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Probability and Random Processes
for Computer Engineering
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Multiple Discrete Random Variables (2)

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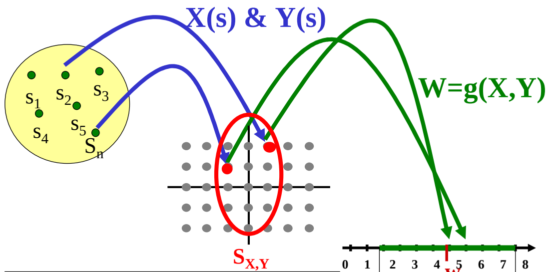
Derived Random Variable Functions of 2 RVs



$$P_W(w) = \sum_{(x,y):g(x,y)=w} P_{X,Y}(x,y)$$

Example

- Fax Sending (text-40sec & graphics-60sec)

$P_{S,T}(s,t)$	t = 40	t = 60
s = 1 sheet	0.15	0.1
s = 2 sheets	0.3	0.2
s = 3 sheets	0.15	0.1

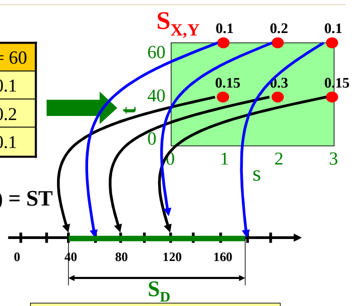
Let D = duration for sending one fax
= g(S,T)
= ST

Find $P_D(d)$, S_D, and E[D]

Example

$P_{S,T}(s,t)$	t = 40	t = 60
s = 1 sheet	0.15	0.1
s = 2 sheets	0.3	0.2
s = 3 sheets	0.15	0.1

D = g(S,T) = ST



S_D = {40,60,80,120,180}

Example

S_D={40,60,80,120,180}

$$P_D(d) = \sum_{(s,t):g(s,t)=d} P_{S,T}(s,t)$$

$$P_D(d) = \begin{cases} 0.15 & d = 40 \\ 0.1 & d = 60 \\ 0.3 & d = 80 \\ 0.15 + 0.2 & d = 120 \\ 0.1 & d = 180 \\ 0 & \text{Otherwise} \end{cases}$$

$$E[D] = \sum_{d \in S_D} P_D(d)$$

= 96 sec

Expectations

- E[W] for W = g(X,Y)
- E[X+Y]
- Var[X+Y] (Variance)
- Cov[X,Y] (Covariance)
- $r_{X,Y}$ (Correlation)
- $\rho_{X,Y}$ (Correlation Coefficient)

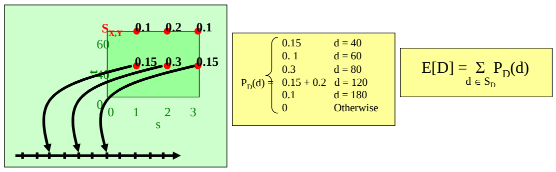
Expected Value of g(X,Y)

Theorem: for W = g(X,Y)

$$E[W] = \sum_{x \in S_X} \sum_{y \in S_Y} g(X,Y)\ P_{X,Y}(x,y)$$

From Last Example

To find the E[D], D = g(S,T) = ST

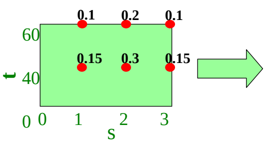


Map g(S,T) → D Find $P_D(d)$ Find E[D]

From Last Example

With the theorem, we can directly find E[D]

$$E[D] = \sum_{s=1}^{3} \sum_{t=40,60} st\ P_{S,T}(s,t)$$



E[D] = 1*40*0.15 + 1*60*0.1 +
2*40*0.3 + 2*60*0.2 +
3*40*0.15 + 3*60*0.1
= 96 sec

For any 2 RVs

Theorem:

$$E[X + Y] = E[X] + E[Y]$$

- Find E[X] and E[Y]
  → Marginal PMF

Var[X+Y]

Definition: $Var[X] = E[(X - \mu_x)^2]$

$Var[X+Y] = E[((X+Y) - \mu_{x+Y})^2]$

$= E[((X+Y) - (\mu_x + \mu_Y))^2]$

$= E[((X-\mu_x) + (Y-\mu_Y))^2]$

$= E[(X-\mu_x)^2 + 2(X-\mu_x)(Y-\mu_Y) + (Y-\mu_Y)^2]$

$= E[(X-\mu_x)^2] + 2E[(X-\mu_x)(Y-\mu_Y)] + E(Y-\mu_Y)^2]$

Theorem:
$Var[X+Y] = Var[X] + Var[Y] + 2E[(X-\mu_x)(Y-\mu_Y)]$

Covariance

Covariance of X and Y

Definition: $Cov[X,Y] = E[(X-\mu_x)(Y-\mu_Y)]$

$Cov[X,Y] = E[XY - \mu_x Y - \mu_Y X + \mu_x \mu_y]$

$= E[XY] - \mu_x E[Y] - \mu_Y E[X] + \mu_x \mu_y$

$= E[XY] - \mu_x \mu_y - \mu_x \mu_y + \mu_x \mu_y$

Theorem: $Cov[X,Y] = E[XY] - \mu_x \mu_y$

Covariance of X and Y

Theorem: $Cov[X,Y] = E[XY] - \mu_x \mu_y$

Correlation

If $X = Y$ ➔ $Cov[X,X] = E[XX] - \mu_x \mu_x$

$= E[X^2] - \mu_x^2$

$= E[X^2 - 2\mu_x^2 + \mu_x^2]$

$= E[X^2 - 2\mu_x X + \mu_x^2]$

$= E[(X - \mu_x)^2]$

$= Var[X]$

If μ_x or $\mu_Y = 0$ ➔ $Cov[X,Y] = E[XY]$

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Correlation

Definition: The correlation of X and Y is $r_{X,Y}$

$r_{X,Y} = E[XY]$

Theorem: $Cov[X,Y] = r_{X,Y} - \mu_x \mu_y$

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More Definition

Definition 1:
X and Y are **Orthogonal** if $r_{X,Y} = 0$ ***;E[XY]=0***

Definition 2:
X and Y are **Uncorrelated** if $Cov[X,Y] = 0$

Definition 3:
Correlation Coefficient of X and Y is

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}} = [-1, 1]$$

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Correlation Coefficient

- $\rho_{X,Y}$
  - Describes the info about X by observing Y
- $\rho_{X,Y} > 0$
  - If X ⇑ (relative to mean) → Y ⇑
  - If X ⇓ (relative to mean) → Y ⇓
- $\rho_{X,Y} < 0$
  - If X ⇑ (relative to mean) → Y ⇓
  - If X ⇓ (relative to mean) → Y ⇑
- Example:
  - X = student's height, Y= student's weight $\rho_{X,Y} > 0$
  - X = cell phone distance, Y= Rx signal Strength $\rho_{X,Y} < 0$

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Uncorrelated

If X and Y are Independent, then

➔ $Cov[X,Y] = 0$ ➔ $\rho_{X,Y} = 0$

➔ X and Y are Uncorrelated

Note:

If X and Y are Uncorrelated,

➔ X and Y **may or may not** Independent

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Conditional Joint PMF by an Event

$$P_{X,Y|B}(x,y) = \frac{P[(X=x,Y=y) \cap B]}{P[B]}$$

If $(X=x, Y=y) \in B$ → $(X=x,Y=y) \cap B = (X=x, Y=y)$

$$P_{X,Y|B}(x,y) = \begin{cases} \dfrac{P[(X=x,Y=y)]}{P[B]} & (x,y) \in B \\ 0 & \text{Otherwise} \end{cases}$$

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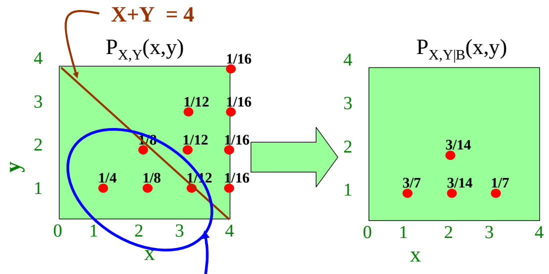
Example $P_{X,Y|B}(x,y)$

Let $B = \{X+Y \le 4\}$ Find $\mathbf{P_{X,Y|B}(x,y)}$



$B = \{X+Y \le 4\}$ → $P[B] = 7/12$

Conditional PMF

- Special case of Conditional Joint PMF by an Event
  →the Event is X=x or Y=y
- $P_{X,Y|B}(x,y)$ when $B = \{Y=y\}$
  → $P_{X,Y|Y=y}(x,y) = P_{X|Y}(x|y)$

Definition: $P_{X|Y}(x|y) = P[X=x \mid Y=y]$

Conditional PMF

$$P_{X|Y}(x|y) = P[X=x \mid Y=y] = \frac{P[X=x,Y=y]}{P[Y=y]} = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

Theorem:
$P_{X,Y}(x,y) = P_{X|Y}(x|y)P_Y(y) = P_{Y|X}(y|x)P_X(x)$

Independent RVs

- From the independent definition
  A and B are independent iff P[AB] = P[A]P[B]
- X and Y are independent RVs if and only if
  {X=x} and {Y=y} are independent for all x,y in $S_{X,Y}$

Definition: $P_{X,Y}(x,y) = P_X(x)P_Y(y)$

Independent RVs

Theorem:

(a) $r_{X,Y} = E[XY] = E[X]E[Y]$
(b) $E[X|Y = y] = E[X]$ for all $y \in S_Y$
(c) $E[Y|X = x] = E[Y]$ for all $x \in S_X$
(d) $Var[X+Y] = Var[X] + Var[Y]$
(e) $Cov[X,Y] = \rho_{X,Y} = 0$

More than 2 RVs

Definition: Joint PMF of discrete RV $X_1,\ldots X_N$ is

$$P_{X_1,\ldots,X_N}(x_1,\ldots,x_N) = P[X_1=x_1, \ldots, X_N=x_N]$$

Homework

1) 2.6.5
2) 2.7.7
3) 2.8.6
4) 2.9.6

1) 3.1.6
2) 3.2.5
3) 3.3.5
4) 3.4.6
5) 3.5.2
6) 3.6.3
7) 3.6.6
8) 3.7.8
9) 3.8.3

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