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Probability and Random Processes
for Computer Engineering
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Stochastic Process (3)

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Stationary Process

- For a random process X(t), normally,
  at t_1: $X(t_1)$ has pdf = $f_{X(t_1)}(x)$ [depends on t_1]
- For a random process X(t),
  at t_1: $X(t_1)$ has pdf = $f_{X(t_1)}(x)$ [not depend on t_1]

Stationary Process
= same random variable at all time
= no statistical properties change with time

$$f_{X(t_1)}(x) = f_{X(t_1+\tau)}(x) = f_X(x)$$

Stationary Process

Definition: A stochastic process X(t) is stationary iif for all sets of time $t_1,\ldots,t_m$ and any time different τ,

$$f_{X(t_1),\ldots,X(t_m)}(x_1,\ldots,x_m) = f_{X(t_1+\tau),\ldots,X(t_m+\tau)}(x_1,\ldots,x_m)$$

Stationary Random Sequence

Definition: A random sequence X_n is stationary iif for any finite sets of time instants $n_1,\ldots,n_m$ and any time different k,

$$f_{X(n_1),\ldots,X(n_m)}(x_1,\ldots,x_m) = f_{X(n_1+k),\ldots,X(n_m+k)}(x_1,\ldots,x_m)$$

Stationary Process

Theorem: A stationary process X(t),

$$\mu_X(t) = \mu_X$$
$$R_X(t,\tau) = R_X(0,\tau) = R_X(\tau)$$
$$C_X(t,\tau) = R_X(\tau) - \mu^2_X = C_X(\tau)$$

Stationary Random Sequence

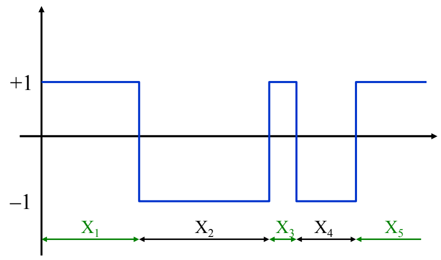
Theorem: A stationary random sequence X_n, for all m

$$E[X_m] = \mu_X$$
$$R_X[m.k] = R_X[0,k] = R_X[k]$$
$$C_X[m,k] = R_X[k] - \mu^2_X = C_X[k]$$

Example

- Telegraph Signal, $X(t)$ take value ± 1
- $X(0) = \pm 1$ with probability $= 0.5$
- Let $X(t)$ toggles the polarity with each occurrence of an event in a Poisson process rate α

Example



Example

- Find Pmf of $X(t)$, $f_{X(t)}(x)$
- $P[X(t)] = P[X(t) \mid X(0) = 1]\, P[X(0) = 1] + P[X(t) \mid X(0) = -1]\, P[X(0) = -1]$
- $P[X(t) \mid X(0) = 1] = P[N(t) = \text{even}]$

$$= \sum_{j=0}^{\infty} \frac{(\alpha t)^{2j}}{(2j)!} e^{-\alpha t}$$

$$= e^{-\alpha t} (1/2) (e^{\alpha t} + e^{-\alpha t})$$

$$= (1/2)(1 + e^{-2\alpha t})$$

Example

- $P[X(t) \mid X(0) = -1] = P[N(t) = \text{odd}]$

$$= \sum_{j=0}^{\infty} \frac{(\alpha t)^{2j+1}}{(2j+1)!} e^{-\alpha t}$$

$$= e^{-\alpha t} (1/2) (e^{\alpha t} - e^{-\alpha t})$$

$$= (1/2)(1 - e^{-2\alpha t})$$

Example

- $P[X(t) = 1]$

$$= P[X(t) \mid X(0) = 1]\, P[X(0) = 1] + P[X(t) \mid X(0) = -1]\, P[X(0) = -1]$$

$$= (1/2)(1 + e^{-2\alpha t})(1/2) + (1/2)(1 - e^{-2\alpha t})(1/2)$$

$$= 1/2$$

- $P[X(t) = -1]$

$$= 1 - P[X(t) = 1] = 1/2$$

$$f_{X(t)}(x) = \begin{cases} 1/2 & X(t) = -1, 1 \\ 0 & \text{Otherwise} \end{cases}$$

Example

- $\mu_X(t) = 1 * P[X(t) = 1] + -1 * P[X(t) = -1] = 1\,(1/2) + (-1)(1/2) = 0$
- $\text{Var}[X(t)] = E[X^2(t)] = 1^2(1/2) + (-1)^2(1/2) = 1$

Autocovariance, $C_X(t,\tau) = E[X(t)X(t+\tau)] - 0 = e^{-2\alpha\tau}$

$$f_{X(t_1),\dots,X(t_m)}(x_1,\dots,x_m) = f_{X(t_1+\tau),\dots,X(t_m+\tau)}(x_1,\dots,x_m)$$

Wide Sense Stationary

Definition: $X(t)$ is a wide sense stationary random process iff for all t,

$$E[X(t)] = \mu_X$$
$$R_X(t,\tau) = R_X(0,\tau) = R_X(\tau)$$

Definition: X_n is a wide sense stationary random sequence iff for all n,

$$E[X_n] = \mu_X$$
$$R_X[n,k] = R_X[0,k] = R_X[k]$$

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Wide Sense Stationary

- For every **stationary** process or sequence, it is also **wide sense stationary**.
- However, if it is a **wide sense stationary** it may or may not be stationary.

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Example

- Let $X_n = \pm 1$ with prob = ½ (n = even)
- For n = odd
  - $X_n = -1/3$ with prob = 9/10
  - $X_n = 3$ with prob = 1/10
- Stationary ?
  - No → Pmf varies with n
- Wide sense stationary ?
  - Mean = 0 for all n
  - $C_X(t,\tau) = 0$ for $\tau > 0$
  - $C_X(t,\tau) = E[X^2] = 1$ for $\tau = 0$
  - Yes , it's wide sense stationary

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Wide Sense Stationary

Theorem: For a wide sense stationary process $X(t)$,

$$R_X(0) \geq 0$$
$$R_X(\tau) = R_X(-\tau)$$
$$|R_X(\tau)| \leq R_X(0)$$

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Wide Sense Stationary

Theorem: For a wide sense stationary sequence X_n,

$$R_X[0] \geq 0$$
$$R_X[k] = R_X[-\tau]$$
$$|R_X[k]| \leq R_X[0]$$

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Average Power

- From Ohm's Law : $V = IR$
- For $v(t)$, $i(t)$, $R\ \Omega$, the instantaneous power dissipated, $P(t)$,

$$P(t) = v^2(t)/R = i^2(t)R$$

- For $R = 1\ \Omega$, $P(t) = v^2(t) = i^2(t)$
- For a voltage or current is a sample function of random process, $x(t,s)$
  - ➔ P across 1 Ω resistor = $x^2(t,s)$

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Average Power

- Define $x^2(t,s)$ as the instantaneous power of $x(t,s)$
- For a $X(t)$, $X^2(t)$ is the instantaneous of power $X(t)$

Definition: Average Power

The average power of a wide sense stationary process $X(t)$,

$$R_X(0) = E[X^2(t)]$$

Homework

- 6.2.3
- 6.3.1
- 6.4.1
- 6.5.4
- 6.7.2
- 6.8.4