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Probability Theory and Random Processes

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Lecture #11
Stochastic Process – II

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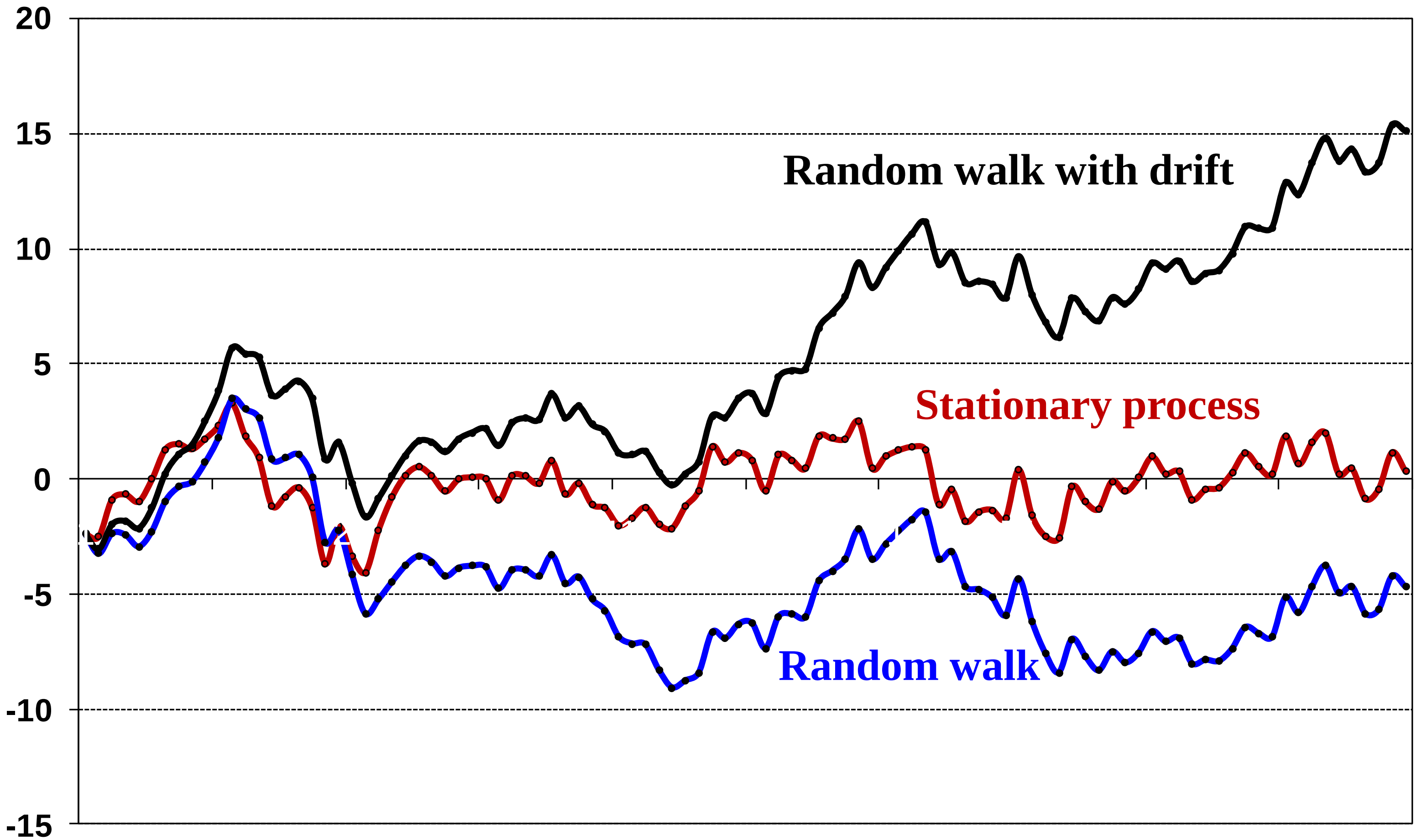
Outline

- Stationary Process
- Wide-sense Stationary Process

Stationary and Non-stationary Process



From Christopher Dougherty 2000–2006 slide

Stationary Process

- For a random process $X(t)$,
  - **At t_1**: $X(t_1)$ has pdf = $\mathbf{f_{X(t_1)}(x)}$
  - Normally, pdf depends on t_1
  - However, pdf may not depend on t_1
- **Stationary Process**
  - Same random variable at all time
  - No statistical properties change with time

$$\boxed{f_{X(t_1)}(x) = f_{X(t_1+\tau)}(x) = f_X(x)}$$

Stationary Process

Definition: A stochastic process X(t) is stationary iff for all sets of time $t_1,\ldots,t_m$ and any time different τ,

$$f_{X(t_1),\ldots,X(t_m)}(x_1,\ldots,x_m) = f_{X(t_1+\tau),\ldots,X(t_m+\tau)}(x_1,\ldots,x_m)$$

Stationary Random Sequence

Definition: A random sequence X_n is stationary iff for any finite sets of time instants $n_1,\ldots,n_m$ and any time different k,

$$f_{X(n_1),\ldots,X(n_m)}(x_1,\ldots,x_m) = f_{X(n_1+k),\ldots,X(n_m+k)}(x_1,\ldots,x_m)$$

Stationary Process

Theorem: A stationary process X(t),

$$\mu_X(t) = \mu_X$$

$$R_X(t,\tau) = R_X(0,\tau) = R_X(\tau)$$

$$C_X(t,\tau) = R_X(\tau) - \mu^2_X = C_X(\tau)$$

autocorrelation
of a stationary process (independent of t)

autocovariance
of a stationary process (independent of t)

Stationary Random Sequence

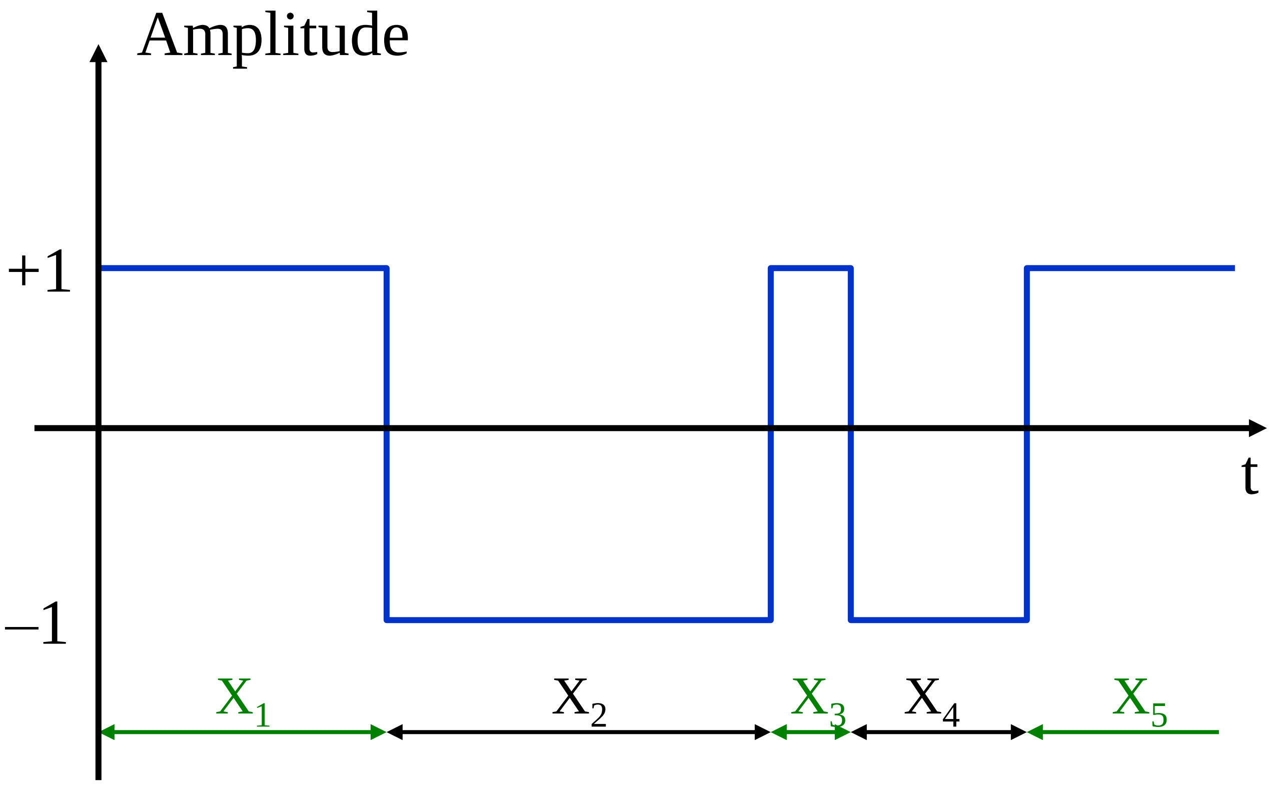
Theorem: A stationary random sequence X_n, for all m

$$E[X_m] = \mu_X$$
$$R_X[m.k] = R_X[0,k] = R_X[k]$$
$$C_X[m,k] = R_X[k] - \mu^2_X = C_X[k]$$

Example

- Telegraph Signal, $X(t)$ take value ± 1
- $X(0) = \pm 1$ with probability $= 0.5$
- Let $X(t)$ toggles the polarity with each occurrence of an event in a Poisson process rate α

Example



Example

- Find PMF of X(t), $f_{X(t)}(x)$

- $P[X(t)=1] = P[X(t) \mid X(0) = 1]\, P[X(0) = 1] + P[X(t) \mid X(0) = -1]\, P[X(0) = -1]$

- $P[X(t) \mid X(0) = 1] = P[N(t) = \text{even}]$

$$= \sum_{j=0}^{\infty} \frac{(\alpha t)^{2j}}{(2j)!} e^{-\alpha t}$$

$$= e^{-\alpha t} (1/2) (e^{-\alpha t} + e^{-\alpha t})$$

$$= (1/2)(1 + e^{-2\alpha t})$$

Example

- $P[X(t) \mid X(0) = -1]$ $= P[N(t) = \text{odd}]$

$$= \sum_{j=0}^{\infty} \frac{(\alpha t)^{2j+1}}{(2j+1)!} e^{-\alpha t}$$

$$= e^{-\alpha t} (1/2) (e^{-\alpha t} - e^{-\alpha t})$$

$$= (1/2)(1 - e^{-2\alpha t})$$

Example

- $P[X(t) = 1]$

  $= P[X(t) \mid X(0) = 1]\, P[X(0) = 1]$

  $+ P[X(t) \mid X(0) = -1]\, P[X(0) = -1]$

  $= (1/2)(1 + e^{-2\alpha t})(1/2) + (1/2)(1 - e^{-2\alpha t})(1/2)$

  $= 1/2$

- $P[X(t) = -1]$

  $= 1 - P[X(t) = 1] = 1/2$

$$f_{X(t)}(x) = \begin{cases} \tfrac{1}{2} & X(t) = -1, 1 \\ 0 & \text{Otherwise} \end{cases}$$

Example

- $\mu_X(t) = 1\,(1/2) + (-\,1)(1/2) = 0$
- $\mathrm{Var}\,[X(t)] = E[X^2(t)]$

$$= 1^2(1/2) + (-1)^2(1/2) = 1$$

$$f_{X(t_1),\ldots,X(t_m)}(x_1,\ldots,x_m) = f_{X(t_1+\tau),\ldots,X(t_m+\tau)}(x_1,\ldots,x_m)$$

Wide Sense Stationary

Definition: $X(t)$ is a wide sense stationary random process iff for all t,

$$E[X(t)] = \mu_X$$

$$R_X(t,\tau) = R_X(0,\tau) = R_X(\tau)$$

Definition: X_n is a wide sense stationary random sequence iff for all n,

$$E[X_n] = \mu_X$$

$$R_X[n,k] = R_X[0,k] = R_X[k]$$

Wide Sense Stationary

- For every **stationary** process or sequence, it is also **wide sense stationary**.
- However, if it is a **wide sense stationary** it may or may not be **stationary**.

Example

- For n = even
  - $X_n = \pm 1$ with prob = 0.5
- For n = odd
  - $X_n = -1/3$ with prob = 0.9
  - $X_n = 3$ with prob = 0.1
- Stationary ?
  - **No**
- Wide sense stationary ?
  - Mean = 0 for all n
  - $C_X(t,\tau) = 0$ for $\tau > 0$
  - $C_X(t,\tau) = 1$ for $\tau = 0$
  - **Yes , it's wide sense stationary**

Wide Sense Stationary

Theorem: For a wide sense stationary process X(t),

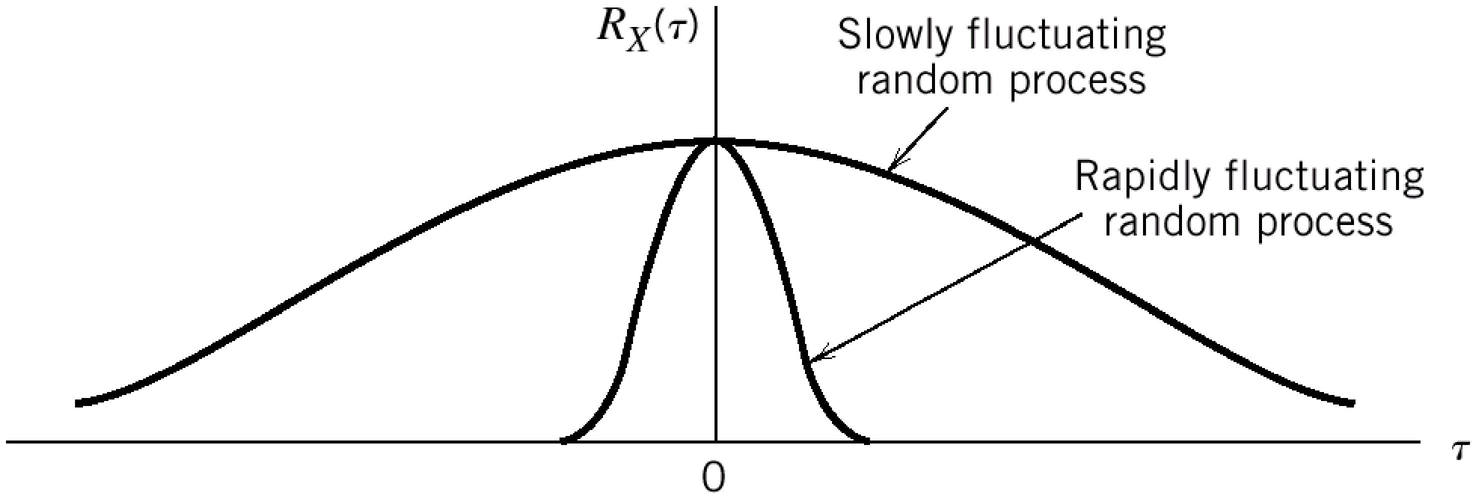
$$R_X(0) \geq 0$$

$$R_X(\tau) = R_X(-\tau)$$

$$|R_X(\tau)| \leq R_X(0) \qquad -R_X(0) \leq R_X(\tau) \leq R_X(0)$$

$R_X(\tau)$

- The $R_X(\tau)$ provides the interdependence information of two random variables obtained from $X(t)$ at times τ seconds apart



http://cc.ee.ntu.edu.tw/~wujsh/PC%20Chapter1.ppt

Wide Sense Stationary

Theorem: For a wide sense stationary sequence X_n,

$$R_X[0] \geq 0$$

$$R_X[k] = R_X[-\tau]$$

$$|R_X[k]| \leq R_X[0]$$

Average Power

- From Ohm's Law : V = IR
- For v(t), i(t), and R
  - The instantaneous power dissipated P(t)

$$P(t) = v^2(t)/R = i^2(t)R$$

- For $R = 1\ \Omega$, $P(t) = v^2(t) = i^2(t)$
- For a voltage or current is a sample function of random process, $x(t,s)$

  ➔ P across $1\ \Omega$ resistor $= x^2(t,s)$

Average Power

- **Define** $x^2(t,s)$
  - as the instantaneous power of $x(t,s)$
- For a $X(t)$,
  - $X^2(t)$ is the instantaneous of power $X(t)$

Definition:

The **average power** of a wide sense stationary process $X(t)$ is

$$R_X(0) = E[X^2(t)]$$