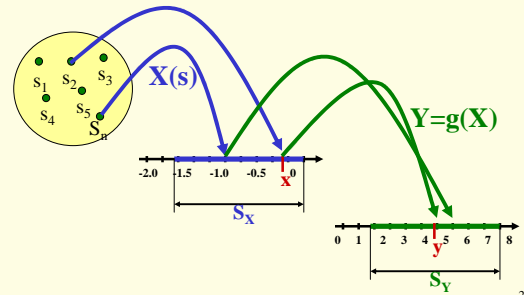


Discrete Random Variable (4)

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Derived Random Variable



Why do we need a Derived Random Variable?

- From sample values of the random variable, use these values to compute other quantities.
- Example:
 - Find a decibel value form signal-to-noise ratio
- $Y = g(X)$

Example-1

- Random Variable $X = \#$ pages in one fax
- $P_X(x) =$ number of pages in each fax
- Charging plan
 - 1st page = 10 Baht
 - 2nd page = 9 Baht
 - ...
 - 5th page = 6 Baht
 - 6 – 10 pages = 50 Baht
- Find the charge in Baht for sending one fax

Example-1

- Random Variable $Y =$ the charge in Baht for sending one fax

$$Y = g(X) = \begin{cases} 10.5X - 0.5X^2 & 1 \leq X \leq 5 \\ 50 & 6 \leq X \leq 10 \end{cases}$$

PMF of Y

Theorem:

$$P_Y(y) = \sum_{x:g(x)=y} P_X(x)$$

$P[Y=y] = \Sigma$ of all outcomes $X = x$ for which $Y = y$

Example-2

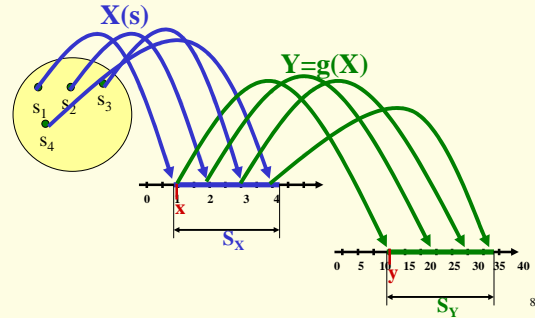
- Suppose all faxes contain 1,2,3, or 4 pages with equal probability
- Find PMF and E[Y]

$$P_X(x) = \begin{cases} \frac{1}{4} & x = 1,2,3,4 \\ 0 & \text{Otherwise} \end{cases}$$

- $S_X = \{1,2,3,4\}$
- $S_Y = \{10,19,27,34\}$

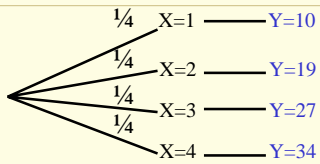
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Example-2



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Example-2



$$P_Y(y) = \begin{cases} \frac{1}{4} & y = 10,19,27,34 \\ 0 & \text{Otherwise} \end{cases}$$

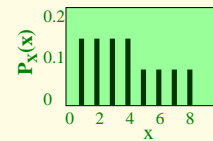
$$E[Y] = (1/4)(10 + 19 + 27 + 34) = 22.7 \text{ Baht}$$

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Example-3

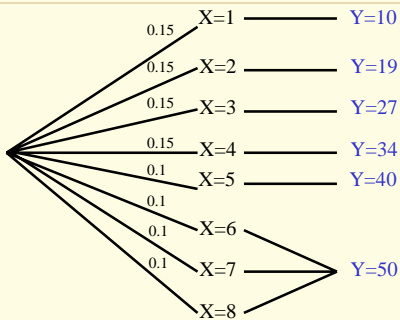
$$P_X(x) = \begin{cases} 0.15 & x = 1,2,3,4 \\ 0.1 & x = 5,6,7,8 \\ 0 & \text{Otherwise} \end{cases}$$

- Find PMF and E[Y]



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Example-3

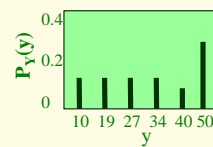


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Example-3

From Theorem

$$P_Y(y) = \begin{cases} 0.15 & y = 10,19,27,34 \\ P_Y(5) = 0.1 & y = 40 \\ P_Y(6)+P_Y(7)+P_Y(8) = 0.3 & y = 50 \\ 0 & \text{Otherwise} \end{cases}$$



$$E[Y] = 0.15(10+19+27+34) + 0.1(40) + 0.3(50) = 32.5 \text{ Baht}$$

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Expected Value of a Derived Random Variable

- To Find $E[Y]$
 - Find $P_Y(y)$
 - Find $E[Y]$
- In case of interesting only $E[Y]$

Theorem:

$$E[Y] = \sum_{x \in S_X} g[X]P_X(x)$$

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Example-4

$$P_X(x) = \begin{cases} 1/4 & x = 1, 2, 3, 4 \\ 0 & \text{Otherwise} \end{cases}$$

$$Y = g(X) = \begin{cases} 10.5X - 0.5X^2 & 1 \leq X \leq 5 \\ 50 & 6 \leq X \leq 10 \end{cases}$$

Theorem: $E[Y] = \sum_{x \in S_X} g[X]P_X(x)$

$$E[Y] = [10.5(1) - 0.5(1)^2][1/4] + [10.5(2) - 0.5(2)^2][1/4] + [10.5(3) - 0.5(3)^2][1/4] + [10.5(4) - 0.5(4)^2][1/4] = 22.5 \text{ Baht}$$

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Important Property of $E[Y]$

Theorem: For $P_X(x)$,

$$E[aX + b] = aE[X] + b$$

Note:

- Linear Transformation
- scale change of quantity (change the unit)
Ex. Celsius → Fahrenheit
- Adding score to every one
→ new $E[X] = \text{old } E[X] + \text{adding value}$
- $Y = X^2 \rightarrow E[Y] \neq (E[X])^2 \rightarrow E[g(X)] \neq g(E[X])$

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Example

$$P_R(r) = \begin{cases} 1/4 & r = 0 \\ 3/4 & r = 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $E[Y]$ for $Y = g(R) = 2R + 4$

$$E[R] = (1/4)(0) + (3/4)(2) = 3/2$$

$$E[Y] = E[g(R)] = E[2R + 4] = 2E[R] + 4 = 2(3/2) + 4 = 7$$

$$E[Y] = \sum g(R)P_R(r) = g(0)(1/4) + g(2)(3/4) = (2*0+4)(1/4) + (2*2+4)(3/4) = 1 + 6 = 7$$

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Important Property of $E[Y]$

From class average, am I doing OK?

Let $Y = g(X) = X - E[X]$

$$E[Y] = E[g(X)]$$

$$= \sum_{x \in S_X} (x - \mu_X) P_X(x)$$

$$= \sum_{x \in S_X} x P_X(x) - \sum_{x \in S_X} \mu_X P_X(x)$$

$$= \mu_X - \mu_X \sum_{x \in S_X} P_X(x)$$

$$= 0$$

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Important Property of $E[Y]$

Theorem: Form any random variable X

$$E[X - \mu_X] = 0$$

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Variance & Standard Deviation

- We knew average, $E[X]$,
Why do we need these Variance & Standard Deviation?
- How far from the average?
- $T = X - \mu_x$
 $E[T] = E[X - \mu_x]$
 $= 0$

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Variance & Standard Deviation

- The useful measurement is $E[|T|]$
- $E[T^2] = E[(X - \mu_x)^2] \rightarrow$ **Variance**

Definition:

$$\text{Var}[X] = E[(X - \mu_x)^2]$$

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Variance & Standard Deviation

Definition:

$$\sigma_x = \sqrt{\text{Var}[X]}$$

Sigma X

- $\sigma_x \rightarrow \mu_x$
- Ex. $\sigma_x = 15$, Score +6 from mean
 \rightarrow OK. Middle of class
- Ex. $\sigma_x = 3$, Score +6 from mean
 \rightarrow V.Good In Top class group

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Variance of X

$$\begin{aligned} \text{Var}[X] &= \sigma_x^2 \\ &= E[(X - \mu_x)^2] \\ &= \sum_{x \in S_x} (x - \mu_x)^2 P_X(x) \\ &= \sum_{x \in S_x} x^2 P_X(x) - 2 \sum_{x \in S_x} \mu_x x P_X(x) + \sum_{x \in S_x} \mu_x^2 P_X(x) \\ &= E[X^2] - 2 \mu_x \sum_{x \in S_x} x P_X(x) + \mu_x^2 \sum_{x \in S_x} P_X(x) \\ &= E[X^2] - 2 \mu_x^2 + \mu_x^2 \end{aligned}$$

$$\text{Var}[X] = E[X^2] - \mu_x^2 = E[X^2] - (E[X])^2$$

Variance of X

Theorem:

- If X always takes the same value,
 $\text{Var}[X] = 0$
- If $Y = X + b$,
 $\text{Var}[Y] = \text{Var}[X]$
- If $Y = aX$,
 $\text{Var}[Y] = a^2 \text{Var}[X]$

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Variance

- (a) Bernoulli - $p \rightarrow \text{Var}[X] = p(1 - p)$
- (b) Geometric - $p \rightarrow \text{Var}[X] = (1 - p)/p^2$
- (c) Binomial - $n, p \rightarrow \text{Var}[X] = np(1 - p)$
- (d) Pascal - $k, p \rightarrow \text{Var}[X] = k(1 - p)/p^2$
- (e) Poisson - $\alpha \rightarrow \text{Var}[X] = \alpha$
- (f) Discrete - k, l
 $\rightarrow \text{Var}[X] = (1 - k)(1 - k + 2)/12$

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Conditional Probability Mass Function

$$P[A|B] = P[X = x|B]$$

Definition: Given event B , $P[B] > 0$

$$P_{X|B}(x) = P[X=x|B]$$

Theorem:
$$P[A] = \sum_{i=1}^n P[A|B_i]P[B_i]$$

Theorem:
$$P_X(x) = \sum_{i=1}^n P_{X|B_i}(x)P[B_i]$$