204312
Probability and Random Processes
for Computer Engineering
1st semester 2004 (June – September)

Continuous Random Variables (4)

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Mixed Random Variable

- Discrete RV → PMF & Summation
- Continuous RV → PDF & Integral
- Mixed RV
  - → **Combination of Discrete & Continuous RV**
  - → **Can use same formulas to describe both RVs**
  - → **Unit impulse function**

Unit Impulse Function

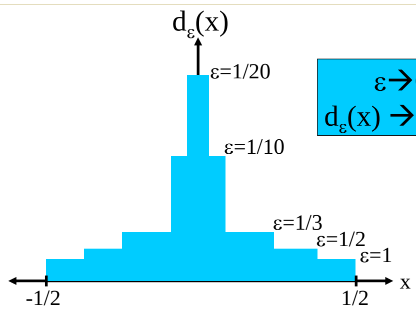
- Delta Function : $\delta(x)$

Definition:

Let $d_\varepsilon(x) = \begin{cases} 1/\varepsilon & -\varepsilon/2 \le x \le \varepsilon/2 \\ 0 & \text{Otherwise} \end{cases}$

Then $\delta(x) = \lim_{\varepsilon \to 0} d_\varepsilon(x)$

Delta Function



$\varepsilon \to 0$
$d_\varepsilon(x) \to \delta(x)$

No mathematically meaning **but very useful**

Delta Function

$$\int_{-\infty}^{\infty} d_\varepsilon(x)\,dx = \int_{-\varepsilon/2}^{\varepsilon/2} \frac{1}{\varepsilon}\,dx = 1$$

As $\varepsilon \to 0$, $d_\varepsilon(x) \to \delta(x)$

→ can conclude that

$$\boxed{\int_{-\infty}^{\infty} \delta(x)\,dx = 1}$$

Sifting Property

1 0

Theorem: (Sifting Property of delta function)

$$\int_{-\infty}^{\infty} g(x)\,\delta(x-x_0)\,dx = g(x_0)$$

Special case of

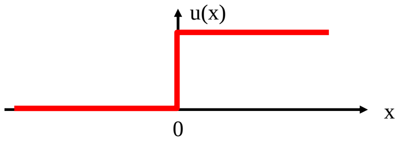
$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

Unit Step Function

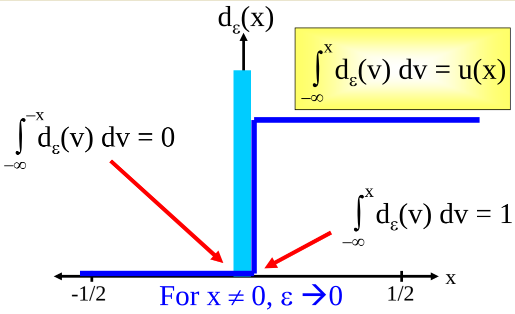
• Delta function ←→ unit step function

Definition:

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



Unit Step & Delta Function



Unit Step & Delta Function

Theorem:

$$\int_{-\infty}^{x} \delta(v)\,dv = u(x)$$

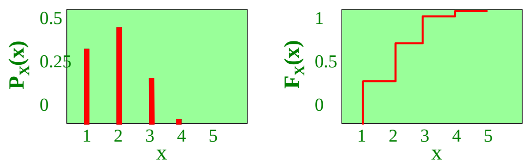
For x = 0 ??

Not exist

$$\delta(x) = \frac{d\,u(x)}{dx}$$

PMF → PDF

$$F_X(x) = \sum_{x_i \in S_X} P_X(x_i)\,u(x-x_i)$$

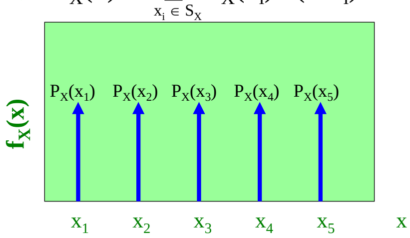


$u(x-x_i) \to u(x)$ shift to x_i

PMF → PDF

$$F_X(x) = \sum_{x_i \in S_X} P_X(x_i)\,u(x-x_i)$$

$$f_X(x) = \sum_{x_i \in S_X} P_X(x_i)\,\delta(x-x_i)$$



PMF → PDF

$$f_X(x) = \sum_{x_i \in S_X} P_X(x_i)\,\delta(x-x_i)$$

$$E[X] = \int_{-\infty}^{\infty} x \sum_{x_i \in S_X} P_X(x_i)\,\delta(x-x_i)\,dx$$

$$= \sum_{x_i \in S_X} \int_{-\infty}^{\infty} x\,P_X(x_i)\,\delta(x-x_i)\,dx$$

$$= \sum_{x_i \in S_X} x_i\,P_X(x_i)$$

Sifting Property

PMF ←→ PDF

Theorem :

For a RV X → the equivalent statements:

- $P[X = x_0] = q$
- $P_X(x_0) = q$
- $F_X(x_0^+) - F_X(x_0^-) = q$ Discontinuity at x_0
- $f_X(x_0) = q\,\delta(0)$

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Mixed Random Variable

Definition: **X** is a mixed RV iff

$\mathbf{f_x(x)}$ = both impulses and nonzero, finite values

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Example

- Observe the period of telephone call
  - 1/3 of calls : never begin (no answer/busy)
  - For the success call, with probability of 2/3, call is uniformly [0,3]
- Find PDF, CDF and Mean of call holding time

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Example

- Y: call holding time
- A: phone was answered → A^c: not answered
- $0 \le y \le 3$
- $F_Y(y) = P[Y \le y]$

$$= \overset{1}{P[Y\le y|A^c]}\,\overset{1/3}{P[A^c]} + \overset{y/3}{P[Y\le y|A]}\,\overset{2/3}{P[A]}$$

$$= (1)(1/3) + (y/3)(2/3)$$

$$= 1/3 + 2y/9$$

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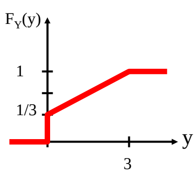
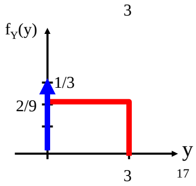
Example

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 1/3 + 2y/9 & 0 \le y \le 3 \\ 0 & \text{Otherwise} \end{cases}$$



$$f_Y(y) = \begin{cases} \delta(y)/3 + 2/9 & 0 \le y \le 3 \\ 0 & \text{Otherwise} \end{cases}$$



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Example

$$E[Y] = \int_{-\infty}^{\infty} y\,(1/3)\delta(y)\,dy + \int_0^3 y\,(2/9)\,dy$$

$$= 0 + 1 \quad = 1$$

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Derive Random Variable

Y = aX

$F_Y(y) = P[aX \le y] = P[X \le y/a] = F_X(y/a)$

$$f_Y(y) = \frac{d\,F_Y(y)}{dy} = (1/a)\, f_X(y/a)$$

Theorem :

- $F_Y(y) = F_X(y/a)$
- $f_Y(y) = (1/a)\, f_X(y/a)$

Derive Random Variable

Y = X + b

$F_Y(y) = P[X + b \le y] = P[X \le y - b] = F_X(y - b)$

$$f_Y(y) = \frac{d\,F_Y(y)}{dy} = f_X(y - b)$$

Theorem :

- $F_Y(y) = F_X(y - b)$
- $f_Y(y) = f_X(y - b)$

Conditioning a continuous RV

$P[A|B] = P[AB] / P[B]$

$$P[x_1 < X \le x_2] = \int_{x_1}^{x_2} f_X(x)\,dx$$

Approx: $P[x < X \le x+dx] = f_X(x)\,dx$

$$f_{X|B}(x)\,dx = P[x < X \le x+dx\,|B] = \frac{P[x < X \le x+dx\,,B]}{P[B]}$$

$$= \frac{P[x < X \le x+dx]}{P[B]} \quad \leftarrow x \in B,\ x+dx \in B$$

$$= \frac{f_X(x)\,dx}{P[B]}$$

Conditioning a continuous RV

$$f_{X|B}(x)\,\cancel{dx} = \frac{f_X(x)\,\cancel{dx}}{P[B]}$$

Definition:

$$f_{X|B}(x) = \begin{cases} \dfrac{f_X(x)}{P[B]} & x \in B \\ 0 & \text{Otherwise} \end{cases}$$

Definition:

$$E[g(X)|B] = \int_{-\infty}^{\infty} g(x)\, f_{X|B}(x)\,dx$$

Example

- Observe the period of telephone call (T) is an **exponential RV** with expected value 3 min.
- Find E[T|T>2]
- **Solution**:

$$f_T(t) = \begin{cases} (1/3)\, e^{-t/3} & t \ge 0 \\ 0 & \text{Otherwise} \end{cases}$$

$$P[T > 2] = \int_2^{\infty} f_T(t)\,dt = e^{-2/3}$$

Example

$$f_{T|T>2}(t) = \begin{cases} f_T(t) / P[T > 2] & t \ge 0 \\ 0 & \text{Otherwise} \end{cases}$$

$$= \begin{cases} (1/3)\, e^{-(t-2)/3} & t \ge 0 \\ 0 & \text{Otherwise} \end{cases}$$

$$E[T \mid T > 2] = \int_2^{\infty} t\,(1/3)\, e^{-(t-2)/3}\,dx$$

$= 5$ min.

Homework

- 4.1.1
- 4.2.2
- 4.3.5
- 4.4.4
- 4.5.6
- 4.6.6
- 4.6.9
- 4.7.6
- 4.7.10
- 4.7.13
- 4.8.2
- 4.8.4

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