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Probability and Random Processes
for Computer Engineering
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Multiple Continuous RV (1)

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Joint CDF

- Pairs of Random Variables
- Discrete:

  Joint PMF $P_{X,Y}(x,y) = P[X=x, Y=y]$
- Continuous:

  $P_{X,Y}(x,y) = 0$ ($P_X(x) = 0$, $P_Y(y) = 0$)

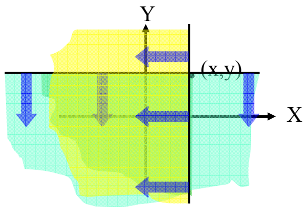
  For 1 RV → interval on real axis

  For 2 RVs → area in a plane

Joint CDF

Definition: Joint CDF of X and Y

$$F_{X,Y}(x,y) = P[X \le x, Y \le y]$$



Interesting Properties

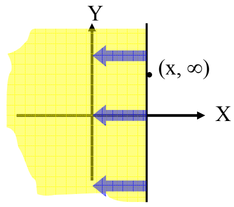
- For Event $\{X \le x\}$

$F_X(x) = P[X \le x]$

$= P[X \le x, Y \le \infty]$

$= \lim_{y \to \infty} F_{X,Y}(x,y)$

$= F_{X,Y}(x, \infty)$



Joint CDF

Theorem :

(a) $0 \le F_{X,Y}(x,y) \le 1$

(b) $F_X(x) = F_{X,Y}(x, \infty)$

(c) $F_Y(y) = F_{X,Y}(\infty, y)$

(d) $F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$

(e) If $x_1 \ge x$ and $y_1 \ge y$

then $F_{X,Y}(x_1,y_1) > F_{X,Y}(x,y)$

(f) $F_{X,Y}(\infty, \infty) = 1$

Joint PDF

Definition: Joint PDF of X and Y is satisfied

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)\, dv\, du$$

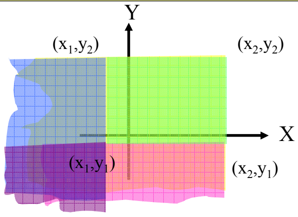
Theorem:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}$$

Joint CDF

Theorem:

$P[x_1 < X \le x_2, y_1 < Y \le y_2]$
$= F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1)$



Joint PDF

Theorem:

(a) $f_{X,Y}(x,y) \ge 0$ for all (x,y)

(b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$

Theorem:

$$P[A] = \iint_A f_{X,Y}(x,y)\,dx\,dy$$

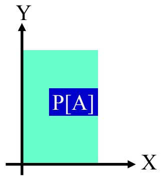
Example

$$f_{X,Y}(x,y) = \begin{cases} c & 0 \le x \le 3,\ 0 \le y \le 5 \\ 0 & \text{Otherwise} \end{cases}$$

Find constant c

$$\int_0^3\int_0^5 c\,dx\,dy = 15c = 1$$

$\rightarrow$ c = 1/15



Find P[A] = P[1 ≤ x ≤ 3, 2 ≤ y ≤ 3]

$$P[A] = \int_1^3\int_2^3 1/15\,dv\,du = 2/15$$

Marginal PDF

Theorem:

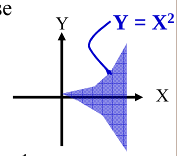
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

Example

$$f_{X,Y}(x,y) = \begin{cases} cx & 0 \le x \le 1,\ |y| < x^2 \\ 0 & \text{Otherwise} \end{cases}$$

Find constant c



$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} cx\,dx\,dy = \int_0^1\left(\int_{-x^2}^{x^2} cx\,dy\right)dx$$

$$= \int_0^1 cx\,(2x^2)\,dx = \left.\frac{cx^4}{2}\right|_0^1$$

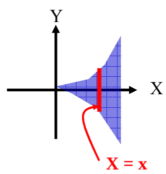
$$= \frac{c}{2} = 1 \quad \rightarrow c = 2$$

Example

Find the marginal PDF $f_X(x)$ and $f_Y(y)$

Fixed x (X = x) then integrate all y

$$f_X(x) = \int_{-x^2}^{x^2} 2x\,dy = 4x^3$$

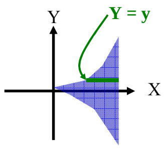


$$f_X(x) = \begin{cases} 4x^3 & 0 \le x \le 1 \\ 0 & \text{Otherwise} \end{cases}$$

Example

Fixed y (**Y = y**) then integrate all x

$$f_{Y,}(y) = \int_{\sqrt{|y|}}^{1} 2x\,dx = 1 - |y|$$



$$f_Y(y) = \begin{cases} 1 - |y| & -1 \le y \le 1 \\ 0 & \text{Otherwise} \end{cases}$$

Functions of 2 RVs

Example:

Wireless based station with 2 antennas. X and Y are RVs of the signal

- Find the strongest signal
  W = X if |X| > |Y| or W = Y otherwise
- Find the addition of 2 signals
  W = X + Y
- Find the addition of 2 signals with weight
  W = aX + bY

Functions of 2 RVs

$$F_W(w) = P[W \le w] = \iint_{g(x,y) \le w} f_{X,Y}(x,y)\,dx\,dy$$

Example

$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \le x \le 3,\ 0 \le y \le 5 \\ 0 & \text{Otherwise} \end{cases}$$

Find PDF of W = max(X,Y)

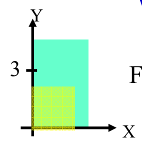
For W = max(X,Y) → {W ≤ w} = {X ≤ w, Y ≤ w}

$$F_W(w) = P[X \le w, Y \le w]$$

$$= \int_{-\infty}^{w}\int_{-\infty}^{w} f_{X,Y}(x,y)\,dx\,dy$$

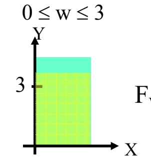
Example

We can divide into 2 cases



$0 \le w \le 3$

$$F_W(w) = \int_0^w\int_0^w 1/15\,dx\,dy = w^2/15$$

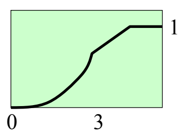


$3 \le w \le 5$

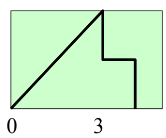
$$F_W(w) = \int_0^w \left(\int_0^3 1/15\,dx \right) dy = w/5$$

Example

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^2/15 & 0 \le w \le 3 \\ w/5 & 3 < w \le 5 \\ 1 & w > 5 \end{cases}$$



$$f_W(w) = \begin{cases} 2w/15 & 0 \le w \le 3 \\ 1/5 & 3 < w \le 5 \\ 0 & \text{Otherwise} \end{cases}$$



Expected Value

Theorem:

$E[W] = E[g(X,Y)]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

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Expected Value

Theorem:

For $g(X,Y) = g_1(X,Y) + \ldots + g_n(X,Y)$

$E[g(X,Y)] = E[g_1(X,Y)] + \ldots + E[g_n(X,Y)]$

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Expected Value

Theorem:

$E[\ X+Y\ ] = E[X] + E[Y]$

Find E[X] → From $f_{X,Y}(x,y)$ **Not the only way**

→ Can find from Marginal PDF $f_X(x)$

So, we can find Var[X+Y], Cov, $\rho_{X,Y}$

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Variance and Covariance

Theorem:

$Cov[\ X,Y\ ] = E[(X - \mu_X)(Y - \mu_Y)]$

$Cov[\ X,Y\ ] = E[XY] - \mu_X\mu_Y)]$

Theorem:

$Var[X+Y] = Var[X] + Var[Y] + 2\ Cov[X,Y]$

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Correlation Coefficient

Theorem:

$$\rho_{X,Y} = \frac{Cov[\ X,Y\ ]}{\sqrt{Var[X]\ Var[Y]}}$$

Theorem:

$-1 \le \rho_{X,Y} \le 1]$

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Conditioning Joint PDF by Event

Definition:

$$f_{X,Y|B}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{P[B]} & (x,y) \in B \\ 0 & \text{Otherwise} \end{cases}$$

For $P[B] > 0$

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Conditional PDF

Definition:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Theorem:

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)\, f_X(x) = f_{X|Y}(x|y)\, f_Y(y)$$

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Conditional Expected Value

Theorem: for $f_Y(y) > 0$

$$E[X|Y=y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\, dx$$

Theorem: for $f_Y(y) > 0$

$$E[g(X,Y)|Y=y] = \int_{-\infty}^{\infty} g(x,y) f_{X|Y}(x|y)\, dx$$

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Independent RVs

Definition: X and Y are independent iff

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

Example:

$$f_{X,Y}(x,y) = \begin{cases} 4xy & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{Otherwise} \end{cases}$$

Are X and Y independent ?

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{Otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{Otherwise} \end{cases}$$

For **all pairs** are true as definition → X and Y are independent

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Jointly Gaussian RV

Definition: Bivariate Gaussian RV

$$f_{X,Y}(x,y) = \frac{\exp\left[-\dfrac{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho\,(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)}\right]}{2\pi\,\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

μ_1 and $\mu_2 \in$ real number, $\sigma_1 > 0$, $\sigma_2 > 0$ and $-1 \le \rho \le 1$

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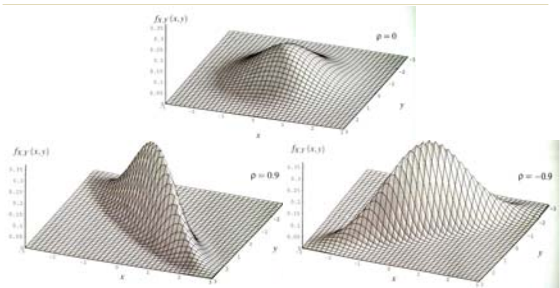
Jointly Gaussian RV



For $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$ and $\rho = 0, 0.9, -0.9$

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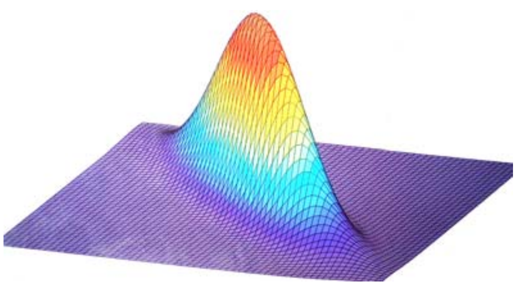
Jointly Gaussian RV



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Jointly Gaussian RV

$$f_{X,Y}(x,y) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{\frac{-(x-\mu_1)^2}{2\sigma_1^2}} \frac{1}{\tilde{\sigma}_2\sqrt{2\pi}} e^{\frac{-(y-\tilde{\mu}_2(x))^2}{2\tilde{\sigma}_2^2}}$$

Theorem:

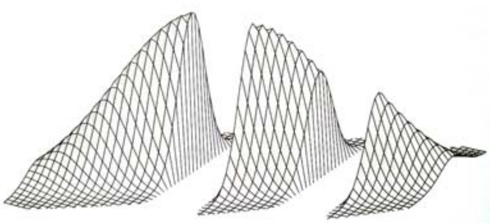
$$f_X(x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{\frac{-(x-\mu_1)^2}{2\sigma_1^2}} \qquad f_Y(y) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{\frac{-(y-\mu_2)^2}{2\sigma_2^2}}$$

Bivariate Gaussian RV

Theorem: Conditional PDF of Y given X

$$f_{Y|X}(y|x) = \frac{1}{\tilde{\sigma}_2\sqrt{2\pi}} e^{\frac{-(y-\tilde{\mu}_2(x))^2}{2\tilde{\sigma}_2^2}}$$

Joint Gaussian PDF



For $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$ and $\rho = 0.9$

$f_{Y|X}(y|x)$ = Gaussian ➔ Bell shape cross section

More Than 2 RVs

- 2 RVs → **Bivariate** Joint PDF
- > 2 RVs → **Multivariate** Joint PDF