400 \text{ min}^2$$

$$N_q = \frac{\lambda^2(\overline{X^2} + \sigma^2)}{2(1 - \rho)} = \frac{(1/30)^2 (24^2 + 400)}{2(1 - 4/5)}$$

$$= 2.711 \text{ customers}$$

# M/G/1 Example

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- For Jook,

$$\lambda = 1/30 \text{ (per min)}$$

$$1/\mu = 25 \text{ min.}$$

$$\rho = \lambda/\mu = 25/30 = 5/6$$

$$\sigma^2 = 2^2 = 4 \text{ min}^2$$

$$N_q = \frac{\lambda^2(\overline{X^2} + \sigma^2)}{2(1 - \rho)} = \frac{(1/30)^2 (25^2 + 4)}{2(1 - 5/6)} = 2.097 \text{ customers}$$

# M/G/1 Example

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- Although working faster on the average,
  - Dang's greater service variability results in an average queue length about **30% greater** than Jook's.
- On the other hand, the proportion of arrivals who would find Dang **idle** and thus experience no delay is  $p_0 = 1 - \rho = 1/5 = \mathbf{20\%}$
- While the proportion who would find Jook **idle** and thus experience no delay is  $p_0 = 1 - \rho = 1/6 = \mathbf{16.7\%}$ .
- On the basis of average **queue length**,  $N_q$ ,
  - Jook wins.

# Busy and Idle period

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- To derive the distribution for the M/G/1 queue
  - the length of the Idle period
  - the length of the busy period



# Busy and Idle period

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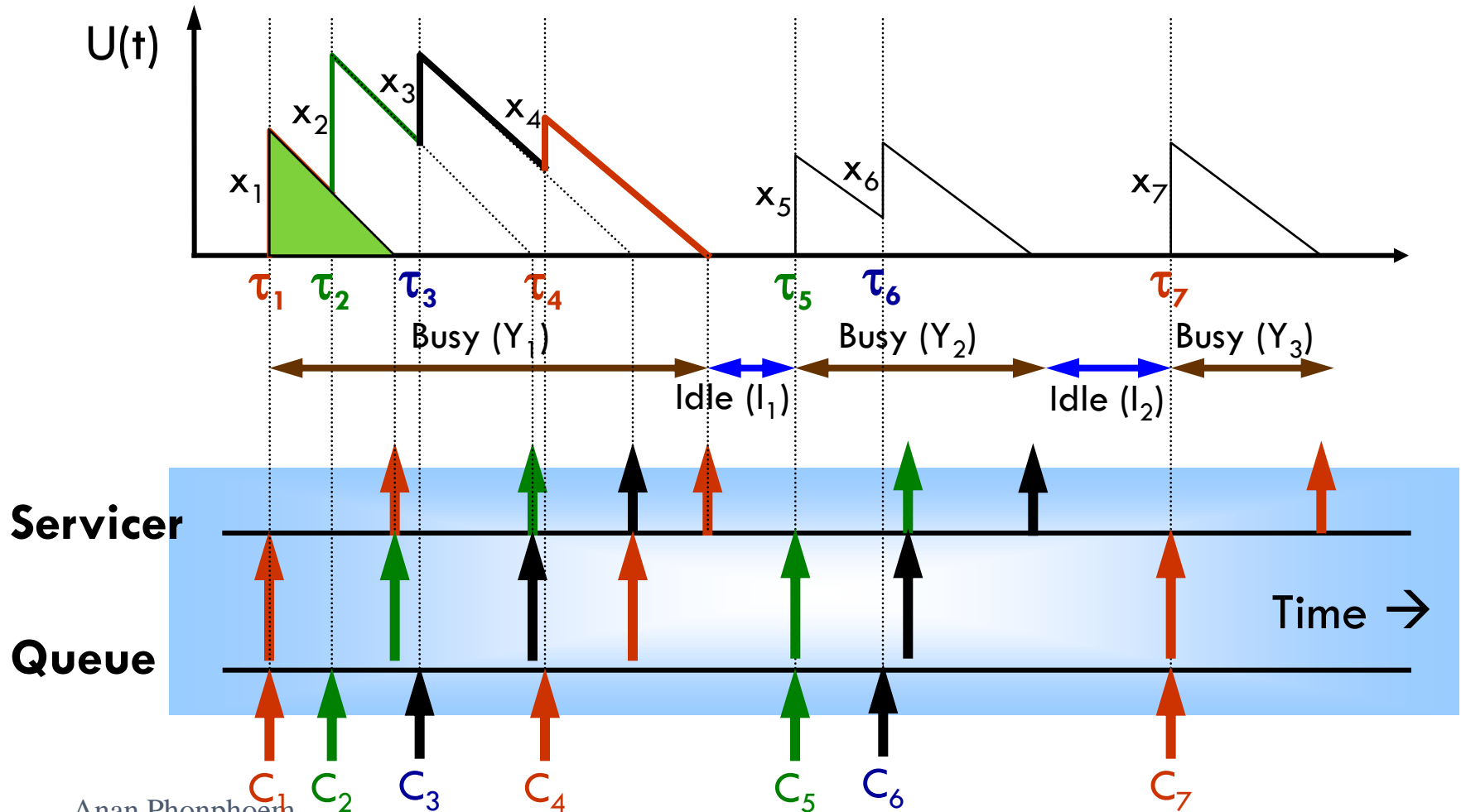
- $C_n$  = the  $n^{\text{th}}$  customer to enter the system
- $\tau_n$  = arrival time of  $C_n$
- $t_n = \tau_n - \tau_{n-1}$  = interarrival time between  $C_{n-1}$  and  $C_n$
- $x_n$  = Service time of  $C_n$

# Busy and Idle period

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- $U(t)$  = Unfinished work in the system  
= Virtual waiting time at time  $t$
- $Y_n$  = Busy period
- $I_n$  = Idle period

# The Unfinished Work (FCFS)



# M/G/1 (FCFS)

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- $A(t) = P[t_n \leq t] = 1 - e^{-\lambda t} \quad t \geq 0$
- $B(x) = P[x_n \leq x]$
- $A(t)$  and  $B(x)$  are independent on  $n$
- $F(y) =$  Idle period distribution  
 $= P[I_n \leq y]$
- $G(y) =$  Busy period distribution  
 $= P[Y_n \leq y]$

# M/G/1 (FCFS)

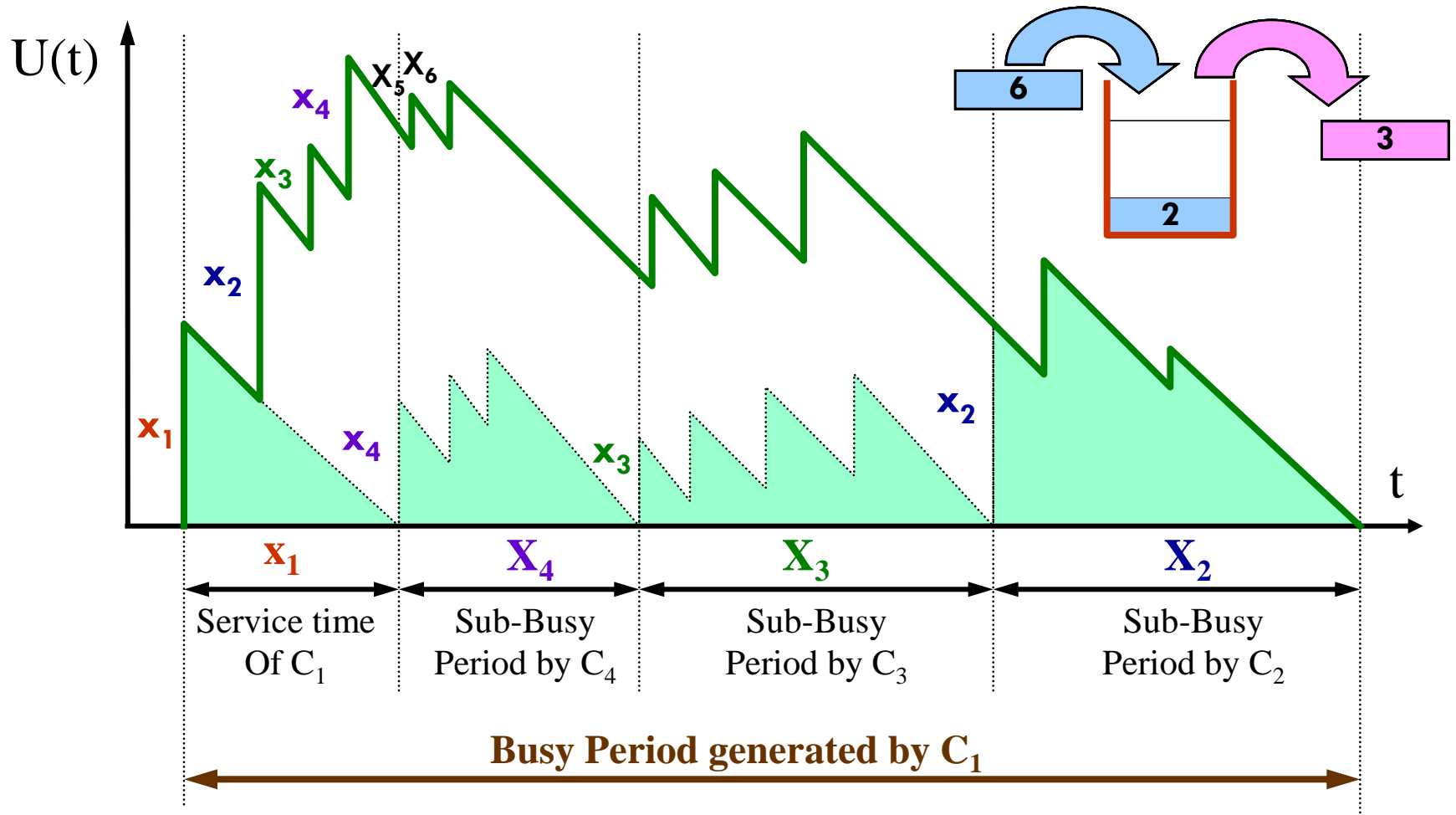
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- For Idle period  $F(y)$ 
  - After busy period  $\rightarrow$  start the idle period
  - A new idle period will stop immediately when the new customer arrives
  - Therefore, from the memoryless distribution

$$F(y) = 1 - e^{-\lambda y} \quad y \geq 0$$

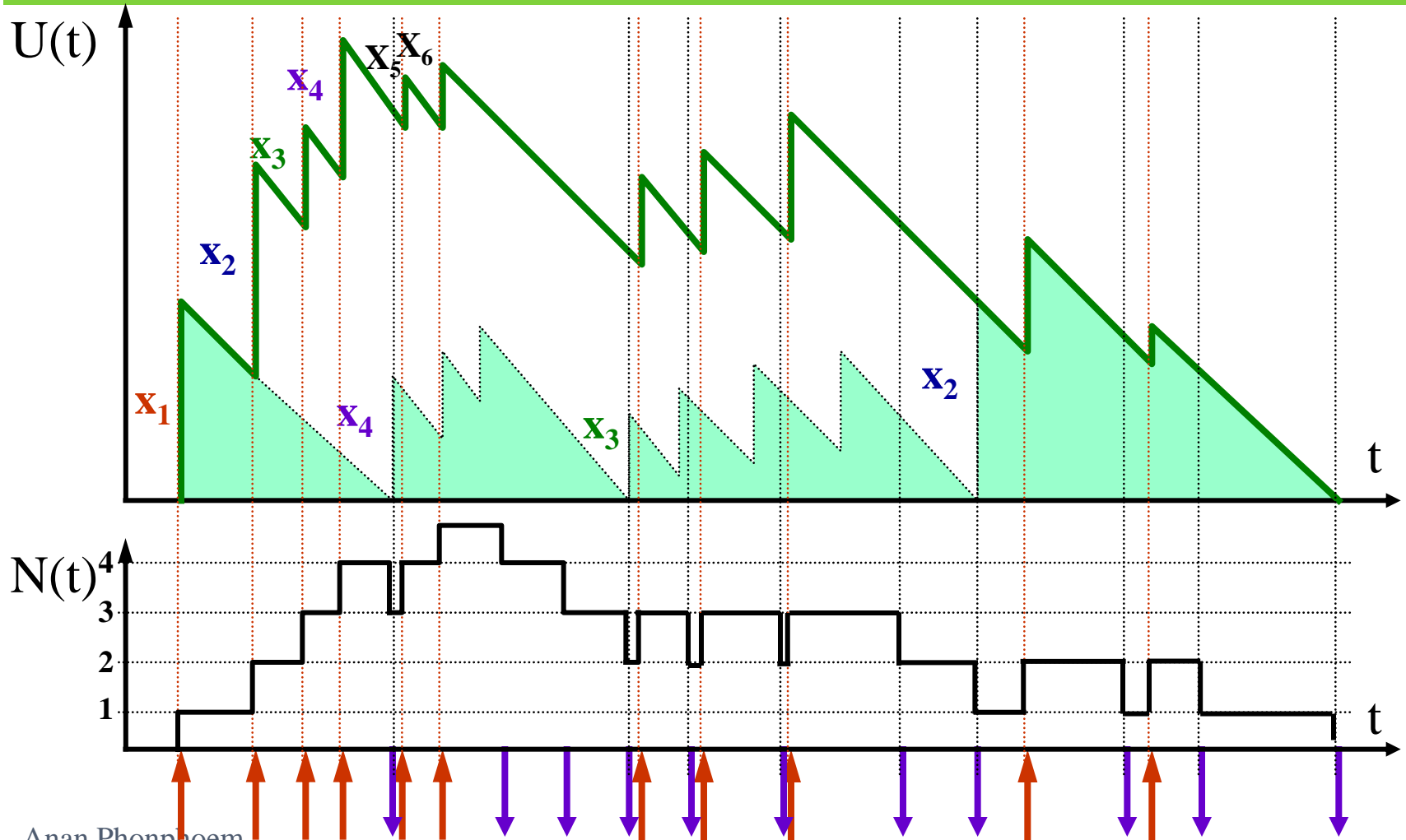
# The Unfinished Work (LCFS)

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# Number of Users (LCFS)

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# M/G/1 (LCFS)

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- Each sub-busy period behaves statistically the same as the major busy period

- The duration of busy period  $Y$

$$Y = X_1 + X_{v+1} + X_{v+2} + \dots + X_3 + X_2$$

- $X_v =$  sub-busy period
- $v =$  an RV = # of customer arrives during  $C_1$  service interval



# M/G/1 (LCFS)

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- For Busy period  $G(y)$

$$G(y) = P[Y_n \leq y] \quad y \geq 0$$

- Transform of M/G/1 busy-period distribution

$$G^*(s) = B^*[s + \lambda - \lambda G^*(s)]$$

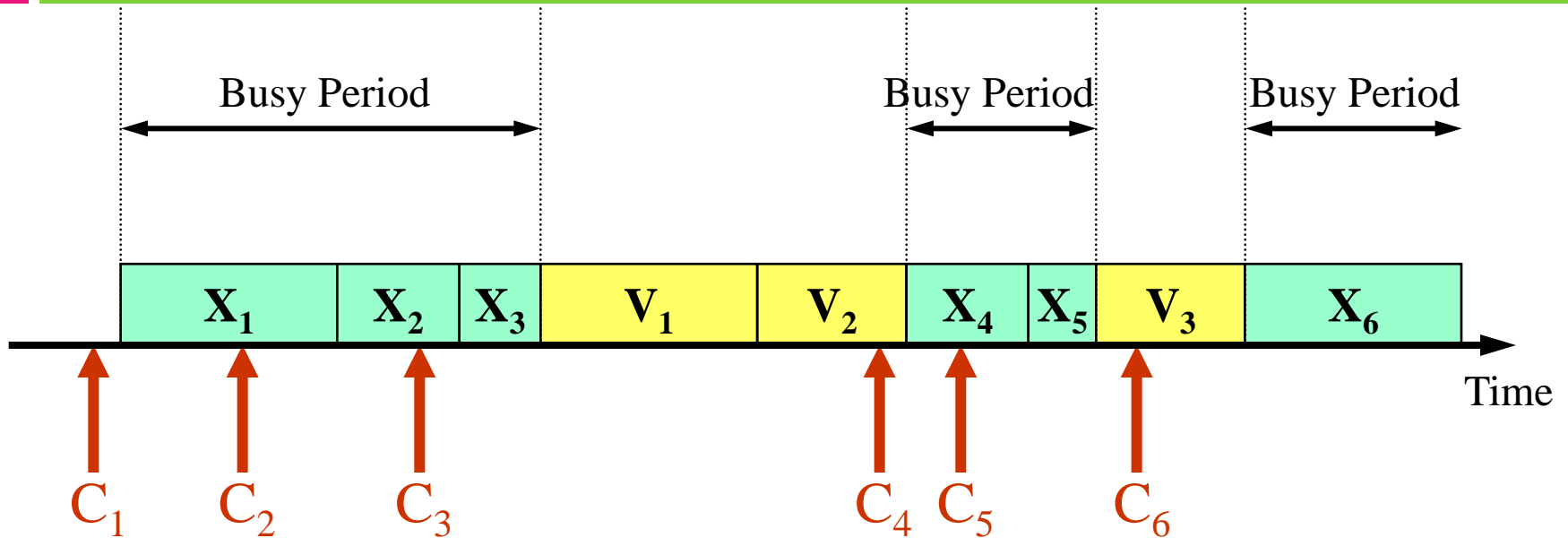
# M/G/1 with Vacations

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- At the end of busy period
  - The server goes on “vacation”
  - The vacation period = random interval of time
  - A new arrive during vacation has to wait until the end of vacation period
  - If the system is idle after vacation, a new vacation starts right away

# M/G/1 with Vacations

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- $V_n =$  Vacation period with  $\bar{V}$  and  $\bar{V}^2$   
= IID random variable and independent of customer interarrival and service time
- $X_n =$  Service period

# M/G/1 with Vacations

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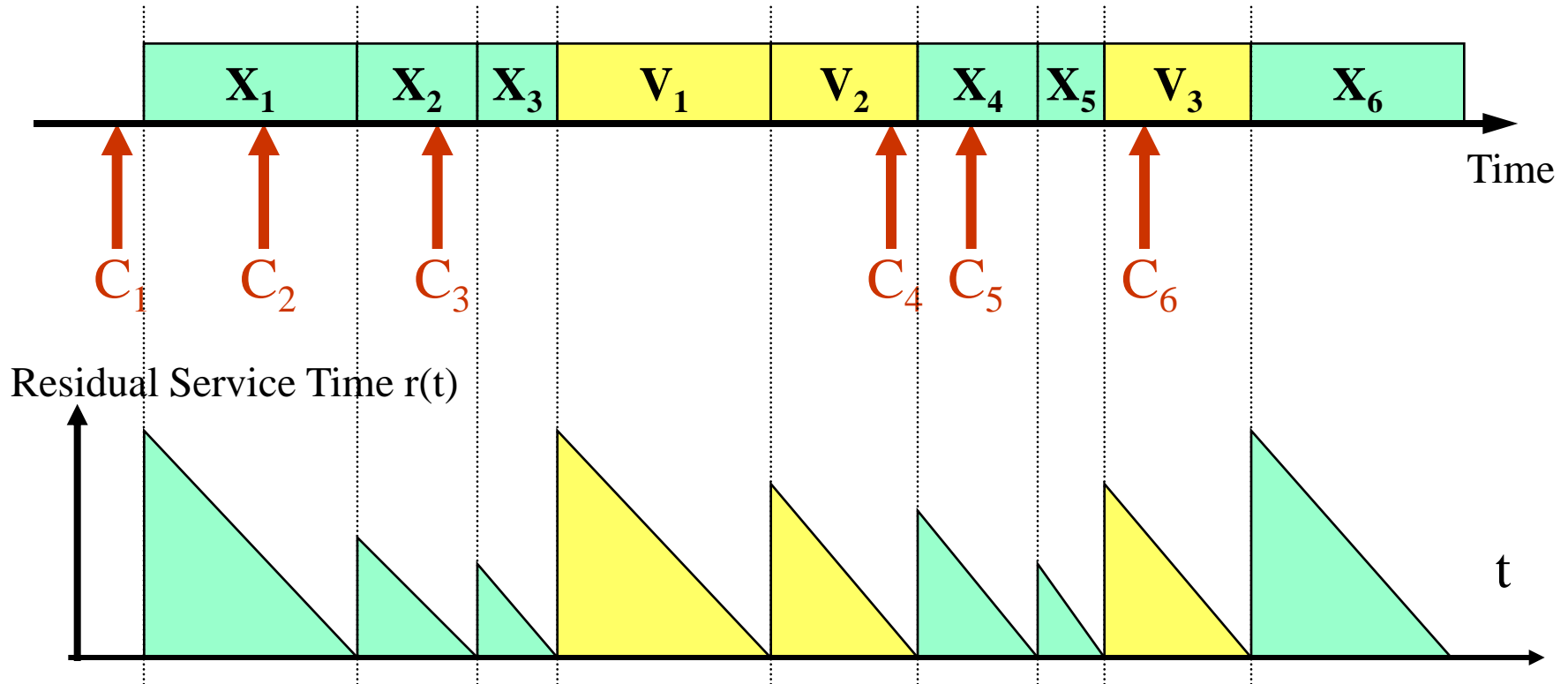
- A new customer is Poisson arrival and service time is general distribution
- The waiting time for customer is  $W$

$$W = \frac{R}{1 - \rho}$$

- $R =$  Residual Time

# M/G/1 with Vacations

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# M/G/1 with Vacations

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$$\begin{aligned} R &= \frac{1}{t} \int_0^t r(\tau) d\tau = \frac{1}{t} \sum_{i=1}^{M(t)} \frac{1}{2} X_i^2 + \frac{1}{t} \sum_{i=1}^{L(t)} \frac{1}{2} V_i^2 \\ &= \frac{M(t)}{t} \frac{\sum_{i=1}^{M(t)} \frac{1}{2} X_i^2}{M(t)} + \frac{L(t)}{t} \frac{\sum_{i=1}^{L(t)} \frac{1}{2} V_i^2}{L(t)} \end{aligned}$$

# M/G/1 with Vacations

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$$t \rightarrow \infty, \quad \frac{M(t)}{t} = \lambda \quad \text{and} \quad \frac{L(t)}{t} = \frac{(1-\rho)}{\overline{V}}$$

$$R = \frac{1}{2} \lambda \overline{X^2} + \frac{(1-\rho)\overline{V^2}}{2\overline{V}}$$

$$W = \frac{\lambda \overline{X^2}}{2(1-\rho)} + \frac{\overline{V^2}}{2\overline{V}}$$