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Probability and Random Processes
for Computer Engineering
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Introduction to Probability (2)

ผศ.ดร. อนันต์ ผลเพิ่ม

Asst.Prof.Anan Phonphoem, Ph.D.
anan@cpe.ku.ac.th
http://www.cpe.ku.ac.th/~anan
Computer Engineering Department
Kasetsart University, Bangkok, Thailand

Conditional Probability

- If we know P[A] before an experiment
  - P[A] ≈ 1
  - P[A] ≈ 0
  - P[A] ≈ ½
- P[A] is a **priori** probability of A

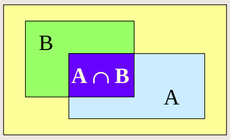
Conditional Probability

- In practice, it maybe impossible to find the precise outcome of an experiment. Rather the outcome s_i itself.
- We know that the outcome is in set B → Event B has occurred (B has many outcomes)

Conditional Probability

- **Notation:** **P[A|B]**
  - "Probability of A given B"
  - The condition probability of the event A given the occurrence of the event B
- **Definition:**

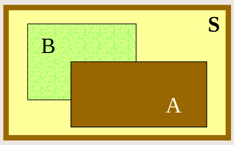
$$P[A|B] = \frac{P[AB]}{P[B]}$$



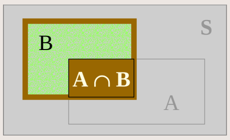
- **Example:**
  - Tiger Woods hits Hole-in-one

More Explanation

$$P[A|S] = \frac{P[AS]}{P[S]} = \frac{P[A]}{1} = P[A]$$

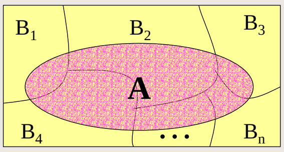


$$P[A|B] = \frac{P[AB]}{P[B]}$$



Law of Total Probability

- Let $B_1, B_2, \ldots, B_n$ be mutual exclusive events whose union equals sample space S
- $P[B_i] > 0$



Theorem: $P[A] = \sum_{i=1}^{n} P[A \cap B_i]$

$P[A] = P[A \cap B_1] + P[A \cap B_2] + \ldots$

$P[A] = P[A|B_1]P[B_1] + P[A|B_2]P[B_2] + \ldots$

Theorem: $P[A] = \sum_{i=1}^{n} P[A|B_i]P[B_i]$

Bayes' Theorem

$$P[B|A] = \frac{P[BA]}{P[A]} \qquad P[A|B] = \frac{P[AB]}{P[B]}$$

$$= \frac{P[A|B]P[B]}{P[A]}$$

Theorem: $$\mathbf{P[B|A] = \frac{P[A|B]P[B]}{P[A]}}$$

2 Independent Events

Definition: Event A and B are independent iff
P[AB] = P[A]P[B]

$$P[A|B] = \frac{P[AB]}{P[B]}$$

$$= \frac{P[A]P[B]}{P[B]}$$

P[A|B] = P[A]

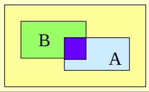
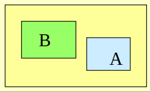
P[B|A] = P[B]

Independent Interpretation

P[A] = 0.3
P[A|B] = 0.3

No matter event B occurs or not, event A is not affected

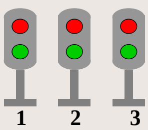
Independent VS. Disjoint

Independent	Disjoint
	
$P[AB] \neq 0$	$P[AB] = 0$
$P[A \cap B] = P[A] * P[B]$	$P[A \cup B] = P[A] + P[B]$

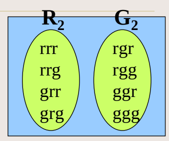
Note: Independent = Disjoint iff P[A]=0 or P[B]=0

Independent Example

- 3 traffic lights, observe a sequence of lights
  - The sequence is equally likely
  - R_1 = The first light was red
  - R_2 = The second light was red
  - G_2 = The second light was green
- Are R_2 and G_2 independent?
- Are R_1 and R_2 independent?



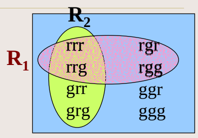
Independent Example



- **Sample Space:**
  - S = {rrr,rrg,rgr,rgg,grr,grg,ggr,ggg}
- **Are R_2 and G_2 independent?**
  - •$P[R_2]$= P[{rrr,rrg,grr,grg}] = 4/8 = ½
  - •$P[G_2]$= P[{rgr,rgg,ggr,ggg}] = ½
  - •$P[R_2G_2] = 0$
  - •$P[R_2]P[G_2]$ = (½)* (½) = ¼
  - →R_2 and G_2 are not independent
  - →R_2 and G_2 are disjoint

Independent Example

- **Are R_1 and R_2 independent?**



- $P[R_1]$= P[{rrr,rrg,rgr,rgg}] = ½
- $P[R_2]$= P[{rrr,rrg,grr,grg}] = ½
- $P[R_1R_2]$ = P[{rrr,rrg}] = 2/8 = 1/4
- $P[R_1]P[R_2]$ = (½) * (½) = ¼

→R_1 and R_2 are independent

→R_1 and R_2 are not disjoint

3 Independent Events

Definition: Event A_1,A_2 and A_3 are independent iff

1) A_1 and A_2 are independent
2) A_2 and A_3 are independent
3) A_1 and A_3 are independent
4) $P[A_1 \cap A_2 \cap A_3] = P[A_1]\, P[A_2]\, P[A_3]$

3 Independent Events

Note:

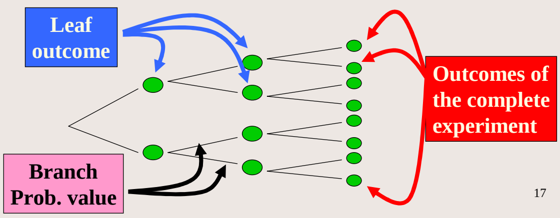
- **Independence in pairs (number 1-3) may not independent**
  - P[A]=P[B]=P[C]=1/5
  - P[AB]=P[AC]=P[BC]=P[ABC] = 1/25
- **Only number 4) is insufficient to guarantee the independence**
  - Ex.: One of the event is Null

Most Common Application of Independence

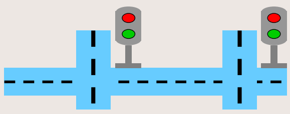
- **Assume that the event of separate experiments are independent**
- Example:
  - Assume that outcome of a coin toss is independent of the outcomes of all prior and all subsequent coin tosses
  - P[H] = P[T] = ½
  - P[HTH] = P[H]P[T]P[H] = 1/2*1/2*1/2 = 1/8

Sequential Experiments

- Experiment → subexperiments →subexperiments
- Each subexp. may depend on the previous one
- Represented by a **Tree Diagram**
- **Model Conditional Prob. → Sequential Experiment**

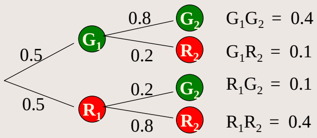


Sequential Example



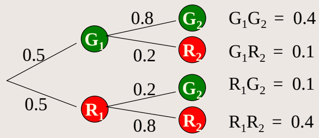
- Timing coordination of 2 traffic lights
  - P[the second light is the same color as the first] = 0.8
  - Assume 1st light is equally likely to be green or red
- Find P[The second light is green] ?
- Find P[wait for at least one light] ?

Sequential Example



- $P[G_1] = P[R_1] = 0.5$
- $P[G_2G_1] = P[G_2|G_1]P[G_1] = (0.8)(0.5) = 0.4$

Sequential Example

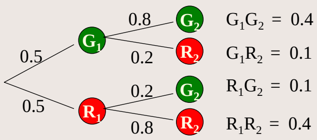


P[The second light is green] ?

$P[G_2] = P[G_2G_1] + P[G_2R_1] = 0.4 + 0.1 = 0.5$

$P[G_2] = P[G_2|G_1]P[G_1] + P[G_2|R_1]P[R_1]$

Sequential Example



P[wait for at least one light] ?

$W = \{G_1R_2 \cup R_1G_2 \cup R_1R_2\}$

$P[W] = P[G_1R_2] + P[R_1G_2] + P[R_1R_2]$

$= 0.1 + 0.1 + 0.4 = 0.6$

Principle of Counting Method

If experiment A has **n** possible outcomes,
and experiment B has **k** possible outcomes,

→Then there are **nk** possible outcomes
when you perform both experiments

Principle of Counting Method

Example: Shuffle a deck and select 3 cards in order
How many outcomes?

1st draw: select 1 out of 52 → 52 outcomes
2nd draw: select 1 out of 51 (one card has been drawn)
→ 51 outcomes
3rd draw: select 1 out of 50 → 50 outcomes

Total outcomes = (52)(51)(50)

k-permutations

Theorem:
The number of **k**-permutations
(order sequence) of **n** distinguishable objects is

$(n)_k = n(n-1)(n-2)\ldots(n-k+1)$

$= n(n-1)(n-2)\ldots(n-k+1)\dfrac{(n-k)!}{(n-k)!}$

$= \dfrac{n(n-1)(n-2)\ldots(n-k+1)\,(n-k)(n-k-1)\ldots(1)}{(n-k)!}$

$$(n)_k = \frac{n!}{(n-k)!}$$

Choose with replacement

Theorem: Given **n** distinguishable objects,
There are **n^k** ways to choose with replacement a sample of **k** objects

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k-combination

Theorem:
The number of ways to choose **k** objects out of **n** distinguishable objects is

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}$$

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Independent Trials

- Perform repeated trials
- **p** = a success probability
- **(1-p)** = a failure probability
- Each trial is independent
- $\mathbf{S_{k,n}}$ = the event that **k** successes in **n** trials

$$P[S_{k,n}] = \binom{n}{k} p^k(1-p)^{n-k}$$

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Independent Trials

Example: In the first round of a programming contest, probability that program will pass the test is 0.8. From 10 candidates, what is the probability that x candidates will pass?
$P[x = 8]$?
Solution:

A = {program pass the test}, $P[A] = 0.8$
Testing a program is an independent trial
$P[A_{x,10}] = \binom{10}{x}(0.8)^x(1-0.8)^{10-x}$
$P[A_{8,10}] = (45)(0.1678)(0.04) = 0.3$

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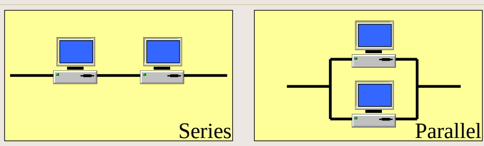
Independent Trials: Reliability



Let probability that a computer works = p
Series: $P[A] = P[A_1A_2] = p^2$
Parallel: $P[B]$ = ?

$$P[B] = 1 – P[B^c]$$
$$= 1 – P[B_1^cB_2^c]$$
$$= 1 – (1 – p)^2$$

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Summary

- Probability meaning
- Sample space, Event, Outcome
- Set Theory
- Probability measurement
- Conditional Probability
- Independence
- Sequential experiments → tree diagram
- Counting Methods
- Independent Trials.

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Homework #1

1) 1.2.2
2) 1.3.2
3) 1.3.5
4) 1.4.3
5) 1.5.5
6) 1.6.6
7) 1.7.5
8) 1.7.10
9) 1.8.6
10) 1.9.7

Due date: Tuesday June 15, In class.